Generating Functions

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   $G(x) = 1 + x + x^2 + x^3.$

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So the answer to the problem is the coefficient of x^{24} in the generating function

$$f(x) = (1+x+x^2+x^3+\cdots+x^{24})^2 \times (1+x^2+x^4+x^6+\cdots+x^{24}) \times (x^6 + x^7 + x^8 + \cdots + x^{24})$$

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$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Arithmetic of Generating Functions

Definition

Suppose we have two generating functions $F(x) = \sum_{n=0}^{\infty} a_n x^n$, $G(x) = \sum_{n=0}^{\infty} b_n x^n$. Then

- the sum of these generating functions is
  $F(x) + G(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n$
- the difference of these generating functions is
  $F(x) - G(x) = \sum_{n=0}^{\infty} (a_n - b_n) x^n$
- the products of these generating functions is
  $G(x).F(x) = \sum_{n=0}^{\infty} (a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0) x^n = \sum_{n=0}^{\infty} (\sum_{k=0}^{n} a_k b_{n-k}) x^n$

Example

$$\frac{e^x}{1-x} = e^x.\frac{1}{1-x} = (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots).(1 + x + x^2 + x^3 + \dots)$$

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$1.1 + (1.1 + 1.1)x + (1.1 + 1.1 + \frac{1}{2!}.1)x^2 + (1.1 + 1.1 + \frac{1}{2!}.1 + \frac{1}{3!}.1)x^3 + \dots$

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$$= 1.1+(1.1+1.1)x+(1.1+1.1+\frac{1}{2!}.1)x^2+(1.1+1.1+\frac{1}{2!}.1+\frac{1}{3!}.1)x^3+\dots$$

$$\frac{e^x}{1-x} = \sum_{n=0}^{\infty} a_n x^n \text{ where } a_n = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!} = \sum_{n=0}^{n} \frac{1}{k!}$$

Exercise

(1) $x^2 + x^3 + x^4 + ... = \sum_{n=2}^{\infty} x^n$

(2) $(1 - x)(1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + ...)$

Definition

If a is a real number and k is a non-negative integer, we define the generalized binomial coefficient $\binom{n}{k}$ as
$\binom{a}{k} = \frac{a(a-1)(a-2)\ldots(a-k+1)}{k!}$ if $k > 0$ and 1 otherwise.

$$\binom{3/2}{4} = \frac{(3/2)(1/2)(-1/2)(-3/2)}{4!} = \frac{3}{128}.$$

If a is a negative integer : $a = -n$ then
$\binom{a}{k} = (-1)^k C(n+k-1, k)$.

Definition

If a is a real number and x is a real number with $|x| < 1$, then

$$(1+x)^a = 1 + \binom{a}{1}x + \binom{a}{2}x^2 + \binom{a}{3}x^3 + \cdots = \sum_{n=0}^{\infty} \binom{a}{n} x^n$$

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$$(1 + x + x^2 + x^3 + \dots)^n = (\frac{1}{1-x})^n = \frac{1}{(1-x)^n} = \sum_{n=0}^{\infty} \binom{n+i-1}{i} x^i$$

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Each subdivition will get at least one new employee.
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$$4^{11} - 4(3^{11}) + 6(2^{11}) - 4(1^{11}) = \sum_{n=0}^{4} (-1)^i \binom{4}{i} (4-i)^{11} \ .$$