Subgraphs and Paths and Cycles

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15 October, 2015

Subgraphs

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C_5 is a subgraph of K_6, and $K_1, K_2, \ldots, K_5$ are all subgraph of K_6.

Graph Isomorphism

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Let $G = (V, E)$, $G' = (V', E')$ two graphs. Suppose $f : V \longrightarrow V'$ is a one-to-one function.

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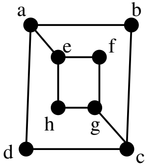
f(a)=c and f(b)=d
f is isomorphism



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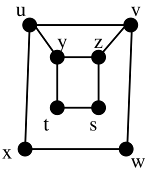

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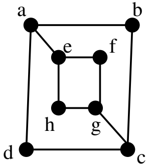

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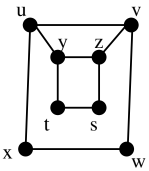

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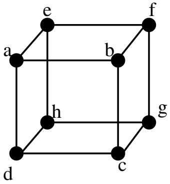

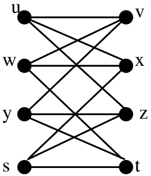

Cycles and Eulerian tour

Definition

Let $G = (V, E)$ be a graph or multigraph with no isolated vertices. G has an **Eulerian** circuit (closed trail) if there is a circuit (closed trail) that traverses each edge exactly once.

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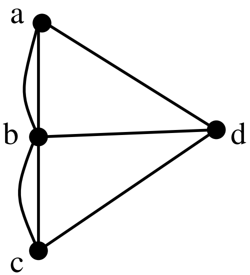

Theorem

Let $G = (V, E)$ be an undirected graph or multigraph with no isolated vertices. Then G has an Euler circuit if and only if G is connected and every vertex in G has even degree.

Proof.

If G has an Eulerian circuit then there is a trail from a to b for every two vertices a, b and hence by previous Theorem there is a path from a to b. This would imply that G is connected.

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The even degree follows from the fact that whenever we enter a vertex v (in the circuit) we leave v. Therefore number of times we visit a vertex is even (visit using different edges).

□

Proof.

Now suppose G is connected and degree of each vertex is even. We use induction on the number of edges of G (for $n = 1, 2$ easy to see).

Select vertex s and construct a circuit C_1 containing s there exists such a circuit (consider a longest trail starting at s).

Remove C_1 and now in the remaining graph C' each vertex has even degree. If C' is not connected it has connected components and each connected components is Eulerian (by induction).

Now we can traverse each circuit of each connected components as we traverse C_1. This way we can pick up all the edges and go back to s.

□

Theorem

*Let $G = (V, E)$ be an undirected graph or multigraph with no isolated vertices. Then G has an Euler trail if and only if G is connected and has exactly two
vertices of odd degree.*

Proof.

Add edge a, b where a, b have odd degree. Now G' is Eulerian and hence it has an Eulerian circuit. By removing edge ab we obtained the desired trail in G

□

Definition

Let $G = (V, A)$ be a directed graph or multigraph. For each vertex v

(a) The *incoming* or *in-degree* of v is the number of arcs coming to v and denoted by $id(v)$.

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Theorem

Let $G = (V, A)$ be an directed graph or multigraph with no isolated vertices. Then G has a directed Euler circuit if and only if G is connected and for each vertex v, $id(v) = od(v)$.

Hamiltonian cycles and paths

Definition

Let $G = (V, E)$ be an undirected graph or multigraph. We say G has a *Hamilton* cycle if there is a cycle in G that contains every vertex of G. We say G has a *Hamilton* path if there is a path in G that contains every vertex of G.

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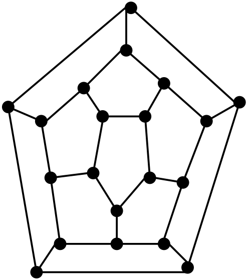

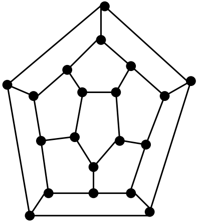

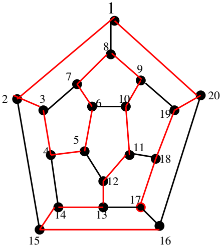
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Theorem

Let T be a tournament (between every pair u, v ($u \neq v$) exactly one of the uv and vu is an arc). Show that T has a directed Hamilton path.

Proof.

Let P be a longest directed path in T. If P is a Hamilton path then we are done. Otherwise suppose there is a vertex u outside P. Let $P = v_1, v_2, \ldots, v_m$. Let $i \leq m$ be a maximum index such that $v_i u$ is an arc. If i does not exists then it means uv_1 is an arc and hence $u, v_1, v_2, \ldots, v_m$ is a longer directed path. If i exists then it means uv_{i+1}, $i < m$ is an arc. If $i = m$ then Pu is a longer directed path, a contradiction. If $i < m$ then
$P' = v_1, v_2, \ldots, v_i, u, v_{i+1}, v_{i+2}, \ldots, v_m$ is a longer path, a contradiction. □

Theorem

Let T be a strongly connected tournament (between every pair (u, v) $(u \neq v)$ there is a directed path). Show that T has a directed Hamilton cycle.

Proof.

Since T is strong it has a closed trail and hence it has a directed cycle. Let $C = v_1, v_2, \ldots, v_m, v_1$ be a longest directed cycle.
If $|C| = |T|$ then we are done. Otherwise suppose there is a vertex $u \in T - C$. If there exists an index i such that $v_i u, u v_{i+1}$ are arcs of T then $C' = v_1, v_2, \ldots, v_i, u, v_{i+1}, v_{i+2}, \ldots, v_m, v_1$ is a longer cycle, a contradiction.
So for every vertex u we have either $V(C) \Rightarrow u$ or $u \Rightarrow V(C)$. Let $I = \{u | u \Rightarrow V(C)\}$ and
$O = \{u | V(C) \Rightarrow u\}$. Note that $V(T) = V(C) \cup I \cup O$. If there is an arc from $u \in O$ to $v \in I$ then $C' = v_1, u, v, v_2, \ldots, v_m, v_1$ is longer than C, contradiction. Therefore $I \Rightarrow O$. But in this case T is not strong, a contradiction. □

Theorem

Let $G = (V, E)$ be a loop free simple graph with $|V| = n \geq 2$. If $deg(x) + deg(y) \geq n - 1$ for all x, y, $x \neq y$, then G has a Hamilton path.

Proof.

We show that G is connected. Suppose G_1 and G_2 are two components of G. Let $v_i \in V(G_i)$, $i = 1, 2$. Now v_1 is adjacent to at most $|G_1| - 1$ vertices and v_2 is adjacent to at most $|G_2| - 1$ vertices and hence $deg(v_1) + deg(v_2) \leq |G_1| + |G_2| - 2 \leq n - 2$, contradiction. Let $P = v_1, v_2, \ldots, v_m$ be a longest path in G. Note that $uv_1 \notin E$ for $u \in G - P$ otherwise uP is a longer path. Similarly $v_m u$ is not an edge as otherwise Pu is a longer path. Thus the neighbors of v_1, v_m are on P.

□

Proof.

We show there exists some i such that $v_1 v_i$ is an edge and $v_{i-1} v_m$ is also an edge. Suppose this is not the case. Then for each neighbor v_j of v_1, v_{j-1} is not neighbor of v_m. We also note that $v_1 v_m$ is not an edge (otherwise we are done) and hence the neighbors of v_1, v_m are between $v_2, v_3, \ldots, v_{m-1}$.

Note that if v_1 has r neighbors on P, v_m has $r-1$ non-neighbors on P. This means v_m has at most $m-2-(r-1)$ neighbors.

Therefore $deg(v_1) + deg(v_m) \leq m-1$, a contradiction since $m < n$.

So there exists such i and hence

$C = v_1, v_2, \ldots, v_{i-1}, v_m, v_{m-1}, \ldots, v_{i+1}, v_i, v_1$ is a cycle in G. Now since G is connected, there is an edge from C to a vertex outside C, say an edge $v_j u$. Now $u, v_j, v_{j+1}, \ldots, v_m, v_1, v_2, \ldots, v_{j-1}$ is a longer path, a contradiction.

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Theorem

Let $G = (V, E)$ be a loop free simple graph with $|V| = n \geq 2$. If $deg(x) + deg(y) \geq n$ for all x, y, $x \neq y$, then G has a Hamilton cycle.

Proof.

By the previous Theorem G has a Hamilton path. Let $P = v_1, v_2, \ldots, v_n$. Now similar to the previous proof there exists i such that $v_1 v_i$ is an edge and $v_{i-1} v_n$ is also an edge. Suppose this is not the case. Then for each neighbor v_j of v_1, v_{j-1} is not neighbor of v_n. We also note that $v_1 v_n$ is not an edge (otherwise we are done) and hence the neighbors of v_1, v_n are between $v_2, v_3, \ldots, v_{n-1}$. Note that if v_1 has r neighbors on P, v_n has $r - 1$ non-neighbors on P. This means v_n has at most $n - 2 - (r - 1)$ neighbors. Therefore $deg(v_1) + deg(v_n) \leq n - 1$, a contradiction. □

Problem

Suppose there are n students in each of the three schools A, B, C. Each student from each of these three universities has at least $n + 1$ friends from the other two universities altogether. Show that there are three students x, y, z from A, B, C (respectively) that are pairwise friend.

Exercises

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Proof.

We consider a graph G with three partite sets A, B, C. We put an edge from x to y if they are friends and in different universities. We need to show there is a triangle in G. Consider a student u from one partite sets that has the maximum number of friends in one other part.

□

Proof.

We may assume $x \in A$ has a set S of r neighbors in B (r maximum). Let $z \in C$ be a neighbor of x. If z is adjacent to a vertex in S then we are done. Otherwise z has at most $n - r$ neighbors in B. Since z has $n + 1$ neighbors in $A \cup B$, we conclude that z must have at least $n + 1 - (n - r) = r + 1$ neighbors is A. Therefore $deg_A(z) = r + 1$, a contradiction to the choice of x. □

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Show that in every tournament T, there exists a vertex x such that for every other vertex $y \neq x$, either xy is an arc or there exists z such that xz, zy are arcs.

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Proof.

Consider a vertex x with maximum out-degree. Let O be the set of out neighbors of x and I be the set of in neighbors of x. Observe that $|T| = |I| + |O| + 1$. Consider an arbitrary vertex $y \in O$. If there exists a vertex $z \in I$ such that zy is an arc then we are done. Therefore for every $z \in O$, yz is an arc. Now $od(y) = |O| + 1$ and hence $od(y) > od(x)$, contradiction. □