Recurrence Relations(continued)

Arash Rafiey

September 24, 2015

Repeated Real Roots

Solve the recurrence relation $a_{n+2} = 4a_{n+1} - 4a_n$ where $n \geq 0$ and $a_0 = 1$, $a_1 = 3$.

Repeated Real Roots

Solve the recurrence relation $a_{n+2} = 4a_{n+1} - 4a_n$ where $n \geq 0$ and $a_0 = 1$, $a_1 = 3$.

We let $a_n = cr^n$ and hence the characteristic equation is :

$r^2 - 4r + 4 = 0$ in which both roots are $r = 2$.

Repeated Real Roots

Solve the recurrence relation $a_{n+2} = 4a_{n+1} - 4a_n$ where $n \geq 0$ and $a_0 = 1$, $a_1 = 3$.

We let $a_n = cr^n$ and hence the characteristic equation is :

$r^2 - 4r + 4 = 0$ in which both roots are $r = 2$.

Now since $2^n, 2^n$ are not independent then we should assume $a_n = g(n)2^n$ where $g(n)$ is not a constant.

Repeated Real Roots

Solve the recurrence relation $a_{n+2} = 4a_{n+1} - 4a_n$ where $n \geq 0$ and $a_0 = 1$, $a_1 = 3$.

We let $a_n = cr^n$ and hence the characteristic equation is :

$r^2 - 4r + 4 = 0$ in which both roots are $r = 2$.

Now since $2^n, 2^n$ are not independent then we should assume $a_n = g(n)2^n$ where $g(n)$ is not a constant.

Thus we have $g(n+2)2^{n+2} = 4g(n+1)2^{n+1} - 4g(n)2^n$ and hence

Repeated Real Roots

Solve the recurrence relation $a_{n+2} = 4a_{n+1} - 4a_n$ where $n \geq 0$ and $a_0 = 1$, $a_1 = 3$.

We let $a_n = cr^n$ and hence the characteristic equation is :

$r^2 - 4r + 4 = 0$ in which both roots are $r = 2$.

Now since $2^n, 2^n$ are not independent then we should assume $a_n = g(n)2^n$ where $g(n)$ is not a constant.

Thus we have $g(n+2)2^{n+2} = 4g(n+1)2^{n+1} - 4g(n)2^n$ and hence

$g(n+2) = 2g(n+1) - g(n)$.

Repeated Real Roots

Solve the recurrence relation $a_{n+2} = 4a_{n+1} - 4a_n$ where $n \geq 0$ and $a_0 = 1,\ a_1 = 3$.

We let $a_n = cr^n$ and hence the characteristic equation is :

$r^2 - 4r + 4 = 0$ in which both roots are $r = 2$.

Now since $2^n, 2^n$ are not independent then we should assume $a_n = g(n)2^n$ where $g(n)$ is not a constant.

Thus we have $g(n+2)2^{n+2} = 4g(n+1)2^{n+1} - 4g(n)2^n$ and hence

$g(n+2) = 2g(n+1) - g(n)$.

It is clear $g(n) = n$ holds for $g(n)$.

Repeated Real Roots

Solve the recurrence relation $a_{n+2} = 4a_{n+1} - 4a_n$ where $n \geq 0$ and $a_0 = 1,\ a_1 = 3$.

We let $a_n = cr^n$ and hence the characteristic equation is :

$r^2 - 4r + 4 = 0$ in which both roots are $r = 2$.

Now since $2^n, 2^n$ are not independent then we should assume $a_n = g(n)2^n$ where $g(n)$ is not a constant.

Thus we have $g(n+2)2^{n+2} = 4g(n+1)2^{n+1} - 4g(n)2^n$ and hence

$g(n+2) = 2g(n+1) - g(n)$.

It is clear $g(n) = n$ holds for $g(n)$.

So another solution would be $n2^n$ (note that $n2^n, 2^n$ are independent)

Repeated Real Roots

Solve the recurrence relation $a_{n+2} = 4a_{n+1} - 4a_n$ where $n \geq 0$ and $a_0 = 1,\ a_1 = 3$.

We let $a_n = cr^n$ and hence the characteristic equation is :

$r^2 - 4r + 4 = 0$ in which both roots are $r = 2$.

Now since $2^n, 2^n$ are not independent then we should assume $a_n = g(n)2^n$ where $g(n)$ is not a constant.

Thus we have $g(n+2)2^{n+2} = 4g(n+1)2^{n+1} - 4g(n)2^n$ and hence

$g(n+2) = 2g(n+1) - g(n)$.

It is clear $g(n) = n$ holds for $g(n)$.

So another solution would be $n2^n$ (note that $n2^n, 2^n$ are independent)

Then we have $a_n = c_1 2^n + c_2 n 2^n$ with $a_0 = 1,\ a_1 = 3$.

Solve the recurrence relation $a_{n+2} = 4a_{n+1} - 4a_n$ where $n \geq 0$ and $a_0 = 1$, $a_1 = 3$.

We let $a_n = cr^n$ and hence the characteristic equation is :

$r^2 - 4r + 4 = 0$ in which both roots are $r = 2$.

Now since $2^n, 2^n$ are not independent then we should assume $a_n = g(n)2^n$ where $g(n)$ is not a constant.

Thus we have $g(n+2)2^{n+2} = 4g(n+1)2^{n+1} - 4g(n)2^n$ and hence

$g(n+2) = 2g(n+1) - g(n)$.

It is clear $g(n) = n$ holds for $g(n)$.

So another solution would be $n2^n$ (note that $n2^n, 2^n$ are independent)

Then we have $a_n = c_1 2^n + c_2 n2^n$ with $a_0 = 1$, $a_1 = 3$.

After all $a_n = 2^n + (1/2)n2^n = 2^n + n2^{n-1}$.

Definition

In general suppose $C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} = 0$ where C_i's are constant and $C_0 \neq 0$ and $C_k \neq 0$ and r is the characteristic root with multiplicity $2 \leq m \leq k$. Then the part of the general solution involving root r has the following form :

$$(A_0 + A_1 n + A_2 n^2 + \cdots + A_{m-1} n^{m-1}) r^n$$

where A_i are arbitrary constant.

Nonhomogeneous Recurrence Relation

Consider the recurrence relations :

(1) $a_n + C_1 a_{n-1} = f(n)$, $n \geq 1$.

(2) $a_n + C_1 a_{n-1} + C_2 a_{n-2} = f(n)$, $n \geq 2$.

Nonhomogeneous Recurrence Relation

Consider the recurrence relations :

(1) $a_n + C_1 a_{n-1} = f(n)$, $n \geq 1$.

(2) $a_n + C_1 a_{n-1} + C_2 a_{n-2} = f(n)$, $n \geq 2$.

There is no general method for solving above recurrence relations.

Nonhomogeneous Recurrence Relation

Consider the recurrence relations :

(1) $a_n + C_1 a_{n-1} = f(n)$, $n \geq 1$.

(2) $a_n + C_1 a_{n-1} + C_2 a_{n-2} = f(n)$, $n \geq 2$.

There is no general method for solving above recurrence relations.

But in some cases there is a way.

Suppose $a_n - a_{n-1} = f(n)$.

Nonhomogeneous Recurrence Relation

Consider the recurrence relations :
(1) $a_n + C_1 a_{n-1} = f(n)$, $n \geq 1$.
(2) $a_n + C_1 a_{n-1} + C_2 a_{n-2} = f(n)$, $n \geq 2$.

There is no general method for solving above recurrence relations.

But in some cases there is a way.
Suppose $a_n - a_{n-1} = f(n)$.
Now by replacement we have
$a_1 = a_0 + f(1)$
$a_2 = a_1 + f(2)$
$a_3 = a_2 + f(3)$
.
.
$a_n = a_{n-1} + f(n)$

Consider the recurrence relations :
(1) $a_n + C_1 a_{n-1} = f(n)$, $n \geq 1$.
(2) $a_n + C_1 a_{n-1} + C_2 a_{n-2} = f(n)$, $n \geq 2$.

There is no general method for solving above recurrence relations.

But in some cases there is a way.
Suppose $a_n - a_{n-1} = f(n)$.
Now by replacement we have

$a_1 = a_0 + f(1)$
$a_2 = a_1 + f(2)$
$a_3 = a_2 + f(3)$
.
.
$a_n = a_{n-1} + f(n)$

Therefore $a_n = a_0 + f(1) + f(2) + \cdots + f(n)$.

Example :

$a_n - a_{n-1} = 3n^2$ where $n \geq 1$ and $a_0 = 7$.

Example :

$a_n - a_{n-1} = 3n^2$ where $n \geq 1$ and $a_0 = 7$.

$a_n = a_0 + \sum_{i=1}^{n} f(i) = 7 + 3\sum_{i=1}^{n} i^2 = 7 + \frac{1}{2}n(n+1)(2n+1)$.

Example :

$a_n - a_{n-1} = 3n^2$ where $n \geq 1$ and $a_0 = 7$.

$a_n = a_0 + \sum_{i=1}^{n} f(i) = 7 + 3\sum_{i=1}^{n} i^2 = 7 + \frac{1}{2}n(n+1)(2n+1)$.

What about the following relation ?

$a_n - 3a_{n-1} = 5(7^n)$, where $n \geq 1$ and $a_0 = 2$.

Example :

$a_n - a_{n-1} = 3n^2$ where $n \geq 1$ and $a_0 = 7$.

$a_n = a_0 + \sum_{i=1}^{n} f(i) = 7 + 3\sum_{i=1}^{n} i^2 = 7 + \frac{1}{2}n(n+1)(2n+1)$.

What about the following relation ?

$a_n - 3a_{n-1} = 5(7^n)$, where $n \geq 1$ and $a_0 = 2$.

$a_n - 3a_{n-1} = 5(3^n)$, where $n \geq 1$ and $a_0 = 2$.

Definition

Consider the nonhomogeneous first-order relation (k constant)

$$a_n + C_1 a_{n-1} = kr^n$$

When r^n is not a solution ($C_1 \neq -r$) for $a_n + C_1 a_{n-1} = 0$ then $a_n = A(-C_1)^n + B(r^n)$ for some constants A, B.
When r^n is a solution for the recurrence, i.e. $(-C_1 = r)$ then $a_n = Ar^n + Bnr^n$ for some constants A, B.

Example :

$a_n - 3a_{n-1} = 5(7^n)$, where $n \geq 1$ and $a_0 = 2$.

Example :

$a_n - 3a_{n-1} = 5(7^n)$, where $n \geq 1$ and $a_0 = 2$.

The root for associated with $a_n - 3a_{n-1} = 0$ is the root of $r - 3 = 0$ and $3 \neq 5$ therefore
$a_n = A(3^n) + B(7^n)$

Example :

$a_n - 3a_{n-1} = 5(7^n)$, where $n \geq 1$ and $a_0 = 2$.

The root for associated with $a_n - 3a_{n-1} = 0$ is the root of $r - 3 = 0$ and $3 \neq 5$ therefore
$a_n = A(3^n) + B(7^n)$

$a_0 = 2$ and $a_1 = 6 + 35 = 41$ therefore
$a_n = (5/4)(7^{n+1}) - (1/4)(3^{n+1})$.

Example :

$a_n - 3a_{n-1} = 5(7^n)$, where $n \geq 1$ and $a_0 = 2$.

The root for associated with $a_n - 3a_{n-1} = 0$ is the root of $r - 3 = 0$ and $3 \neq 5$ therefore
$a_n = A(3^n) + B(7^n)$

$a_0 = 2$ and $a_1 = 6 + 35 = 41$ therefore
$a_n = (5/4)(7^{n+1}) - (1/4)(3^{n+1})$.

Example :

$a_n - 3a_{n-1} = 5(3^n)$, where $n \geq 1$ and $a_0 = 2$.
The root for associated with $a_n - 3a_{n-1} = 0$ is the root of $r - 3 = 0$ and $3 = 3$ therefore
$a_n = A(3^n) + Bn(3^n)$

Example :

$a_n - 3a_{n-1} = 5(7^n)$, where $n \geq 1$ and $a_0 = 2$.

The root for associated with $a_n - 3a_{n-1} = 0$ is the root of $r - 3 = 0$ and $3 \neq 5$ therefore
$a_n = A(3^n) + B(7^n)$

$a_0 = 2$ and $a_1 = 6 + 35 = 41$ therefore
$a_n = (5/4)(7^{n+1}) - (1/4)(3^{n+1})$.

Example :

$a_n - 3a_{n-1} = 5(3^n)$, where $n \geq 1$ and $a_0 = 2$.
The root for associated with $a_n - 3a_{n-1} = 0$ is the root of $r - 3 = 0$ and $3 = 3$ therefore
$a_n = A(3^n) + Bn(3^n)$

$a_0 = 2$, $a_1 = 18$ therefore $a_n = (2 + 5n)3^n$.

Definition

Consider the nonhomogeneous second-order relation (k constant)

$$a_n + C_1 a_{n-1} + C_2 a_{n-2} = kr^n$$

With homogeneous relation $(h) : a_n + C_1 a_{n-1} + C_2 a_{n-2} = 0$. If

1. r^n is not a solution for (h) then $a_n = Ar^n + B(r_1)^n + C(r_2)^n$
2. r^n is a solution for (h) and (h) has other solution r_1^n, $(r \neq r_1)$ then $a_n = (A + Bn)r^n + C(r_1)^n$.
3. the characteristic equation $r^2 + C_1 r + C_2 = 0$ has $r_1 = r_2 = r$ solution then $a_n = Ar^n + Bnr^n + Cn^2 r^n$.

Example :

What is the number of binary sequences of length n with no "100" ?

Example :

What is the number of binary sequences of length n with no "100" ?

Let a_n be the number of such sequences.

Example :

What is the number of binary sequences of length n with no "100" ?

Let a_n be the number of such sequences.

If the last symbol is 1 then the first $n-1$ symbols is a binary sequences of length $n-1$ with no "100". Therefore we have a_{n-1} of such sequences.

Example :

What is the number of binary sequences of length n with no "100" ?

Let a_n be the number of such sequences.

If the last symbol is 1 then the first $n-1$ symbols is a binary sequences of length $n-1$ with no "100". Therefore we have a_{n-1} of such sequences.

If the last symbol is 0 and the $(n-1)$-th symbol is 1 then the first $n-2$ symbols is a binary sequences of length $n-2$ with no "100". Therefore we have a_{n-2} of such sequences.

Example :

What is the number of binary sequences of length n with no "100" ?

Let a_n be the number of such sequences.

If the last symbol is 1 then the first $n-1$ symbols is a binary sequences of length $n-1$ with no "100". Therefore we have a_{n-1} of such sequences.

If the last symbol is 0 and the $(n-1)$-th symbol is 1 then the first $n-2$ symbols is a binary sequences of length $n-2$ with no "100". Therefore we have a_{n-2} of such sequences.

If the last symbol is 0 and the $(n-1)$-th symbol is 0 then all the previous symbols must be 0 (one such sequence).

Example :

What is the number of binary sequences of length n with no "100" ?

Let a_n be the number of such sequences.

If the last symbol is 1 then the first $n-1$ symbols is a binary sequences of length $n-1$ with no "100". Therefore we have a_{n-1} of such sequences.

If the last symbol is 0 and the $(n-1)$-th symbol is 1 then the first $n-2$ symbols is a binary sequences of length $n-2$ with no "100". Therefore we have a_{n-2} of such sequences.

If the last symbol is 0 and the $(n-1)$-th symbol is 0 then all the previous symbols must be 0 (one such sequence).

Therefore $a_n = a_{n-1} + a_{n-2} + 1$ with $a_1 = 2$ and $a_2 = 4$.

Example :

What is the number of binary sequences of length n with no "100" ?

Let a_n be the number of such sequences.

If the last symbol is 1 then the first $n-1$ symbols is a binary sequences of length $n-1$ with no "100". Therefore we have a_{n-1} of such sequences.

If the last symbol is 0 and the $(n-1)$-th symbol is 1 then the first $n-2$ symbols is a binary sequences of length $n-2$ with no "100". Therefore we have a_{n-2} of such sequences.

If the last symbol is 0 and the $(n-1)$-th symbol is 0 then all the previous symbols must be 0 (one such sequence).

Therefore $a_n = a_{n-1} + a_{n-2} + 1$ with $a_1 = 2$ and $a_2 = 4$.

$a_n = a_n^h + a_n^p$ where $a_n^p = A$ (constant) and (a_n^h is the homogenous part)

$a_n = c_1(\frac{1+\sqrt{5}}{2})^n + c_2(\frac{1-\sqrt{5}}{2})^n + A.$

Problem 1. What is the number of binary sequences of length n with no "101" ?