Recurrence Relations

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But the recurrence $a_{n+1} = 3a_na_{n-1}$ is not linear.

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Set $b_n = a_n^2$ then $b_{n+1} = 5b_n$ and $b_0 = 4$.

Therefore $b_n = 4(5^n)$ and hence $a_n = 2(\sqrt{5})^n$ and $a_{12} = 2(\sqrt{5})^{12} = 2(25^3) = 31250$.

General First Order Relation

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$a_n = a_{n-1} + (n-1)$, $n \geq 2$, $a_1 = 0$.
$a_n = 1 + 2 + \cdots + (n-1) = [(n-1)n/2]$.

$$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} = f(n),\ n \geq k.$$

When $f(n) = 0$ for all n the function is called homogenous otherwise it is called nonhomogeneous.

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Let r_1, r_2 be the roots of $C_0 r^2 + C_1 r + C_2 = 0.$ There are three cases :

1. r_1, r_2 are distinct real numbers
2. r_1, r_2 are complex number (conjugate of each other)
3. $r_1 = r_2$ is a real number

In all cases r_1, r_2 are called the characteristic roots.

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Fibonacci Numbers

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Therefore $c_1 = \frac{1}{\sqrt{5}}$ and hence

$$F_n = \frac{1}{\sqrt{5}}[(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n].$$

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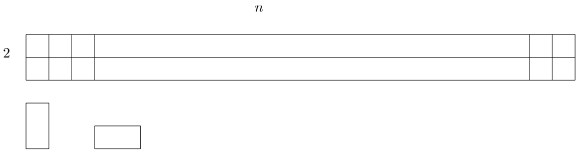

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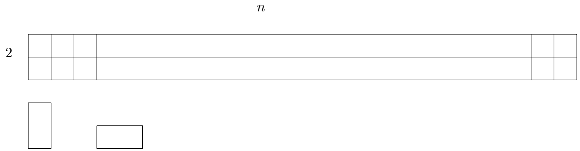

Let b_n be the number of ways :

Either we put the last domino vertically(2 by 1) and then in the remaining we have b_{n-1} ways or

We put two dominos horizontally (2 of 1 by 2) and then in the remaining we have b_{n-2} ways.

Therefore $b_n = b_{n-1} + b_{n-2}$, $b_1 = 1$ and $b_2 = 2$.

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Now the solution is the $n+2$-th Fibonacci's number.

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Later we will learn how to solve this recurrence relation! However, we can write a computer program to compute a_n (not a recursive function!!). Just a for loop!