Inclusion and Exclusion

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$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.

In general $|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{i,j}^{i \neq j} |A_i \cap A_j| + \sum_{i,j,k}^{i \neq j \neq k} |A_i \cap A_j \cap A_k| + \ldots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|$.

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$|A_3 \cap A_4|$ number of ones divisible by 12.

$|A_3 \cap A_{10}|$ number of ones divisible by 30.

$|A_4 \cap A_{10}|$ number of ones divisible by 20.

$|A_3 \cap A_4 \cap A_{10}|$ number of ones divisible by 60 which is 2014/60.

$|A_3| = 2014/3$, $|A_4| = 2014/4$, $|A_{10}| = 2014/10$,

$|A_3 \cap A_4| = 2014/12$, $|A_3 \cap A_{10}| = 2014/30$, $|A_4 \cap A_{10}| = 2014/20$.

$671 + 503 + 201 - 167 - 67 - 100 + 33 = 1074$. Therefore $2014 - 1074 = 940$

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Show that if five points are randomly selected inside or on an equilateral triangle of side 1, then there will be two of these points whose distance is not more than $\frac{1}{2}$.

Show that from a set of $n+1$ integer which none of them is divisible by n, the difference of two of them is a multiple of n.

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