Permutation and Combinations

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Theorem

The number of k-permutations from n distinct objects is denoted by $P(n, k)$ and we have

$$P(n,k) = n(n-1)(n-2)...(n-k+1) = \frac{n!}{(n-k)!}.$$

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Proof.

In the first position we have n possibilities, in the second position we have $n-1$ and in the k-position we have $n-k+1$ possibilities. □

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The number of k-combinations taken from a set of n distinct objects $(0 \leq k \leq n)$ is denoted by $C(n,k)$ and we have $C(n,k) = \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!}$ $\left(C(n,k) = \binom{n}{k}\right)$

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Show that if we multiply m consecutive positive integers then the result is a multiple of $m!$.

Let $n+1, \ldots, n+m$ be the m consecutive numbers. We need to show that $m! | (m+n)(m+n-1)...(n+1)$. This is because $\binom{m+n}{m} = \frac{(m+n)(m+n-1)...(n+1)}{m!}$

Binomial Theorem

Theorem

The number of subsets of a set with n elements is $C(n,0) + C(n,1) + \cdots + C(n,n) = 2^n$.

Recall that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

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Proof.

$(x+y)^n = (x+y)(x+y)....(x+y)$ (n times) $x^k y^{n-k}$ arise from choosing k $x's$ from one of the n- terms (x, y)'s . This means we choose k x from n possible terms (object). Therefore there are $C(n, k) = \binom{n}{k}$ ways and hence the coefficient of $x^k y^{n-k}$ is $\binom{n}{k}$. □

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$(1-1)^m = 1^m + \binom{m}{1}1^{m-1}(-1)^1 + \binom{m}{2}1^{m-2}(-1)^2 + ...$

Multinomial Coefficients

Theorem

Assume that there are n_1 objects of type 1, n_2 objects of type 2, ..., n_k objects of type k and $n = n_1 + n_2 + \cdots + n_k$. The number of distinguishable permutations of these n objects is :

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1!n_2!...n_k!}$$

This number is also the number of ways to place n distinct objects into k distinguished group with n_1 objects in the first group, n_2 in the second group,..., n_k in the last group.

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Proof.

There are $n! = (n_1 + n_2 + ... + n_k)!$ permutation of these objects. But for each type i there are $n_i!$ permutations (permuting the object of the same types) that are the same. Therefore we should divide $n!$ by $n_1!n_2!\ldots n_k!$.

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From the remanning $n - n_1$ objects choose n_2 slots to be filled with type 2 objects, there are $\binom{n-n_1}{n_2}$ ways to do so and continue this way.

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Thus we have $\binom{n}{n_1}.\binom{n-n_1}{n_2}.\binom{n-n_1-n_2}{n_3}.\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}$
$= \frac{n!}{n_1!n_2!\ldots n_k!}$

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Multinomial Theorem

Theorem

If $x_1, x_2, \ldots, x_r$ *are numbers, and* n *is a nonnegative integer, then*

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{k_1+k_2+\cdots+k_r=n} \binom{n}{k_1, k_2, \ldots, k_r} x_1^{k_1} x_2^{k_2} \ldots x_r^{k_r}$$

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Example :

$(x + 2y + 3z)^3 = x^3 + (2y)^3 + (3z)^3 + 3x^2(2y) + 3x(2y)^2 + 3x^2(3z) + 3x(3z)^2 + 3(2y)^2 3z + 3(2y)(3z)^2 + 36xyz.$

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Exercise :

What is the coefficient of $a^6 b^3 c^3 d^2$ in the expansion of $(2a - 3b + 4c - d)^{14}$.

What is the number of solutions for $x_1 + x_2 + \cdots + x_k = n$ where $1 \leq x_i \leq n$

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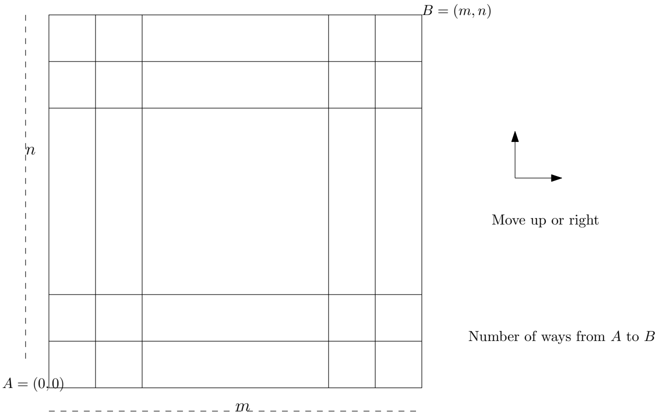
Theorem

The number of ways of distributing n identical objects to d different places is $\binom{n+d-1}{d-1}$.

1) What is the number of ways from $(0, 0)$ to (m, n) using one step up and one step right at a time ?



Exercises

2) Show that $\binom{n}{n-k} = \binom{n}{k}$, $0 \leq k \leq n$.

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Suppose we want to choose k objects from a set of n objects. It is like not choosing $n-k$ objects from a set of n objects. We can relate each subset of k objects to a subset of $n-k$ objects and vice versa.

3) Show that $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$

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Choosing k elements from A means $\binom{n}{k}$ and choosing $n-k$ elements from B means $\binom{n}{n-k}$.

Since $0 \leq k \leq n$, we have the sum in the left side.

4) Show that $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$

5.1) Expand $(x + 2y + 3z)^4$

5.2) What is the coefficient of $x^4y^6z^8w^{24}$ in the expansion of $(x + 2y + 3z^2 + w^4)^{20}$.

How many onto functions f are there with the following domains and codomanis?

(a) $f : \{1, 2, 3, 4, 5\} \rightarrow \{1, 2, 3, 4, 5, 6\}$

(b) $f : \{1, 2, 3, 4, 5, 6\} \rightarrow \{1, 2, 3, 4, 5, 6\}$

(c) $f : \{1, 2, 3, 4, 5, 6\} \rightarrow \{1, 2, 3, 4, 5\}$

(d) $f : \{1, 2, 3, 4, 5, 6, 7\} \rightarrow \{1, 2\}$