An introduction to graph theory

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Definition of Graph

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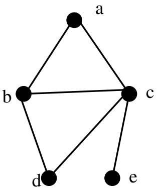
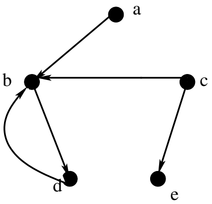
When the order does not matter (relation is symmetric) we have a graph $G = (V, E)$ and $E(G) \subseteq \{\{u, v\} | u, v \in V(G)\}$



V={a,b,c,d,e}

E={{a,b},{a,c},{b,c},{b,d},{d,c},{c,e}}



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A cycle is a *simple closed* walk (no vertex repeated except the beginning).

Theorem

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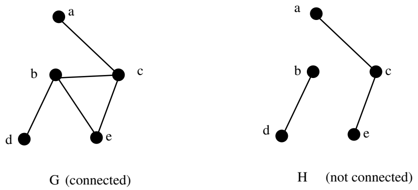
Proof.

Consider the shortens trail $a, x_1, x_2, \ldots, x_n, b$ in G ($ax_1, x_1x_2, x_2x_3, \ldots, x_{n-1}x_n, x_nb$ are edges). If this trail is not a path then there exist k, m where $x_k = x_m$ ($x_0 = a$, $x_{m+1} = b$, $k < m$). But then we can contract and get a shorter trail $a, x_1, \ldots, x_k, x_{m+1}, \ldots, x_n, b$ from a to b.

□

Definition

A graph G is called **connected** if there is a path between any two distinct vertices of G.



Definition

If G is not connected then it can be partitioned into pieces where each piece is a connected graph and is called a **connected component**.
The number of connected components of G is denoted by $\kappa(G)$.

Example :

Let $G = (V, E)$ be an undirected graph whose vertices are binary n-sequences and the two vertices x, y are adjacent if they differ in exactly two positions.
Find $\kappa(G)$.

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If v is a vertex of graph G, then the degree of v, denoted $deg(v)$ ($d_G(v)$, or d_v) is the number of edges incident to v. Is the number of neighbors of v. The self-loop is counted twice.

If G is a simple graph and each vertex has degree k then G is called a k-regular graph.

Theorem (Handshaking)

For any graph $G = (V, E)$ *we have*

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If an edge e has two end points u and v then it will contribute one to each of $deg(u)$, $deg(v)$. If e is a self loop incident to vertex u then it will contribute two to $deg(u)$. In any case each edge contributes two in the sum and identity follows. □

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Corollary

In any graph G the number of vertices of odd degree must be even.

Some families of Graphs

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4) For every $n \geq 2$, the n-hypercube, denoted by Q_n has vertex set $V(Q_n) = \{length \;\; \text{n} \;\; bit \;\; string\}$ and two vertices are adjacent if their bit strings differ in exactly one position.

Bipartite Graphs

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A graph $G = (V, E)$ is bipartite if V can be partitioned into U, W such that each edge of G has one end point in U and one end point in W.

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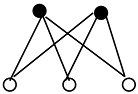
For which value of n, P_n is bipartite ?

For which value of n, Q_n is bipartite ?

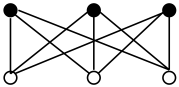
Definition

For positive integers n, m, the complete bipartite graph $K_{n,m}$ has the following vertex and edge sets :

- $V(K_{n,m}) = \{u_1, u_2, \dots, u_n\} \cup \{v_1, v_2, \dots, v_m\}$
- $E(K_{n,m}) = \{u_i v_j | 1 \leq i \leq n, 1 \leq j \leq m\}$



$K_{2,3}$



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$(2, 2, 2)$ is graphic but $(2, 3, 4, 3, 2, 3)$ is not graphic.

Theorem

Suppose $\pi = (d_1, d_2, \ldots, d_n)$ *is a sequence with* $n > d_1 \geq d_2 \geq \cdots \geq d_n$.

1. *If* π *is graphic then there is a graph* G *with* $V(G) = \{v_1, v_2, \ldots, v_n\}$ *and* $deg(v_i) = d_i$ *and the neighbors of* v_1 *are* $v_2, v_3, \ldots, v_{d_1+1}$.
2. π *is graphic if and only if* $(d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$ *is graphic.*

Proof.

Proof of (1). Since π is graphic, there is a graph G with vertices $v_1, v_2, \ldots, v_n$ and degree sequence π.
We may assume that $G = (V, E)$ is such a graph that $N(v_1) \cap S$ is maximum where $S = \{v_2, v_3, \ldots, v_{d_1+1}\}$ ($N(v_1)$ is the neighborhood of v_1). If v_1 is adjacent to all the elements in S then we are done. Otherwise there exists some v_k such that $v_1 v_k \notin E$ and hence there exits $\ell > d_1 + 1$ where $v_1 v_\ell \in E$.

□

Proof.

Since $k < \ell$, v_k has as many neighbors as v_ℓ and hence v_k has a neighbor v_j that is not neighbor of v_ℓ. Now create new graph G' by removing edge $v_1 v_\ell$ and adding edge $v_1 v_k$ and removing edge $v_k v_j$ and adding edge $v_j v_\ell$. G' has the same degree sequence as G but $N(v_1) \cap S$ increases in G', a contradiction.

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Proof of (2). If π is graphic then by (1) there is G with vertices $v_1, v_2, \ldots, v_n$ and degree sequence π where v_1 is adjacent to $v_2, v_3, \ldots, v_{d_1+1}$. If we remove v from G then we have a sequence $(d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$. Conversely, if we start with graphic sequence $(d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$ associated with graph G' then we add a new vertex v and connect it to the vertices of G' with degrees $d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1$. This way we obtain G with degree sequence π.

$\square$

Graphic $(n > d_1 \geq d_2 \geq d_3 \geq \cdots \geq d_n)$

1. **while** $d_1 > 0$
2. Set $(d'_1, d'_2, \ldots, d'_{n-1})$ be a non-decreasing permutation of $(d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$
3. Set $n = n - 1$ and $(d_1, d_2, \ldots, d_n) = (d'_1, d'_2, \ldots, d'_n)$
4. **if** $d_i < 0$ **then** output NO **exit**
5. **else if** $d_1 = 0$ **then** output YES **exit**

Example

Initial Sequence : (4,4,3,3,2,2)

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Sort : (3,2,2,2,1)

(2) : (1,1,1,1)

(3) : (0,1,1)

Sort (1,1,0)

(4) : (0,0) True

Is the sequence $(5, 5, 3, 5, 3, 3, 3, 3)$ is graphic ?

Subgraphs

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C_5 is a subgraph of K_6, and $K_1, K_2, \ldots, K_5$ are all subgraph of K_6.

Graph Isomorphism

Definition

Let $G = (V, E)$, $G' = (V', E')$ two graphs. Suppose $f : V \longrightarrow V'$ is a one-to-one function.

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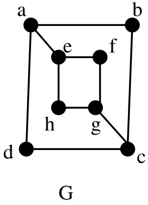
f(a)=c and f(b)=d
f is isomorphism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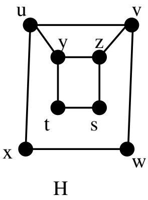
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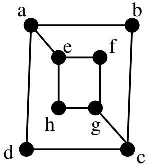

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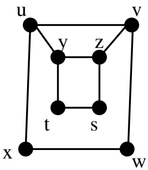

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