Matching in Bipartite Graphs

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17 November 2015

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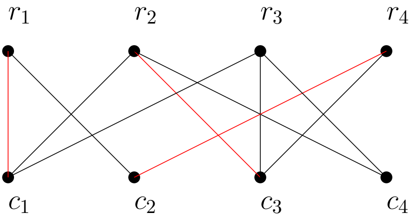
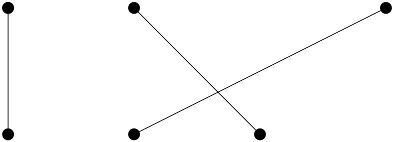
Our goal is to assign (some of) the resources to customers such that each customer receives one resource from its neighborhood.

In other words, we would like to match each customer with a resource.

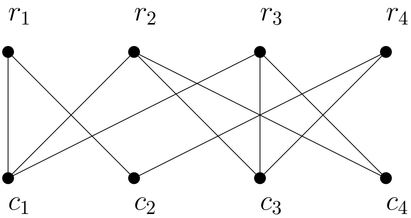
A matching in graph G is the set $M \subseteq E$ s.t for every $e, f \in M$, e, f do not have a node in common.

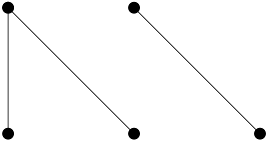




$$M_1 = \{c_1r_1, c_2r_4, c_3r_2\}$$

M_1 is a matching.

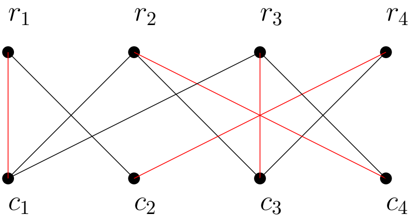




$$M_2 = \{c_1r_1, c_2r_1, c_3r_2\}$$

M_2 is not a matching.

We say a matching M is perfect (with respect to C) if it contains all the vertices in C.

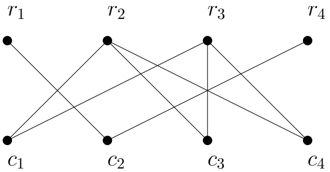


We say a matching M is maximum (with respect to C) if $|C \cap M|$ is maximum (has maximum number of edges).

A necessary condition for having a perfect matching

Hall's condition

If there is a set of customer C' such that in their neighborhood there is not enough resources. Then there is no perfect matching.



$C' = \{c_1, c_3, c_4\}$ $\qquad N(C') = \{r_2, r_3\}$

In the Figure $|C'| = 3$, $N(C') = \{r_2, r_3\}$ and $|C'| > |N(C')|$. Therefore there is no perfect matching.

If the Hall's condition satisfies for every subset of C then there is a perfect matching in G.

$$\forall C' \subseteq C, |C'| \leq |N(C')|$$

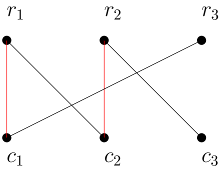
$N(C')$ is the neighborhood of C'. All the elements that have at least one neighbor in C'.

An exchanging and expanding idea

Suppose we have a matching M and it has $c_1 r_1$ and $c_2 r_2$. Suppose $c_3 r_2, c_2 r_1, c_1 r_3$ are edges of G.

Then we can start from c_3 and go to r_2 then to c_2 and then to r_1 and then to c_1 and then to r_3.

We can replace $r_1 c_1$ and $r_2 c_2$ by $c_3 r_2, c_2 r_1, c_1 r_3$.



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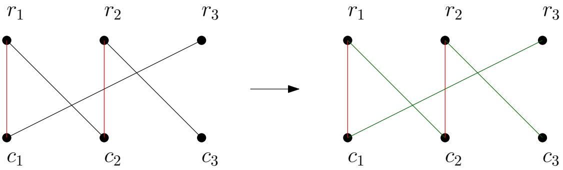

$M = \{c_1 r_1, c_2 r_2\}$ $\qquad$ $M_{new} = \{c_3 r_2, c_2 r_1, c_1 r_3\}$

Let M be a matching. We say a vertex is matched if it belongs to an edge $e \in M$.

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We start with an unmatched node u and find an alternating path P from u to another unmatched node v such that :

First edge of P is not in M and second edge of P is in M and the third edge of P is not in M and so on. The last edge of P is not in M.

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If P exists then we set $M' = (P - M) \cup (M - P)$.
M' has one edge more than M.

Matching(G=(C,R,E))

1) Start with a matching with one edge. As long as you can add edge into it (without violating the definition)

2) Start with a vertex v in C that is not matched in M

3) Find an alternating path P form v to some other un matched vertex u (use modified version of DFS algorithm)

4) If P exists then set $M' = (P - M) \cup (M - P)$ and go to (2)

5) Else there is no perfect matching

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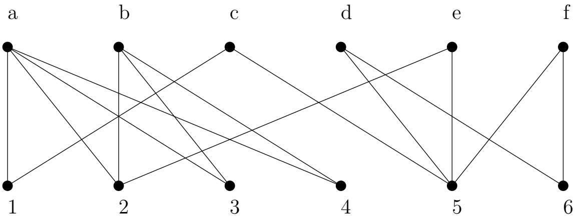
Running time $\mathcal{O}(nm)$ (m is the number of edges) when we use link list. The reason is we consider every unmatched vertex v (there are at most n nodes) and for that vertex we run DFS with $\mathcal{O}(m)$.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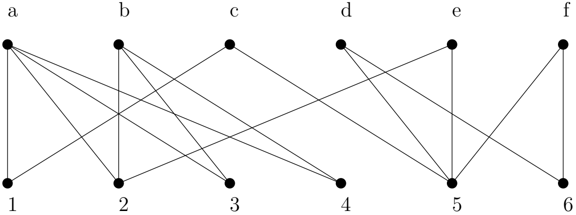
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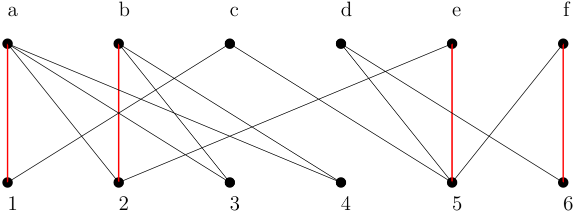
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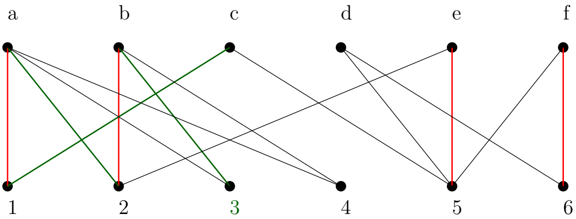
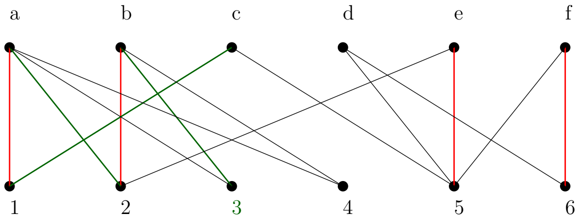


Start with unmatched 3 P=3,b,2,a,1,c



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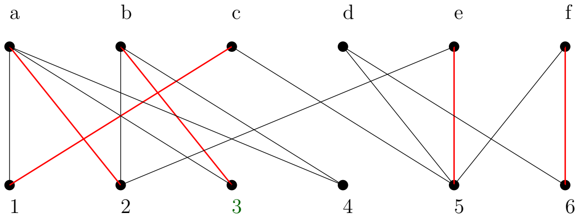
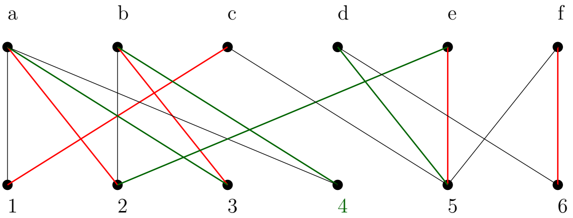
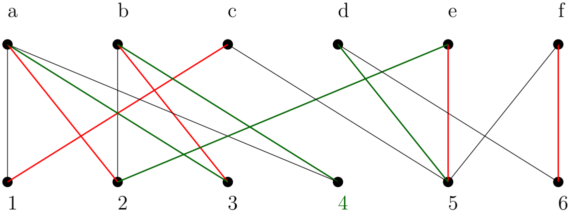


Apply $M = (M - P) \cup (P - M)$

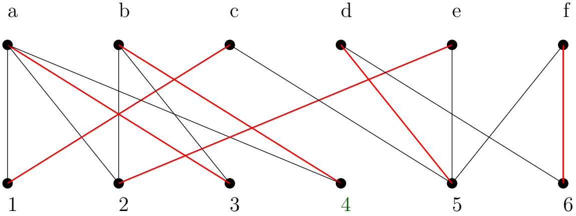


Start with unmatched 4

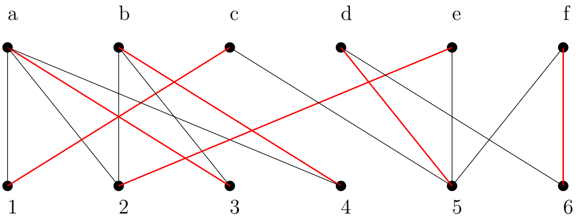
P=4,b,3,a,2,e,5,d



Start with unmatched 4 P=4,b,3,a,2,e,5,d



Appy $M = (P - M) \cup (M - P)$



M is perfect

Matching(G=(C,R,E)

0. Set $M = \emptyset$.

1. **for** (every edge $uv \in E$)
2. **if** ($u \notin M$ and $v \notin M$)
3. $M = M \cup \{uv\}$.

4. **for** (every $u \in C$ where $u \notin M$)
5. FOUND=0; $P = \emptyset$;
6. Au-DFS(u,1);
7. **if** (FOUND==1)
8. $M = (M - P) \cup (P - M)$

Au-DFS (u,flag)

if (flag==2 and $u \notin M$)
    add u to P and set FOUND=1; // a path found
    exit;
Explore[u]=1; add u to P.
if (flag==1)
    for (every $w \in N(u)$ && $uw \notin M$)
      if (Explore[w]==0)
        Au-DFS(w,2);
        remove w from P.
if (flag==2)
    for (every $w \in N(u)$ && $uw \in M$)
      if (Explore[w]==0)
        Au-DFS(w,1);
        remove w from P.

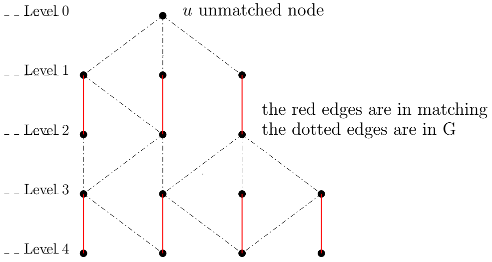
Theorem

Bipartite graph $G = (C, R, E)$ has a perfect matching with respect to C if and only if the Hall's condition is satisfied, i.e. $\forall X \subseteq C, |X| \leq |N(X)|$.

Proof :

If there exists a perfect matching in G w.r.t. C then clearly for each subset C' of C, we have $|C'| \leq |N(C')|$. Now suppose the Hall's condition satisfied for every subset C' of C. We show that at each step of the algorithm there exists an augmenting (alternating) path. Suppose there exits no augmenting path P for unmatched vertex u with partial matching M.

Au-DFS algorithm explores a tree T from node u. u is on level 0. All the neighbors of u (on level 1) are matched with a unique edge in M otherwise there exists P. The vertices on level 3 are also matched by new edges in M. In general, for each new node w on an odd level of T there exists a new edge in M that matches w. Since there exists no path P, the number of nodes on even levels is less than the number of nodes on odd levels. Set C' be the set of vertices on even levels of T and $N(C')$ be the vertices of odd levels of T. As argued, we have $|C'| > |N(C')|$, a contradiction.



Excercise

1) We are given a k-regular bipartite graph H (every vertex has degree k). Show that we can color the edges of H with k-colors such that the edges incident to a same vertex receive different colors.

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2) We are given a bipartite graph $H = (C, R, E)$. We want to find an assignment of the resources to the customers such that each customer receives at least b resources from R. What is the necessary condition ?