Recurrence Relations (review and examples)

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Let r_1, r_2 be the roots of $C_0 r^2 + C_1 r + C_2 = 0$. There are three cases :

1. r_1, r_2 are distinct real numbers
2. r_1, r_2 are complex numbers (conjugate of each other)
3. $r_1 = r_2$ is a real number

In all cases r_1, r_2 are called the characteristic roots.

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Definition

In general suppose $C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} = 0$ where C_i's are constant and $C_0 \neq 0$ and $C_k \neq 0$ and r is the characteristic root with multiplicity $2 \leq m \leq k$. Then the part of the general solution involving root r has the following form :

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$a_0 = 5 = A_0$ and $a_1 = 12 = (5 + A_1)3$ and hence $A_1 = -1$.

nonhomogeneous first-order relation

Definition

Consider the nonhomogeneous first-order relation (k constant)

$$a_n + C_1 a_{n-1} = kr^n$$

When r^n is not a solution ($C_1 \neq -r$) for $a_n + C_1 a_{n-1} = 0$ then $a_n = A(-C_1)^n + B(r^n)$ for some constants A, B.
When r^n is a solution for the recurrence, i.e. ($-C_1 = r$) then $a_n = Ar^n + Bnr^n$ for some constants A, B.

Definition

Consider the nonhomogeneous second-order relation (k constant)

$$a_n + C_1 a_{n-1} + C_2 a_{n-2} = kr^n$$

With homogeneous relation $(h) : a_n + C_1 a_{n-1} + C_2 a_{n-2} = 0$. If

1. r^n is not a solution for (h) then $a_n = Ar^n + B(r_1)^n + C(r_2)^n$
2. r^n is a solution for (h) and (h) has other solution r_1^n, $(r \neq r_1)$ then $a_n = (A + Bn)r^n + C(r_1)^n$.
3. the characteristic equation $r^2 + C_1 r + C_2 = 0$ has $r_1 = r_2 = r$ solution then $a_n = Ar^n + Bnr^n + Cn^2 r^n$.

Theorem

Consider the recurrence relation
$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + f(n)$. *Suppose*
$f(n) = (b_s n^s + b_{s-1} n^{s-1} + \cdots + b_1 n + b_0)\lambda^n$.

1. *If* λ *is not a characteristic root then* $a_n^p = (d_s n^s + d_{s-1} n^{s-1} + \cdots + d_1 n + d_0)\lambda^n$ *(nonhomogeneous solution)*
2. *If* λ *is a characteristic root with multiplicity* m *then* $a_n^p = n^{m-1}(d_s n^s + d_{s-1} n^{s-1} + \cdots + d_1 n + d_0)\lambda^n$

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$\lambda = 2$ and $m = 2$ therefore $a_n = n(d_1 n + d_0)2^n + c2^n$.

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$a_0 = 1 = c$ and $a_1 = 2 = 2(d_1 + d_0) + 2$ and hence $d_1 = -d_0$,
$a_2 = 12 = 2(2d_1 - d_1)4 + 4$

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b_n is the n-th Fibonacci's number.

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$a_n = a_n^h + a_n^p$ where $a_n^p = A$ (constant) and (a_n^h is the homogenous part)

$a_n = c_1(\frac{1+\sqrt{5}}{2})^n + c_2(\frac{1-\sqrt{5}}{2})^n + A.$

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$a_n = (1 + 3n - 2n^2)(-1)^n$.

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$a_0 = 1 = c_1 + c_2$, $a_1 = 7 = 4c_1 + c_2$. Therefore $c_1 = 2$ and $c_2 = -1$.

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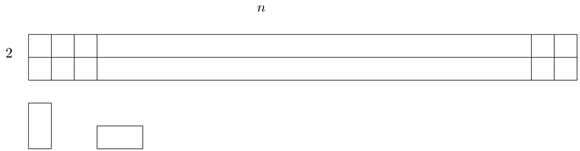

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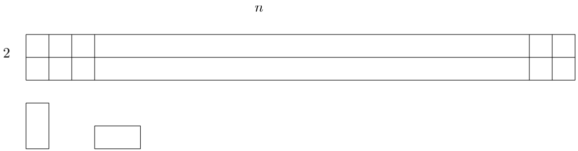

Let b_n be the number of ways :

Either we put the last domino vertically(2 by 1) and then in the remaining we have b_{n-1} ways or

We put two dominos horizontally (2 of 1 by 2) and then in the remaining we have b_{n-2} ways.

Therefore $b_n = b_{n-1} + b_{n-2}$, $b_1 = 1$ and $b_2 = 2$.

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Let a_n be the number of such sequences.

If the last symbol is 1 then the first $n-1$ symbols is a binary sequences of length $n-1$ with no consecutive $0's$ thus we have a_{n-1} of such sequences.

If the last symbol is 0 then the $(n-1)$-th symbol is 1 and the first $n-2$ symbols is a binary sequences of length $n-2$ with no consecutive $0's$ thus we have a_{n-2} of such sequences.

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Now the solution is the $n+2$-th Fibonacci's number.

Definition : We say a sequence S of $0, 1$ is **nice** if the number of ones and the number of zeros are the same and in every prefix of S the number of ones is not less than the number of zero.

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Consider the first index i that the number of 1's and the number of 0's (from 1 to $2i$) are the same.

Then we can write :

$$b_n = \sum_{i=1}^{i=n} b_{i-1} b_{n-i}$$

$b_0 = 1,\ b_1 = 1.$

$$b_{n+1} = b_0 b_n + b_1 b_{n-1} + \cdots + b_{n-1} b_1 + b_n b_0$$

$$\sum_{n=0}^{\infty} b_{n+1} x^{n+1} = \sum_{n=0}^{\infty} (b_0 b_n + b_1 b_{n-1} + \cdots + b_{n-1} b_1 + b_n b_0) x^{n+1}$$

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$(f(x) - b_0) = x \sum_{n=0}^{\infty} (b_0 b_n + b_1 b_{n-1} + \cdots + b_{n-1} b_1 + b_n b_0) x^n = x[f(x)]^2$.

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$x[f(x)]^2 - f(x) + 1 = 0$ and hence $f(x) = [1 \pm \sqrt{1-4x}]/(2x)$.

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$\sqrt{1-4x} = (1-4x)^{1/2} = \binom{1/2}{0} + \binom{1/2}{1}(-4x) + \binom{1/2}{2}(-4x)^2 + \ldots$

The coefficient of x^n, $n \geq 1$ is

$$\binom{1/2}{n}(-4)^n = \frac{(1/2)(1/2-1)(1/2-2)\ldots((1/2)-n+1)}{n!}(-4)^n$$

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which is $\frac{(-1)}{(2n-1)}\binom{2n}{n}$. Then

$f(x) = \frac{1}{2x}[1 - [1 - \sum_{n=1}^{\infty} \frac{1}{(2n-1)}\binom{2n}{n}x^n]]$,

and b_n the coefficient of x^n in $f(x)$ is half of the coefficient of x^{n+1} in

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$b_n = \frac{1}{2}[\frac{1}{2(n+1)-1}]\binom{2(n+1)}{n+1} = \frac{1}{(n+1)}\binom{2n}{n}$

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Case 1. Suppose the last symbol is 2. Then the first $n-1$ symbols is a good sequence of length $n-1$ and we have a_{n-1} of such sequences.

Let b_n^0 be the number of good sequences ending with 0 and b_n^1 be the number of good sequences ending with 1.

Note that $b_n^0 = b_n^1 = b_n$ and we have $a_n = a_{n-1} + 2b_n$.

Case 2. Suppose the last symbol is 0. Then symbol $n-1$ is 2 or 1.

Therefore $b_n = a_{n-2} + b_{n-1}$ (a_{n-2} for when $n-1$-symbol is 2).

Now we have :

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Therefore $(a_n - a_{n-1}) = (a_{n-1} - a_{n-2}) + 2a_{n-2}$ and hence

$$a_n = 2a_{n-1} + a_{n-2}$$

$\mathcal{O}$, θ notations

For a given function $g(n)$, $\Theta(g(n))$ denotes the set

$$\Theta(g(n)) = \{f(n) : \begin{array}{l} \text{there exist positive constants} \\ c_1, c_2, n_0 \text{ such that} \\ c_1 \cdot g(n) \le f(n) \le c_2 \cdot g(n) \\ \text{for all } n \ge n_0\} \end{array}$$

Intuition: $f(n)$ belongs to the family $\Theta(g(n))$ if $\exists$ constants c_1, c_2 s.t. $f(n)$ can fit between $c_1 \cdot g(n)$ and $c_2 \cdot g(n)$, for all n sufficiently large.

Correct notation: $f(n) \in \Theta(g(n))$

Usually used: $f(n) = \Theta(g(n))$.

We also say that "$f(n)$ is in $\Theta(g(n))$".

Examples of Θ-notation:

$f(n) = 2n^2 = \Theta(n^2)$

because with $g(n) = n^2$ and $c_1 = 1$ and $c_2 = 2$ we have

$0 \leq c_1 g(n) \leq f(n) = 2 \cdot n^2 \leq c_2 \cdot g(n)$.

Examples of Θ-notation:

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$0 \le c_1 g(n) \le f(n) = 2 \cdot n^2 \le c_2 \cdot g(n)$.

$f(n) = 8n^5 + 17n^4 - 25 = \Theta(n^5)$
because $f(n) \ge 7 \cdot n^5$ for n large enough

n	$8n^5 + 17n^4 - 25$	n^5	$7n^5$
1	$8 \cdot 1 + 17 \cdot 1 - 25 = 0$	1	7
2	$8 \cdot 32 + 17 \cdot 16 - 25 = 503$	32	224

and $f(n) \le 8n^5 + 17n^5 = 25n^5$, thus $c_1 = 7$, $c_2 = 25$ and $n_0 = 2$ are good enough.

Big-O-notation

When we're interested in **asymptotic upper bounds** only, we use O-notation (read: "big-O").
For given function $g(n)$, define $O(g(n))$ (read: "big-O of g of n" or also "order g of n") as follows:

$$O(g(n)) = \{f(n) : \text{there exist positive constants } c, n_0 \text{ such that } f(n) \leq c \cdot g(n) \text{ for all } n \geq n_0\}$$

We write $f(n) = O(g(n))$ to indicate that $f(n)$ is member of set $O(g(n))$.
Obviously, $f(n) = \Theta(g(n))$ implies $f(n) = O(g(n))$; we just drop the left inequality in the definition of $\Theta(g(n))$.

Big-Omega-notation

Like O-notation, but for lower bounds

For a given function $g(n)$, $\Omega(n)$ denotes the set

$$\Omega(g(n)) = \{f(n) : \begin{array}{l}\text{there exist positive constants} \\ c, n_0 \text{ such that} \\ c \cdot g(n) \leq f(n) \\ \text{for all } n \geq n_0\}\end{array}$$

Saying $T(n) = \Omega(n^2)$ means growth of $T(n)$ is at least the of n^2.

Clearly, $f(n) = \Theta(g(n))$ iff $f(n) = \Omega(g(n))$ and $f(n) = O(g(n))$.

o-notation

Similar to O

$f(n) = O(g(n))$ means we can upper-bound the growth of f by the growth of g (up to a constant factor)

$f(n) = o(g(n))$ is the same, **except** we require the growth of f to be **strictly** smaller than the growth of g:

For a given function $g(n)$, $o(n)$ denotes the set

$$o(g(n)) = \{f(n) : \begin{array}{l} \text{for } \textbf{any} \text{ pos constant } c \\ \text{there exists a pos constant } n_0 \\ \text{such that} \\ f(n) < c \cdot g(n) \\ \text{for all } n \geq n_0\} \end{array}$$

omega-notation

For a given function $g(n)$, $\omega(n)$ denotes the set

$$\omega(g(n)) = \{f(n) : \begin{array}{l} \text{for } \textbf{any} \text{ pos constant } c \\ \text{there exists a pos constant } n_0 \\ \text{such that} \\ c \cdot g(n) < f(n) \\ \text{for all } n \geq n_0\} \end{array}$$

In other words:

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$$

if the limit exists.

I.e., $f(n)$ becomes **arbitrarily** large relative to $g(n)$.

Heuristics that can help to find a good guess.

- One way would be to have a **look at first few terms.** Say if we had $T(n) = 2T(n/2) + 3n$, then

$$
\begin{aligned}
T(n) &= 2T(n/2) + 3n \\
&= 2(2T(n/4) + 3(n/2)) + 3n \\
&= 2(2(2T(n/8) + 3(n/4)) + 3(n/2)) + 3n \\
&= 2^3T(n/2^3) + 2^2 3(n/2^2) + 2^1 3(n/2^1) + 2^0 3(n/2^0)
\end{aligned}
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  We can do this $\log n$ times

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We can do this $\log n$ times

$$\begin{aligned} & 2^{\log n} \cdot T(n/2^{\log n}) + \sum_{i=0}^{\log(n)-1} 2^i 3(n/2^i) \\ = \; & n \cdot T(1) + 3n \cdot \sum_{i=0}^{\log(n)-1} 1 \\ = \; & n \cdot T(1) + 3n \log n = \Theta(n \log n) \end{aligned}$$

After guessing a solution you'll have to **prove the correctness**.

Theorem

Let a, b, c *be positive integers with* $b \geq 2$ *and let* $f : Z^+ \rightarrow R$. *If* $f(1) = c$ *and* $f(n) = af(n/b) + c$ *for* $n = b^k$
then for all $n = 1, b, b^2, b^3, \ldots,$

1. $f(n) = c(\log_b^n + 1)$, *when* $a = 1$
2. $f(n) = \frac{c(an^{\log_b^a} - 1)}{a-1}$, *when* $a \geq 2$.

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  For $T(n) = 2T(\lfloor n/2 \rfloor) + n$ we see

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  From there, we can perhaps "converge" on asymptotically tight bound $\Theta(n \log n)$.

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Now rename $m = \log n \Leftrightarrow 2^m = n$. We know $\sqrt{n} = n^{1/2} = (2^m)^{1/2} = 2^{m/2}$ and thus obtain

$$T(2^m) = 2T(2^{m/2}) + m$$

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Looks familiar. We know the solution $S(m) = \Theta(m \log m)$.

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Looks familiar. We know the solution $S(m) = \Theta(m \log m)$. Going back from $S(m)$ to $T(n)$ we obtain

$$T(n) = T(2^m) = S(m) = \Theta(m \log m) = \Theta(\log n \log \log n)$$

The "Master Method"

Theorem

Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the nonnegative integers by the recurrence

$$T(n) = aT(n/b) + f(n),$$

where we interpret n/b to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then $T(n)$ can be bounded asymptotically as follows.

1. *If $f(n) = \mathcal{O}(n^{(\log_b a)-\epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.*
2. *If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \cdot \log n)$.*
3. *If $f(n) = \Omega(n^{(\log_b a)+\epsilon})$ for some constant $\epsilon > 0$, and if $a \cdot f(n/b) \leq c \cdot f(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$.*

2. If $f(n) = \Theta(n^{\log_b a})$, then
   $T(n) = \Theta(n^{\log_b a} \cdot \log n)$.

Note 1: Although it's looking rather scary, it really isn't. For instance, $T(n) = 2T(n/2) + \Theta(n)$ we have
$n^{\log_b a} = n^{\log_2 2} = n^1 = n$, and we can apply case 2.
The result is therefore $\Theta(n^{\log_b a} \cdot \log n) = \Theta(n \log n)$.

Notes on Master's Theorem

2. If $f(n) = \Theta(n^{\log_b a})$, then
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1. If $f(n) = \mathcal{O}(n^{(\log_b a)-\epsilon})$ for some constant $\epsilon > 0$, then
   $T(n) = \Theta(n^{\log_b a})$.

Note 2: In case 1,

$$f(n) = n^{(\log_b a)-\epsilon} = n^{\log_b a}/n^{\epsilon} = o(n^{\log_b a}),$$

so the ϵ **does** matter. This case is basically about "small" functions f. But it's not enough if $f(n)$ is just asymptotically smaller than $n^{\log_b a}$ (that is $f(n) \in o(n^{\log_b a})$), it must be *polynomially smaller*!

3. If $f(n) = \Omega(n^{(\log_b a)+\epsilon})$ for some constant $\epsilon > 0$, and if $a \cdot f(n/b) \leq c \cdot f(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$.

3. If $f(n) = \Omega(n^{(\log_b a)+\epsilon})$ for some constant $\epsilon > 0$, and if $a \cdot f(n/b) \leq c \cdot f(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$.

Note 3: Similarly, in case 3,

$$f(n) = n^{(\log_b a)+\epsilon} = n^{\log_b a} \cdot n^{\epsilon} = \omega(n^{\log_b a}),$$

so the ϵ **does** matter again. This case is basically about "large" functions n. But again, $f(n) \in \omega(n^{\log_b a})$ is not enough, it must be *polynomially larger*. And in addition $f(n)$ has to satisfy the "**regularity condition**":

$$af(n/b) \leq cf(n)$$

for some constant $c < 1$ and $n \geq n_0$ for some n_0.

Using the master theorem

Simple enough. Some examples:

$T(n) = 9T(n/3) + n$

We have $a = 9$, $b = 3$, $f(n) = n$. Thus, $n^{\log_b a} = n^{\log_3 9} = n^2$. Clearly, $f(n) = \mathcal{O}(n^{\log_3(9)-\epsilon})$ for $\epsilon = 1$, so case 1 gives $T(n) = \Theta(n^2)$.

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$$T(n) = T(2n/3) + 1$$

We have $a = 1$, $b = 3/2$, and $f(n) = 1$, so $n^{\log_b a} = n^{\log_{2/3} 1} = n^0 = 1$.

Apply case 2 ($f(n) = \Theta(n^{\log_b a}) = \Theta(1)$, result is $T(n) = \Theta(\log n)$.

$T(n) = 3T(n/4) + n \log n$

We have $a = 3$, $b = 4$, and $f(n) = n \log n$, so $n^{\log_b a} = n^{\log_4 3} = \mathcal{O}(n^{0.793})$.

Clearly, $f(n) = n \log n = \Omega(n)$ and

thus also $f(n) = \Omega(n^{\log_b(a)+\epsilon})$

for $\epsilon \approx 0.2$. Also, $a \cdot f(n/b) = 3(n/4)\log(n/4) \leq (3/4)n \log n = c \cdot f(n)$ for any $c = 3/4 < 1$.

Thus we can apply case 3 with result $T(n) = \Theta(n \log n)$.

Exercises

True or False

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2) $\frac{n^2 \log n}{(5n+4)} = \mathcal{O}(n)$

3) $3^{n+2} = \Omega(3^n)$

Exercises

True or False

1) $\frac{(n^4+10n)}{2n^2+4} = \mathcal{O}(n^2)$

2) $\frac{n^2 \log n}{(5n+4)} = \mathcal{O}(n)$

3) $3^{n+2} = \Omega(3^n)$

4) $(\log n)^{\log n} = \Omega(n/\log n)$

Exercises

Problem

Give asymptotic upper and lower bounds for $T(n)$ in each of the following recurrences. Assume that $T(n)$ is constant for $n \leq 2$.

(a) $T(n) = 2T(n/2) + n^3$

(b) $T(n) = T(n-1) + n$

(c) $T(n) = T(\sqrt{n}) + 1$

(d) $T(n) = 16T(n/4) + n^2$

(e) $T(n) = 2T(n-1) + \log n$