# Bi-Arc Digraphs and Conservative Polymorphisms 

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#### Abstract

We introduce the class of bi-arc digraphs, and show they coincide with the class of digraphs that admit a conservative semi-lattice polymorphism, i.e., a min ordering. Surprisingly this turns out to be also the class of digraphs that admit totally symmetric conservative polymorphisms of all arities. We give an obstruction characterization of, and a polynomial time recognition algorithm for, this class of digraphs. The existence of a polynomial time algorithm was an open problem due to Bagan, Durand, Filiot, and Gauwin.

We also discuss a generalization to $k$-arc digraphs, which has a similar obstruction characterization and recognition algorithm.

When restricted to undirected graphs, the class of bi-arc digraphs is included in the previously studied class of bi-arc graphs. In particular, restricted to reflexive graphs, bi-arc digraphs coincide precisely with the well known class of interval graphs. Restricted to reflexive digraphs, they coincide precisely with the class of adjusted interval digraphs, and restricted to bigraphs, they coincide precisely with the class of two directional ray graphs. All these classes have been previously investigated as analogues of interval graphs. We believe that, in a certain sense, bi-arc digraphs are the most general digraph version of interval graphs with nice algorithms and characterizations.


## 1 Introduction

In this paper we investigate a class of digraphs that can be viewed as a digraph generalization of interval graphs. A graph $H$ is an interval graph if it is the intersection graph of a family of intervals on the real line, i.e., if there exist intervals $I_{v}, v \in V(H)$, such that $u v \in A(H)$ if and only if $I_{u} \cap I_{v} \neq \emptyset$.

Interval graphs are one of the most popular and useful graph classes; they admit efficient recognition algorithms, elegant obstruction characterizations, and frequently occur in practice $[4,6,16,17,18,30]$. The classical digraph versions of interval graphs [35] lack many of these desirable attributes, although the authors have (in an paper joint with T. Feder and J. Huang) proposed a version of interval digraphs that shares with interval graphs many nice properties; that version only applies to reflexive digraphs (that is, digraphs in which every vertex has a loop) [12].

[^0]Specifically, a reflexive digraph $H$ is an adjusted interval digraph if there are two families of real intervals $I_{v}, J_{v}, v \in V(H)$, and with the same left endpoint, such that $u v \in A(H)$ if and only if $I_{u} \cap J_{v} \neq \emptyset$. Adjusted interval digraphs have efficient recognition algorithms, forbidden structure characterizations, and certain desirable algorithmic properties [12]. This suggested they should be the analogue of interval graphs amongst reflexive digraphs. The question of which general digraphs should be viewed as an appropriate extension of interval graphs remained open.

There is, however, a natural concept for what interval digraphs should be. To motivate this concept, we introduce the notion of a polymorphism. Given digraphs $G$ and $H$, a homomorphism of $G$ to $H$ is a mapping $f: V(G) \rightarrow V(H)$ such that $u v \in A(G)$ implies $f(u) f(v) \in A(H)$. A product of digraphs $G$ and $H$ has the vertex set $V(G) \times V(H)$ and arc set $E(G \times H)$ consisting of all pairs $(u, x)(v, y)$ such that $u v \in A(G)$ and $x y \in A(H)$. The product of $k$ copies of the same graph $H$ is denoted by $H^{k}$. A polymorphism of $H$ of order $k$ is a homomorphism of $H^{k}$ to $H$. In other words, it is a mapping $f$ from the set of $k$-tuples over $V(H)$ to $V(H)$ such that if $x_{i} y_{i} \in A(H)$ for $i=1,2, \ldots, k$, then $f\left(x_{1}, x_{2}, \ldots, x_{k}\right) f\left(y_{1}, y_{2}, \ldots, y_{k}\right) \in A(H)$. Polymorphisms of $H$ play a pivotal role in recognizing digraphs (and more general relational systems) $G$ that admit a homomorphism to $H$ [3], cf. also [20].

A polymorphism $f$ is conservative if each value $f\left(x_{1}, x_{2}, \ldots, x_{k}\right)$ is one of the arguments $x_{1}, x_{2}, \ldots$ , $x_{k}$. A binary (order two) $f$ polymorphism that is conservative and commutative $(f(x, y)=f(y, x)$ for all vertices $x, y$ ) is called a CC polymorphism. If $f$ is additionally associative $(f(f(x, y), z)=$ $f(x, f(y, z))$ for all vertices $x, y, z)$ it will be called a conservative semi-lattice or an CSL polymorphism. A CSL polymorphism of $H$ naturally defines a binary relation $x \leq y$ on the vertices of $H$ by $x \leq y$ if and only if $f(x, y)=x$; by associativity, the relation $\leq$ is a linear order on $V(H)$, which we call a min ordering of $H$. In other words, an ordering of vertices $v_{1}<v_{2}<\cdots<v_{n}$ of $H$ is a $\min$ ordering if and only if $u v \in A(H), u^{\prime} v^{\prime} \in A(H) \Longrightarrow \min \left(u, u^{\prime}\right) \min \left(v, v^{\prime}\right) \in A(H)$. Yet another way to state this is as follows. The ordering $v_{1}<v_{2}<\cdots<v_{n}$ of $V(H)$ is a min ordering if and only if $u v \in A(H), u^{\prime} v^{\prime} \in A(H)$ and $u<u^{\prime}, v^{\prime}<v$ implies that $u v^{\prime} \in A(H)$. It is also clear that, conversely, a min ordering $<$ of $H$ defines a CSL polymorphism $f: H^{2} \rightarrow H$ by $f(x, y)=\min (x, y)$.

Interval graphs are known to have an ordering characterization: $H$ is an interval graph if and only if $V(H)$ can be linearly ordered $v_{1}<v_{2}<\cdots<v_{n}$ so that $u<v<w$ and $u w \in A(H)$ imply that $u v \in A(H)$. However, it is easy to check, see [24], that a reflexive graph $H$ (every vertex has a loop) is an interval graph if and only if it has a min ordering. (Note that it is reasonable to define an interval graph to be reflexive, since any interval intersects itself.) It is also known that a reflexive digraph is an adjusted interval digraph if and only if it has a min ordering [12]. Moreover, min ordering on bigraphs (bipartite graphs, or more precisely bipartite digraphs with all edges between the parts oriented in the same direction) characterizes the class complements of circular arc graphs [23], and equivalently the class of two-dimensional ray graphs, a natural generalization of interval graphs in the class of bipartite graphs [19, 32]. Thus, it has long been believed that digraphs with min ordering are the right notion for the general version of interval digraphs. However, it was not known whether this class of digraphs can be recognized in polynomial time, whether it has an obstruction characterization, and whether it has any geometric meaning. We remedy the situation on all three fronts. We give a geometric representation of the class of digraphs with a min ordering, we give an obstruction characterization of it, and we also give a certifying polynomial time recognition algorithm for the class. The existence of such an algorithm was posed as an open problem in [1, 26]; it is somewhat unexpected, since for more general relational structures the
recognition problem is known to be NP-complete [5]. We note that for structures with two binary relations (digraphs with two kinds of arcs), the recognition problem of having a min ordering is NP-complete, via a reduction (from a preliminary version of [1]) similar to that in the proof of Theorem 4.9 [26].

Other questions about the existence of polymorphisms of various kinds have turned out to also be interesting $[2,5,13,22,25,31]$. In particular, the existence of conservative polymorphisms is a hereditary property (if $H$ has a particular kind of conservative polymorphism, then so does any induced subgraph of $H$ ). Thus these questions present interesting problems in graph theory. In particular, we note that there are forbidden induced substructure characterizations for the existence of conservative majority [25] and conservative Maltsev [7, 25] polymorphisms.

A polymorphism $f$ of $H$ of order $k$ is totally symmetric if $f\left(x_{1}, x_{2}, \ldots, x_{k}\right)=f\left(y_{1}, y_{2}, \ldots, y_{k}\right)$ whenever the sets $\left\{y_{1}, y_{2}, \ldots, y_{k}\right\}$ and $\left\{x_{1}, x_{2}, \ldots, x_{k}\right\}$ are the same. A set polymorphism of $H$ is a mapping $f$ of the non-empty subsets of $V(H)$ to $V(H)$, such that $f(S) f(T) \in A(H)$ whenever $S, T$ are non-empty subsets of $V(H)$ with the property that for each $s \in S$ there is a $t \in T$ with st $\in A(H)$ and also for every $t \in T$ there is an $s \in S$ with $s t \in A(H)$. It is easy to see, cf. [5, 14], that $H$ has a conservative set polymorphism if and only if it has conservative totally symmetric polymorphisms of all orders $k$. In such a case we say $H$ has a CTS polymorphism.

We note that a digraph $H$ that admits a CSL polymorphism also admits a CTS polymorphism: the conservative set function that assigns to each set $S$ the minimum under the min ordering. Moreover, a CTS polymorphism applies to all orders, including order two, whence it implies a CC polymorphism. Thus the class of digraphs with a min ordering is included in the class of digraphs with a conservative set polymorphism, which is included in the class of digraphs with a CC polymorphism. We will give forbidden induced structure characterizations for all three of these digraph classes, from which it will follow that (surprisingly) the first two classes coincide. In all three cases, the characterizations yield polynomial time recognition algorithms. Although this was known for CC polymorphisms [14] (even known to be in non-deterministic logspace [5]), it was open for CSL and CTS polymorphisms [1, 5, 26]. We emphasize that our results are specific to digraphs; for more general relational structures it is known, for instance, that the existence of CSL polymorphisms is NP-complete, even with two binary relations [1, 5].

## 2 Preliminaries

A digraph $H$ consists of a vertex set $V(H)$ and an arc set $A(H)$. Each arc is an ordered pair of vertices. We say that $u v \in A(H)$ is an arc from $u$ to $v$. Sometimes we emphasize this by saying that $u v$ is a forward arc of $H$, and also say $v u$ is a backward arc of $H$. We say that $u, v$ are adjacent in $H$ if $u v$ is a forward or a backward arc of $H$. A walk in $H$ is a sequence $P=x_{0}, x_{1}, \ldots, x_{n}$ of consecutively adjacent vertices of $H$; note that a walk has a designated first and last vertex. A path $P=x_{0}, x_{1}, \ldots, x_{n}$ is a walk in which all $x_{i}$ are distinct. A walk $P=x_{0}, x_{1}, \ldots, x_{n}$ is closed if $x_{0}=x_{n}$ and a cycle if all other $x_{i}$ are distinct. A walk is directed if all its arcs are forward. A vertex $u^{\prime}$ is said to be reachable from a vertex $u$ in $H$ if there is a directed walk from $u$ to $u^{\prime}$ in $H$; a set $U^{\prime}$ is reachable from a set $U$ if some vertex of $U^{\prime}$ is reachable from some vertex of $U$. Note that every vertex is reachable from itself, by a directed path of length zero.

For walks $P$ from $a$ to $b$, and $Q$ from $b$ to $c$, we denote by $P+Q$ the walk from $a$ to $c$ which is the concatenation of $P$ and $Q$, and by $P^{-1}$ the walk $P$ traversed in the opposite direction, from $b$
to $a$. We call $P^{-1}$ the reverse of $P$. For a closed walk $C$, we denote by $C^{a}$ the concatenation of $C$ with itself $a$ times.

The net length of a walk is the number of forward arcs minus the number of backward arcs. A closed walk is balanced if it has net length zero; otherwise it is unbalanced. Note that in an unbalanced closed walk we may always choose a direction in which the net length is positive (or negative). A digraph is unbalanced if it contains an unbalanced closed walk (or equivalently an unbalanced cycle); otherwise it is balanced. It is easy to see that a digraph is balanced if and only if it admits a labeling of vertices by non-negative integers so that each arc goes from a vertex with a label $i$ to a vertex with a label $i+1$. The height of $H$ is the maximum net length of a walk in $H$. Note that an unbalanced digraph has infinite height, and the height of a balanced digraph is the greatest label in a non-negative labeling in which some vertex has label zero.

For a walk $P=x_{0}, x_{1}, \ldots, x_{n}$ and any $i \leq j$, we denote by $P\left[x_{i}, x_{j}\right]$ the walk $x_{i}, x_{i+1}, \ldots, x_{j}$, and call it a prefix of $P$ if $i=0$. Suppose $P=x_{0}, x_{1}, \ldots, x_{n}$ is a walk in $H$ of net length $k \geq 0$. We say that $H$ is constricted from below if the net length of any prefix $P\left[x_{0}, x_{j}\right]$ is non-negative, and is constricted from above if the net length of any prefix is at most $k$. We also say that $P$ is constricted if it is constricted both from below and from above. Moreover, we say that $P$ is strongly constricted from below or above, if the corresponding net lengths are strictly positive or smaller than $k$. For walks $P$ of net length $k<0$, we say that $P$ is (strongly or not) constricted below, or above, or both, if the above definitions apply to the reverse walk $P^{-1}$.

Consider a cycle $C$ in $H$ of non-zero net length $k$. A vertex $v$ is extremal in $C$ if either $k>0$ and traversing $C$, starting at $v$ in the positive direction yields a walk constricted from below, or $k<0$ and traversing $C$ (starting at $v$ ) in the negative direction yields a walk constricted from above. We observe that a cycle $C$ of positive net length $k$ has $k$ extremal vertices, since starting at any vertex $x$ the net length of the prefix $C[x, v]$ varies from 0 to a possibly negative minimum $m$, but ending with $k>0$. We can let $v_{0}$ be the last vertex with the net length of $C\left[x, v_{0}\right]$ equal to the minimum $m$ (possibly $v_{0}=x$ if $m=0$ ). We can let $v_{i}$ be the last vertex with the net length of $C\left[v_{0}, v_{i}\right]$ equal to $i, i=1,2, \ldots, k-1$. Note that each walk $C\left[v_{i}, v_{i+1}\right]$ is constricted from below and has net length one. We also note for future reference that any other extremal vertex of $C$ has a walk of net length zero to one of $v_{0}, v_{1}, \ldots, v_{k-1}$. Similar observations apply when $C$ has a negative net length. A cycle of $H$ is induced if $H$ contains no other arcs on the vertices of the cycle. In particular, an induced cycle with more than one vertex does not contain a loop.

The following lemma is well known. (For a proof, see [21, 37] or Lemma 2.36 in [24]).
Lemma 2.1 Let $P_{1}$ and $P_{2}$ be two constricted walks of net length $r$. There exists a constricted path $P$ of net length $r$ that admits a homomorphism $f_{1}$ to $P_{1}$ and a homomorphism $f_{2}$ to $P_{2}$, such that each $f_{i}, i=1,2$ takes the starting vertex of $P$ to the starting vertex of $P_{i}$ and the ending vertex of $P$ to the ending vertex of $P_{i}$.

We call $P$ a common pre-image of $P_{1}$ and $P_{2}$. In particular, we emphasize that we use the term pre-image of a path $P^{\prime}$ to be a path $P$ that admits a homomorphism to $P^{\prime}$ taking the first vertex of $P$ to the first vertex of $P^{\prime}$ and the last vertex of $P$ to the last vertex of $P^{\prime}$.

We visualize $P$ as following the arcs of $P_{1}$ (respectively $P_{2}$ ), starting where $P_{1}$ (respectively $P_{2}$ ) starts and ending where $P_{1}$ (respectively $P_{2}$ ) ends, but possibly taking intermediate back and forth steps. When we mention pre-images of walks, we shall always assume that the pre-image starts at the starting vertex and ends at the ending vertex of the original walk.

## 3 Warm-up: Obstructions to CC polymorphisms

In this section we introduce a construction that will be used throughout the paper. We also illustrate the techniques on the easy case of CC polymorphisms. As mentioned earlier, the existence of CC polymorphisms is well understood; it is solvable by 2-SAT [14, 15], so it is both known to be decidable in polynomial time and characterized by forbidden substructures [33]. In fact, it is shown in [5] that it can be decided in non-determinstic logspace. Nevertheless, we present our obstructions to the existence of CC polymorphism because they illuminate the general obstructions to CSL polymorphisms, and underscore the relationship between the two types of polymorphisms and their obstructions.

Suppose $f$ is a CC polymorphism of $H$. If $x x^{\prime}, y y^{\prime} \in A(H)$ but $x y^{\prime} \notin A(H)$, then $f(x, y)=x$ implies $f\left(x^{\prime}, y^{\prime}\right)=x^{\prime}$. A similar situation arises if $x^{\prime} x, y^{\prime} y \in A(H)$ but $y^{\prime} x \notin A(H)$ : again $f(x, y)=$ $x$ implies $f\left(x^{\prime}, y^{\prime}\right)=x^{\prime}$.

We define two walks $P=x_{0}, x_{1}, \ldots, x_{n}$ and $Q=y_{0}, y_{1}, \ldots, y_{n}$ in $H$ to be congruent, if they follow the same pattern of forward and backward arcs, i.e., $x_{i} x_{i+1}$ is a forward (backward) arc if and only if $y_{i} y_{i+1}$ is a forward (backward) arc (respectively). Suppose the walks $P, Q$ as above are congruent. We say an arc $x_{i} y_{i+1}$ is a faithful arc from $P$ to $Q$, if it is a forward (backward) arc when $x_{i} x_{i+1}$ is a forward (backward) arc (respectively), and we say an arc $y_{i} x_{i+1}$ is a faithful arc from $Q$ to $P$, if it is a forward (backward) arc when $x_{i} x_{i+1}$ is a forward (backward) arc (respectively). We say that $P$ avoids $Q$ if there is no faithful arc from $P$ to $Q$ at all.

We note for future reference that if two congruent walks $P, Q$ avoid each other, then the same is true for any common pre-images $P^{\prime}, Q^{\prime}$. (Note that if $P$ avoids $Q$, it is not necessarily true that $P^{\prime}$ avoids $Q^{\prime}$ because of the back steps involved in the pre-images.)

We define the pair digraph $H^{+}$as follows. The pairs of $H^{+}$are all ordered pairs $(x, y)$ of distinct vertices of $H$, and $(x, y)\left(x^{\prime}, y^{\prime}\right) \in A\left(H^{+}\right)$just if

- $x x^{\prime}, y y^{\prime} \in A(H)$ but $x y^{\prime} \notin A(H)$, or
- $x^{\prime} x, y^{\prime} y \in A(H)$ but $y^{\prime} x \notin A(H)$.

In the former case we call the $\operatorname{arc}(x, y)\left(x^{\prime}, y^{\prime}\right) \in A\left(H^{+}\right)$a positive arc, and the the second case we call it a negative arc. We say positive arc $(x, y)\left(x^{\prime}, y^{\prime}\right)$ is symmetric if $x x^{\prime}, y y^{\prime} \in A(H)$ but $x y^{\prime}, y x^{\prime} \notin A(H)$ We say negative arc $(x, y)\left(x^{\prime}, y^{\prime}\right)$ is symmetric if $x^{\prime} x, y^{\prime} y \in A(H)$ but $y^{\prime} x, x^{\prime} y \notin$ $A(H)$. A path in $H^{+}$is symmetric if all its arcs are symmetric.

Note that in $H^{+}$we have an arc from $(x, y)$ to $\left(x^{\prime}, y^{\prime}\right)$ if and only if there is an arc from $(y, x)$ to $\left(y^{\prime}, x^{\prime}\right)$. We call this the skew property of $H^{+}$.

Note that a directed walk $W$ in $H^{+}$corresponds precisely to a pair of congruent walks $P, Q$ in $H$ such that $P$ avoids $Q$. The net value of the directed walk $W$ is defined to be the net length of the walk $P($ or $Q)$. It is the difference between the number of positive and negative arcs of $W$. We say that $W$ has constricted values if the walk $P$ (or $Q$ ) is constricted, i.e., if each initial segment of $W$ has net value between zero and the net value of $W$. Walks with values constricted below or above are defined similarly.

According to our observation, having a directed walk in $H^{+}$from $(x, y)$ to $\left(x^{\prime}, y^{\prime}\right)$ means that $f(x, y)=x$ implies that $f\left(x^{\prime}, y^{\prime}\right)=x^{\prime}$ in any CC polymorphism $f$ of $H$. In particular, for any strong component $C$ of $H^{+}$, and any CC polymorphism $f$ of $H$, either all pairs $(x, y) \in C$ are
mapped by $f$ to the first coordinate or all are mapped to the second coordinate. Moreover, if $C_{2}$ is reachable from $C_{1}$ in $H^{+}$, and $f$ maps pairs in $C_{1}$ to the first coordinate, then it also maps pairs in $C_{2}$ to the first coordinate. An invertible pair of $H$ is a vertex $(x, y)$ of $H^{+}$such that $(x, y)$ and $(y, x)$ are in the same strong component of $H^{+}$. It is easy to see, using the skew property of $H^{+}$, that if one vertex of a strong component of $H^{+}$is invertible, then so are all others, and, that if $H$ has no invertible pairs, then each component $C$ has a corresponding dual component $C^{\prime}$ such that $(x, y) \in C$ if and only if $(y, x) \in C^{\prime}$. In fact, constructing a CC polymorphism for $H$ amounts to selecting, one strong component from each pair $C, C^{\prime}$ of dual strong components, to map $f$ to the first coordinate, in such a way, that if $C_{2}$ is reachable from $C_{1}$ in $H^{+}$, and $C_{1}$ was selected, then $C_{2}$ is also selected. This is easy to do, for instance by the following algorithm.

We say that a strong component $C$ of a digraph is ripe if no other strong component is reachable from it. The algorithm begins by selecting a ripe strong component $C$ of $H^{+}$, and deleting it and its dual $C^{\prime}$ from $H^{+}$, continuing the same way with the remaining digraph.

This algorithm clearly selects exactly one pair from $(x, y),(y, x)$ for each $x \neq y$. It remains to prove that if $(x, y)$ is selected and $(x, y)\left(x^{\prime}, y^{\prime}\right)$ is an arc of $H^{+}$, then $\left(x^{\prime}, y^{\prime}\right)$ is also selected. This is clear if $(x, y),\left(x^{\prime}, y^{\prime}\right)$ are from the same strong component of $H^{+}$. Otherwise, suppose for a contradiction, that $(x, y)$ was selected in a component $C$ after $\left(x^{\prime}, y^{\prime}\right)$ was deleted in a component $D^{\prime}$, where $D$ was selected (before $C$ ). By the skew property of $H^{+}$, we see that $D$ has an arc to $C^{\prime}$, so it was selected when it was not yet ripe.

Theorem 3.1 $A$ digraph $H$ admits a CC polymorphism if and only if no strong component of $H^{+}$ contains an invertible pair.

## 4 Min Ordering and Geometric Representation

In this section we introduce a geometric representation of digraphs that admit a min-ordering. Let $C$ be a circle with two distinguished points (the poles) $N$ and $S$, and let $H$ be a digraph. Let $I_{v}, v \in V(H)$ and $J_{v}, v \in V(H)$ be two families of arcs on $C$ such that each $I_{v}$ contains $N$ but not $S$, and each $J_{v}$ contains $S$ but not $N$. We say that the families $I_{v}$ and $J_{v}$ are consistent if they have the same clockwise order of their clockwise ends, i.e., if the clockwise end of $I_{v}$ precedes in the clockwise order the clockwise end of $I_{w}$ if and only if the clockwise end of $J_{v}$ precedes in the clockwise order the clockwise end of $J_{w}$. Suppose two families $I_{v}, J_{v}$ are consistent; we define an ordering $<$ on $V(H)$ where $v<w$ if and only if the clockwise end of $I_{v}$ precedes in the clockwise order the clockwise end of $I_{w}$; we call < the ordering generated by the consistent families $I_{v}, J_{v}$.

A bi-arc representation of a digraph $H$ is a consistent pair of families of circular arcs, $I_{v}, J_{v}, v \in$ $V(H)$, such that $u v \in A(H)$ if and only if $I_{u}$ and $J_{v}$ are disjoint. A digraph $H$ is called a bi-arc digraph if it has a bi-arc representation.

Theorem 4.1 A digraph $H$ admits a min ordering if and only if $H$ is a bi-arc digraph.

Proof: Suppose $I_{v}, J_{v}$ form a bi-arc representation of $H$. We claim that the ordering < generated by $I_{v}, J_{v}$ is a min ordering of $H$. Indeed, suppose $u<u^{\prime}$ and $v^{\prime}<v$ have $u v, u^{\prime} v^{\prime} \in A(H)$. Then $I_{u^{\prime}}$ spans the area of the circle between $N$ and the clockwise end of $I_{u}$, and $J_{v}$ spans the area of the circle between $S$ and the clockwise end of $J_{v^{\prime}}$. (See Figure 1.) This implies that $I_{u}$ and $J_{v^{\prime}}$ are disjoint:


Figure 1: Illustration for the proof of Lemma 4.1
indeed, the counterclockwise end of $I_{u}$ is blocked from reaching $J_{v^{\prime}}$ by $J_{v}$ (since $u v \in A(H)$ ), and the counterclockwise end of $J_{v^{\prime}}$ is blocked from reaching $I_{u}$ by $I_{u^{\prime}}$ (since $u^{\prime} v^{\prime} \in A(H)$ ). (The clockwise ends are fixed by the ordering $<$.)

Conversely, suppose < is a min ordering of $H$. We construct families of $\operatorname{arcs} I_{v}$ and $J_{v}$, with $v \in V(H)$, as follows. The intervals will $I_{v}$ contain $N$ but not $S$, the intervals $J_{v}$ will contain $S$ but not $N$. The clockwise ends of $I_{v}$ are arranged in clockwise order according to $<$, as are the clockwise ends of $J_{v}$. The counterclockwise ends will now be organized so that $I_{v}, J_{v}, v \in V(H)$ becomes a bi-arc representation of $H$. For each vertex $v \in V(H)$, let $o_{v}$ denote the last out-neighbour of $v$, and let $i_{v}$ denote the last in-neighbour of $v$, where "last" refers to the order $<$. Then we extend each $I_{v}$ counterclockwise as far as possible without intersecting $J_{o_{v}}$, and extend each $J_{v}$ counterclockwise as far as possible without intersecting $I_{i_{v}}$. We claim this is a bi-arc representation of $H$. Clearly, if $w>o_{v}$, then $I_{v}$ intersects $J_{w}$ by the construction, and similarly for $u>i_{v}$ we have $J_{v}$ intersecting $I_{u}$. This leaves disjoint all pairs $I_{u}, J_{v}$ such that $u \leq i_{v}$ and $v \leq o_{u}$; since $u o_{u}, i_{v} v$ are arcs of $H$, the definition of min ordering implies that $u v$ is in fact an arc of $H$, as required.

If $<$ is a min ordering of $H$, then min (with respect to $<$ ) is a CC polymorphism of $H$, so much of the above observations apply verbatim. If $x x^{\prime}, y y^{\prime} \in A(H)$ but $x y^{\prime} \notin A(H)$, (or if $x^{\prime} x, y^{\prime} y \in A(H)$ but $y^{\prime} x \notin A(H)$ ), then $x<y$ implies $x^{\prime}<y^{\prime}$. (Note that otherwise $x<y, y^{\prime}<x^{\prime}$ would violate the min property).

A circuit in $H^{+}$is a set of pairs $\left(x_{0}, x_{1}\right),\left(x_{1}, x_{2}\right), \ldots,\left(x_{n-1}, x_{n}\right),\left(x_{n}, x_{0}\right)$ of $H^{+}$. Note that an invertible pair of $H$ is a circuit with $n=1$ in a strong component of $H^{+}$.

If a strong component of $H^{+}$contains a circuit, then $H$ cannot have a min ordering, since $x_{0}<x_{1}$ implies $x_{0}<x_{1}<x_{2}<\cdots<x_{n}<x_{0}$ (and similarly for $x_{0}>x_{1}$ ), contradicting the transitivity of $<$. We have proved one direction of the following result.

Theorem 4.2 $A$ digraph $H$ admits a min ordering if and only if no strong component of $H^{+}$ contains a circuit.

This nicely complements Theorem 3.1, highlighting the difference in the obstructions.
We now single out a particular situation in which a circuit occurs in one strong component of the pair digraph $H^{+}$.

Theorem 4.3 If $H$ contains an induced cycle of net length greater than one, then a strong component of $\mathrm{H}^{+}$contains a circuit.

Proof: Suppose $C$ is an induced cycle of net length $k>1$. Recall that the cycle $C$ has $k$ extremal vertices $v_{0}, v_{1}, \ldots, v_{k-1}$ with each $C\left[v_{i}, v_{i+1}\right]$ (subscript addition modulo $k$ ) constricted from below and of net length one.

We shall show that $\left(v_{0}, v_{1}\right),\left(v_{1}, v_{2}\right), \ldots,\left(v_{k-1}, v_{0}\right)$ belong to the same strong component of $H^{+}$. Indeed, for any $i=1, \ldots, k-1$, we shall exhibit a directed walk in $H^{+}$from $\left(x_{i-1}, x_{i}\right)$ to $\left(x_{i}, x_{i+1}\right)$. These directed walks in $H^{+}$will be constructed out of pairs of walks on the cycle $C$.

Assume first that the height of $C\left[v_{i-1}, v_{i}\right]$ is at most the height of $C\left[v_{i}, v_{i+1}\right]$. To be able to use Lemma 2.1, we consider the last vertex $h$ of $C\left[v_{i}, v_{i+1}\right]$ maximizing the net length of $C\left[v_{i}, h\right]$, and the first vertex $h^{\prime}$ of $C\left[v_{i}, v_{i+1}\right]$ such that $C\left[h^{\prime}, h\right]$ has net length one.

Now $C\left[v_{i-1}, h^{\prime}\right]$ and $C\left[v_{i}, h\right]$ are constricted and have the same net length. Thus by Lemma 2.1 they have a common pre-image $A$. Also $C^{-1}\left[h^{\prime}, v_{i}\right]$ and $C\left[h, v_{i+1}\right]$ are constricted and have the same net length; thus they also have a common pre-image $B$. Let $X$ be the walk in $C$ from $v_{i-1}$ to $v_{i}$ corresponding to $A+B$, and let $Y$ be the walk in $C$ from $v_{i}$ to $v_{i+1}$ corresponding to $A+B$. We claim that $X$ avoids $Y$. Consider the $j$-th vertex $u$ of $X$, and the $(j+1)$-st vertex $v$ of $Y$. Note that the net lengths of $C\left[v_{0}, u\right]$ and $C\left[v_{0}, v\right]$ differ by two; since $C$ is an induced cycle of net length greater than one, there can be no faithful arc between $u$ and $v$. This implies that there is a directed path in $H^{+}$from $\left(v_{i-1}, v_{i}\right)$ to $\left(v_{i}, v_{i+1}\right)$.

If the height of $C\left[v_{i-1}, v_{i}\right]$ is greater than the height of $C\left[v_{i}, v_{i+1}\right]$, we argue analogously. We denote by $P$ the infinite walk obtained by continuously following $C$ in the positive direction. Let $h$ be the last vertex of $C\left[v_{i-1}, v_{i}\right]$ maximizing the net length of $C\left[v_{i-1}, h\right]$, and let $h^{\prime}$ be the first vertex of $P$ after $v_{i+1}$ such that $P\left[h, h^{\prime}\right]$ has net length zero. Now Lemma 2.1 can be applied to the walks $C\left[v_{i-1}, h\right]$ and $C\left[v_{i+1}, h^{\prime}\right]$, and to the walks $C\left[h, v_{i}\right]$ and $P^{-1}\left[h^{\prime}, v_{i+1}\right]$, yielding a common pre-image $A$ for the former pair and a common pre-image $B$ for the latter pair. The walk $X$ in $C$ from $v_{i-1}$ to $v_{i}$ corresponding to $A+B$ again avoids the walk $Y$ in $P$ from $v_{i}$ to $v_{i+1}$ corresponding to $A+B$.

We note that the avoidance of the walks didn't quite need that the cycle $C$ be induced. It should just not have certain chords. We state a useful version as follows.

Corollary 4.4 Suppose $C$ is a closed walk in $H$ of net length greater than one, and $x, y$ are two extremal vertices of $C$ such that the net length of $C[x, y]$ is positive. Let $P_{x}$ be the infinite walk starting at $x$, obtained by continuously following the cycle $C$ in the positive direction and let $P_{y}$ be obtained the same way starting at $y$. Let $X, Y$ be two congruent walks such that $X$ avoids $Y$ and $X$ and $X$ is a common pre-image of $P_{x}, P_{y}$ and $Y$ is also a common pre-image of $P_{x}, P_{y}$. Then some strong component of $H^{+}$contains a circuit.

## 5 The Algorithm

We are now ready to claim the converse of Theorem 4.2.
Theorem 5.1 A digraph $H$ admits a min ordering if and only if no strong component of the pair digraph $H^{+}$contains a circuit.

To prove the converse, we introduce an algorithm to construct a min ordering < of $H$, provided no strong component of the pair digraph $H^{+}$contains a circuit. This will prove Theorem 5.1.

Thus we shall assume that no strong component of $\mathrm{H}^{+}$contains a circuit.
At each stage of the algorithm, there are some pairs of $H^{+}$that have been chosen, and others that have been discarded. Let $V_{c}$ denote the set of chosen pairs, and $V_{d}$ the set of discarded pairs; the pairs in the set $S=V\left(H^{+}\right) \backslash\left(V_{c} \cup V_{d}\right)$ are called the remaining pairs. Initially we will have $V_{c}=V_{d}=\emptyset$, and throughout the algorithm we will maintain the following properties:

- $(a, b) \in V_{c}$ if and only if $(b, a) \in V_{d}$;
- if $(a, b) \in V_{c}$ and $(a, b)\left(a^{\prime}, b^{\prime}\right) \in A\left(H^{+}\right)$then $\left(a^{\prime}, b^{\prime}\right) \in V_{c}$;

Consequently, we will always have $V_{c} \cap V_{d}=\emptyset$, and each strong component of $H^{+}$lies entirely in one of the three sets $V_{c}, V_{d}, S$.

Moreover, at the end of the algorithm the set $S$ will be empty and we will have the following additional property:

- if $(a, b) \in V_{c}$ and $(b, c) \in V_{c}$ then $(a, c) \in V_{c}$

A step of the algorithm will consist of selecting one vertex $(a, b) \in S$ to be chosen (moving $(a, b)$ to $V_{c}$ ), and choosing all pairs $\left(a^{\prime}, b^{\prime}\right) \in S$ that can be reached from $(a, b)$ in $H^{+}$. (It is easy to check that a vertex that can be reached from $(a, b) \in S$ in $H^{+}$cannot be in $V_{d}$ : otherwise $(b, a) \in V_{c}$ can reach $\left(b^{\prime}, a^{\prime}\right)$, thus $\left(b^{\prime}, a^{\prime}\right) \in V_{c}$, contradicting $\left(a^{\prime}, b^{\prime}\right) \in S$.) So we use the term selected vertex for this special vertex, and the term selected strong component for the strong component of $H^{+}$ containing the selected vertex.

Recall that a strong component of $H^{+}$is balanced if every closed directed walk has net value zero, i.e., the same number of positive and negative arcs. A pair of $H^{+}$is called balanced if it lies in a balanced strong component.

Let $H^{*}$ be the sub-digraph of $H^{+}$induced by balanced pairs. The pairs in $H^{*}$ can be assigned levels so that if $(a, b)(c, d)$ is a positive arc in $H^{+}$, then the level of $(c, d)$ is one more than the level of $(a, b)$, and if $(a, b)(c, d)$ is a negative arc in $H^{+}$, then the level of $(c, d)$ is one less than the level of $(a, b)$.

Recall that a strong component of $H^{+}$is balanced if every closed directed walk has net value zero, i.e., the same number of positive and negative arcs. Let $H^{*}$ be the set of balanced components in $H^{+}$.

A pair $(x, y)$ in $H^{+}$is called extremal if the following hold.

- $(x, y)$ lies on a direct cycle $D=(X, Y)$ in $H^{+}$
- $X, Y$ are closed walks in $H$ such that $X$ avoids $Y$.
- $x, y$ are extremal vertices on $X, Y$ respectively.

A vertex $a \in V(H)$ is a source for a set $S, S \subseteq V\left(H^{+}\right)$, if for every vertex $b \in V(H)$ we have the following property:

- no $(c, a) \in S$ is reachable from any $(a, b) \in S$ in $H^{+}$

```
Algorithm 1 Algorithm to find a min ordering of input digraph \(H\)
    function MinOrdering( \(H\) )
        Construct \(H^{+}\)and compute its (strong) components
        if a component of \(H^{+}\)contains a circuit then return False
        if \(H^{+}\)has an unbalanced component then
            Set \(S=\) the set of all extremal pairs \(\triangleright\) extremal pairs are in unbalanced components
            Set \(V_{c}=V_{d}=\emptyset\)
        while \(S \neq \emptyset\) do
            Find a source \(p\) for \(S\)
            while there is a pair \((p, q) \in S\) do
                move all pairs from \(S\) reachable from \((p, q)\) to \(V_{c}\)
                move all pairs from \(S\) that can reach \((q, p)\) to \(V_{d}\)
        Construct \(H^{*}\), set \(R=V\left(H^{*}\right)\), and compute the levels of all pairs
        while \(R \neq \emptyset\) do
            Set \(S=\) the set of all pairs at the lowest level \(\ell\) of \(R\), and remove them from \(R\)
            while \(S \neq \emptyset\) do
                Find a vertex \(p \in V(H)\) that respects transitivity in \(V\left(H^{*}\right) \cap V_{c}\)
                while there is a pair \((p, q) \in S\) do
                    move all pairs from \(S\) reachable from \((p, q)\) to \(V_{c}\)
                    move all pairs from \(S\) that can reach \((q, p)\) to \(V_{d}\)
```

Let $B \subseteq V\left(H^{+}\right)$. We say $S$ has a source $p$ that respects transitivity in $B$ if for no $(p, q) \in S$ there is a chain of pairs $\left(q, p_{1}\right),\left(p_{1}, p_{2}\right), \ldots,\left(p_{r-1}, p_{r}\right),\left(p_{r}, p\right)$ all in $B$.

Note that the moves in lines $10,11,18,19$ mean remove the moved vertices from the set $S$.
We will show that a source exists for any of the sets $S$ encountered by the algorithm. Proposition 9.4 shows this for the first phase of the algorithm, handling unbalanced strong components (lines 411). It also shows that that throughout this phase the set $V_{c}$ does not contain a circuit. Proposition 10.1 similarly handles the second phase (lines 12-19), handling the remaining balanced components of $H^{+}$. In other words, the set $V_{c}$ does not contain a circuit throughout the execution of the algorithm.

At the end of the algorithm we exactly one pair from each $(x, y),(y, x)$ is chosen (present in $V_{c}$ ). Recall that the choices ensure that if $(x, y) \in V_{c}$ and it has an arc to $\left(x^{\prime}, y^{\prime}\right)$ in $H^{+}$, then $\left(x^{\prime}, y^{\prime}\right) \in V_{c}$ as well. We define a binary relation $<$ by setting $x<y$ if $(x, y) \in V_{c}$. Since there is no circuit among the chosen pairs, the relation $<$ is transitive, and hence a total order. Moreover, $<$ is a min ordering. Indeed, suppose $x<x^{\prime}, y^{\prime}<y$ and $x y, x^{\prime} y^{\prime} \in A(H)$ but $x y^{\prime} \notin A(H)$. Note that ( $x, x^{\prime}$ ) has an arc to $\left(y, y^{\prime}\right)$ in $H^{+}$; thus since $x<x^{\prime}$, the pair $\left(x, x^{\prime}\right) \in V_{c}$, and so ( $y, y^{\prime}$ ) should have been in $V_{c}$ as well, contrary to $y^{\prime}<y$.

In what follows we assume that there is no circuit in a strong component of $\mathrm{H}^{+}$.

## 6 Structural properties of the walks in $\mathbf{H}$

For simplicity, for two pairs $(u, v),\left(u^{\prime}, v^{\prime}\right)$ let $(u, v) \rightsquigarrow\left(u^{\prime}, v^{\prime}\right)$ denote that $\left(u^{\prime}, v^{\prime}\right)$ is reachable from $(u, v)$ in $H^{+}$and also denote a direct path in $H^{+}$from $(u, v)$ to $\left(u^{\prime}, v^{\prime}\right)$. If there is not direct path in $H^{+}$from $(u, v)$ to $\left(u^{\prime}, v^{\prime}\right)$ then we write $(u, v) \nLeftarrow \rightarrow\left(u^{\prime}, v^{\prime}\right)$.

In this section we formulate some useful facts. The first two deal with a situation where four walks $A, B, C, D$ start in four distinct vertices, but the end vertices of walks $B$ and $C$ coincide. Note that this means that $B$ does not avoid $C$, and vice versa. (At the last step, there is a faithful arc.) The statements serve to identify situations in which all the other pairs avoid each other, i.e., $A$ avoids $B, C, D ; B$ avoids $A, D ; C$ avoids $A, D$; and $D$ avoids $A, B, C$. In the first lemma, we start with congruent walks $A, B, C, D$, while in the second lemma only $A, B$ are congruent, and $C, D$ are congruent; but on the other hand, all four walks are constricted and have the same height, so we can replace them by their common pre-images by Lemma 2.1.

Lemma 6.1 Let $A, B, C, D$ be four congruent walks in $H$, from $p, q, r, s$ to $a, b, b, d$ respectively, such that $A$ avoids $B$ and $C$ avoids $D$.

Suppose in $\mathrm{H}^{+}$

- $(p, q) \nLeftarrow(a, d)$ and $(p, q) \nLeftarrow(d, b)$,
- $(r, s) \nsim(a, d)$ and $(r, s) \nLeftarrow(b, a)$

Then all pairs from $A, B, C, D$ avoid each other, except the pair $B, C$.

Proof: Let $A$ be the walk $p=a_{1}, a_{2}, \ldots, a_{n}=a, B$ the walk $q=b_{1}, b_{2}, \ldots, b_{n}=b, C$ the walk $r=c_{1}, c_{2}, \ldots, c_{n}=b$, and $D$ the walk $s=d_{1}, d_{2}, \ldots, d_{n}=d$.

Let $S_{i}$ denote the statement that all pairs from $A\left[a_{i+1}, a\right], B\left[b_{i+1}, b\right], C\left[c_{i+1}, b\right], D\left[d_{i+1}, d\right]$ avoid each other, except possibly $B\left[b_{i+1}, b\right], C\left[c_{i+1}, b\right]$. The Lemma claims that $S_{0}$ holds, while $S_{n-1}$ holds vacuously. Therefore, let $i, 0 \leq i \leq n-1$ be the first index such that $S_{i}$ holds.

Note that $a_{i} d_{i+1}$ is not a faithful arc. Otherwise, $(p, q) \rightsquigarrow(d, b)$ in $H^{+}$by combining two walks in $H$, namely the walk $A\left[p, a_{i}\right]$ concatenated with $D\left[d_{i+1}, d\right]$, and the walk $B$. This implies that $b_{i} d_{i+1}$ is also not a faithful arc, since otherwise $(p, q) \rightsquigarrow(a, d)$ by combining the walks $A$ and $B\left[q, b_{i}\right]$ concatenated with $D\left[d_{i+1}, d\right]$. (This uses the fact that $a_{i} d_{i+1}$ is not a faithful arc.)

By a similar line of reasoning, we conclude that

- $c_{i} a_{i+1}$ is not a faithful arc (as otherwise $(r, s) \rightsquigarrow(a, d)$ ), and then $d_{i} a_{i+1}$ is not a faithful arc (otherwise $(r, s) \rightsquigarrow(b, a)$ ).
- $d_{i} b_{i+1}$ is not a faithful arc (as otherwise $(p, q) \rightsquigarrow(a, c)$ ), and $b_{i} a_{i+1}$ is not a faithful arc (as otherwise $(r, s) \rightsquigarrow(a, d)$ ).

Together with the fact that $a_{i} b_{i+1}$ and $c_{i} d_{i+1}$ are not faithful $\operatorname{arcs}$ (corresponding to the assumption that $A$ avoids $B$ and $C$ avoids $D$ ), we obtain a contradiction with the minimality of $i$; therefore $i=0$, and the lemma is proved.

A similar result applies to walks that are not all congruent, as long as they are constricted and have the same net length. (Of course the pairs of walks that one avoids another one must be congruent by definition.)

Corollary 6.2 Let $A, B, C, D$ be four constricted walks of the same net length, from $p, q, r, s$ to $a, b, b, d$ respectively, such that $A, B$ are congruent and $A$ avoids $B$, and $C, D$ are congruent and $C$ avoids $D$.

Suppose in $H^{+}$

- $(p, q) \nLeftarrow \rightarrow(a, d)$ and $(p, q) \nLeftarrow(d, b)$,
- $(r, s) \nsim(a, d)$ and $(r, s) \nrightarrow(b, a)$

Then there exist congruent walks $A^{\prime}, B^{\prime}, C^{\prime}, D^{\prime}$ that are pre-images of $A, B, C, D$ respectively, such that all pairs from $A^{\prime}, B^{\prime}, C^{\prime}, D^{\prime}$ avoid each other, except the pair $B^{\prime}, C^{\prime}$ and hence $A, B$ avoid each other and $C, D$ avoid each other.

Proof: Let $A$ be the walk $p=a_{1}, a_{2}, \ldots, a_{n}=a, B$ the walk $q=b_{1}, b_{2}, \ldots, b_{n}=b, C$ the walk $r=c_{1}, c_{2}, \ldots, c_{m}=b$, and $D$ the walk $s=d_{1}, d_{2}, \ldots, d_{m}=d$. We prove the Corollary by induction on the sum of the lengths $m+n$. If $m+n=0$, i.e., $m=n=0$, this holds trivially.

Suppose first that $A, B, C, D$ are strongly constricted from below. This means that the first two arcs in each walk are forward arcs, and the walks $A-p, B-q, C-r, D-s$ are also constricted walks of the same net length, with the first two congruent and the last two congruent. Moreover, in $H^{+}$, neither $(a, d)$ nor $(d, b)$ is reachable from $\left(a_{2}, b_{2}\right)$, otherwise they would also be reachable from $(p, q)$ because $A$ is assumed to avoid $B$. Similarly, neither $(a, d)$ nor $(d, b)$ is reachable from $\left(c_{2}, d_{2}\right)$. By the induction hypothesis, $A-p, B-q, C-r, D-s$ have congruent pre-images $A^{\prime \prime}, B^{\prime \prime}, C^{\prime \prime}, D^{\prime \prime}$ in which all pairs except $B^{\prime \prime}, C^{\prime \prime}$ avoid each other. Noting that $A^{\prime \prime}$ starts in $a_{2}$, we let $A^{\prime}$ consist of $p$ concatenated with $A^{\prime \prime}$ (i.e., $A^{\prime}=a_{1} a_{2}+A^{\prime \prime}$ ), let $B^{\prime}$ be $q$ concatenated with $B^{\prime \prime}$, and similarly for $C^{\prime}$ and $D^{\prime}$. Since $A^{\prime}, B^{\prime}, C^{\prime}, D^{\prime}$ are all congruent, we can apply Lemma 6.1 , and conclude that all pairs avoid each other, except the pair $B^{\prime}, C^{\prime}$.

In the rest of the proof we will show that $B$ also avoids $A$, and that $D$ also avoids $C$; in other words, that $A, B$ avoid each other and that $C, D$ avoid each other. By repeated application of Lemma 2.1 we conclude that there exist congruent walks $A^{\prime}, B^{\prime}, C^{\prime}, D^{\prime}$ from $p, q, r, s$ to $a, b, b, d$ that are pre-images of $A, B, C, D$ respectively. Since $A$ and $B$ are congruent and avoid each other, the walks $A^{\prime}, B^{\prime}$ follow the same sequence of back and forth steps inside $A, B$, and also avoid each other. (Note that if $A^{\prime}, B^{\prime}$ take a backward step along $A, B$ we can only conclude $A^{\prime}$ avoids $B^{\prime}$ if we know that also $B$ avoids $A$.) Similarly, $C^{\prime}, D^{\prime}$ also avoid each other. Therefore we can now apply Lemma 6.1 to $A^{\prime}, B^{\prime}, C^{\prime}, D^{\prime}$ and conclude that all pairs from $A^{\prime}, B^{\prime}, C^{\prime}, D^{\prime}$ avoid each other, except the pair $B^{\prime}, C^{\prime}$.

Since we have already considered the case when all four walks $A, B, C, D$ are strongly constricted from below, we may assume, up to symmetry, that $A, B$ are not strongly constricted from below, i.e., that there exists a subscript $j>1$ such that $A\left[p, a_{j}\right]$ and $B\left[q, b_{j}\right]$ have net length zero. We take the subscript $j$ is as large as possible, therefore $A\left[a_{j}, a\right], B\left[b_{j}, b\right]$ are strongly constricted from below and have the same net length as $C, D$. We now apply the induction hypothesis to $A\left[a_{j}, a\right], B\left[b_{j}, b\right], C, D$ and conclude that $A\left[a_{j}, a\right], B\left[b_{j}, b\right], C, D$ have congruent pre-images that pairwise avoid each other (except for the pre-images of $B\left[b_{j}, b\right], C$ ). This implies that $A\left[a_{j}, a\right], B\left[b_{j}, b\right]$ also avoid each other, and $C, D$ also avoid each other. If $C, D$ were also not strongly constricted from below, we could draw the similar conclusion that $A, B$ avoid each other, as claimed. However, in general $C, D$ may happen to be strongly constricted from below, and we proceed more carefully as follows:
recall that our goal is to prove that $A$ and $B$ avoid each other. Noting that by the definition of $j$ the $\operatorname{arcs} a_{j-1} a_{j}$ and $b_{j-1} b_{j}$ are backward arcs; moreover, the first arcs $c_{1} c_{2}, d_{1} d_{2}$ of $C, D$ are forward arcs. Since $A$ avoids $B$ and $C$ and $D$ avoid each other, we can apply Lemma 6.1 to $A\left[a_{j-1}, a\right], B\left[b_{j-1}, b\right], c_{2} c_{1}+C, d_{2} d_{1}+D$ and conclude that $A\left[a_{j-1}, a\right], B\left[b_{j-1}, b\right]$ have pre-images that avoid each other, and hence that $A\left[a_{j-1}, a\right], B\left[b_{j-1}, b\right]$ also avoid each other. The idea of the proof is to continue this way backwards on $A, B$ until proving that they avoid each other in their entirety.

Thus let $i \leq j$ be the minimum subscript such that $A\left[a_{i}, a\right], B\left[b_{i}, b\right]$ avoid each other and there exists an $\ell$ such that $A\left[a_{i}, a_{j}\right]$ has a common pre-image with a walk $W_{C}$ in $C$ that starts in some vertex $c_{\ell}$ and ends in $c_{1}$; then $A\left[a_{i}, a\right]$ has a common pre-image with $W_{C}+C$. (Of course, this is also a pre-image of $B\left[b_{i}, b\right]$ and $W^{\prime}+D$ where $W_{D}$ is congruent to $W_{C}$ and starts in $d_{\ell}$ and ends in $d_{1}$.) We have just shown that $i \leq j-1$ and for $i=j-1$ we can take $\ell=2$. We claim that $i=1$, which means in particular that $A, B$ avoid each other. We proceed by contradiction.

Let $X, Y, Z, U$ be congruent walks that are pre-images of $A\left[a_{i}, a\right], B\left[b_{i}, b\right], W_{C}+C, W_{D}+D$ respectively, and denote by $x_{t}, y_{t}, z_{t}, u_{t}$ the $t$-th vertices of these walks respectively.

Suppose first that $a_{i-1} a_{i}$ is a forward arc. We would like to show that $A\left[a_{i-1}, a\right], B\left[b_{i-1}, b\right]$ avoid each other and have a common pre-image with $W_{C}^{\prime}+C$ that starts in some $c_{\ell^{\prime}}$. If $c_{\ell+1} c_{\ell}$ is also a forward arc, we can set $\ell^{\prime}=\ell+1$ and argue as above, adding $a_{i-1} a_{i}$ to $X$ and $c_{\ell+1} c_{\ell}$ to $Z$. Otherwise, we claim there is another vertex $\ell^{\prime \prime}$ and another walk $W_{C}^{\prime \prime}$ in $C$ that starts in $c_{\ell^{\prime \prime}}$ and ends in $c_{1}$ and has a common pre-image with $A\left[a_{i}, a_{j}\right]$, and such that $c_{\ell+1} c_{\ell}$ is a forward arc (so we can proceed as above). Note that we have $x_{1}=a_{i}$; we let $t$ be the last subscript such that $X\left[x_{1}, x_{t}\right]$ is constricted from below and has net length zero. Then it is easy to see that $x_{t} x_{t+1}$ is a backward arc. Indeed, the net length of $A\left[a_{i}, a_{j}\right]$ is strictly negative, as $a_{i-1} a_{i}$ is a forward arc, $A$ is constricted, and $A\left[a_{1}, a_{j}\right]$ has net length zero. Since $X$ and $Z$ are congruent, $z_{t} z_{t+1}$ is also a backward arc, and we can set $c_{\ell^{\prime \prime}}=z_{t}$. It remains to construct a walk $W_{C}^{\prime \prime}$ in $C$, from $c_{\ell^{\prime \prime}}$ to $c_{1}$ that has a common pre-image with $A\left[a_{i}, a_{j}\right]$. Note that the walk $X\left[x_{1}, x_{t}\right]$ from $x_{1}=a_{i}$ to $x_{t}$ is congruent with the walk $Z\left[z_{1}, z_{t}\right]$ from $z_{1}=c_{\ell}$ to $z_{t}=c_{\ell^{\prime \prime}}$. Both are constricted from below, and have the same maximum net length of a subwalk; say $X\left[x_{1}, x_{s}\right]$ and $Z\left[z_{1}, z_{s}\right]$ are of maximum net length. Then applying Lemma 2.1 twice (once to $X\left[x_{1}, x_{s}\right], Z^{-1}\left[z_{t}, z_{s}\right]$ and once to $X^{-1}\left[x_{s}, x_{1}\right], Z\left[z_{t}, z_{s}\right]$, we obtain congruent walks $X^{*}, Z^{*}$ from $a_{i}$ to $a_{i}$ and from $c_{\ell^{\prime \prime}}$ to $c_{\ell}$ respectively. Then the concatenation $W_{C}^{\prime \prime}=Z^{*}+Z$ is a walk in $C$ from $c_{\ell^{\prime \prime}}$ to $c_{1}$ that has a common pre-image $X^{*}+X$ with $A\left[a_{i}, a_{j}\right]$, as required.

When $a_{i-1} a_{i}$ is a backward arc, the proof is similar. If $c_{\ell+1} c_{\ell}$ is also a backward arc, we proceed as usual. Otherwise, we let $t$ be the last subscript such that $X\left[x_{1}, x_{t}\right]$ is constricted from above and of net length zero. This again means that $x_{t} x_{t+1}$ and hence also $z_{t} z_{t+1}$ is a forward arc, and we set $c_{\ell^{\prime \prime}}=z_{t}$. Then choosing $x_{s}$ so that the net length of $X\left[x_{1}, x_{s}\right]$ is minimized, and applying Lemma 2.1 twice - to $X\left[x_{1}, x_{s}\right], Z^{-1}\left[z_{t}, z_{s}\right]$ and to $X^{-1}\left[x_{s}, x_{1}\right], Z\left[z_{t}, z_{s}\right]$, we obtain congruent walks $X^{*}, Z^{*}$ from $a_{i}$ to $a_{i}$ and from $c_{\ell^{\prime \prime}}$ to $c_{\ell}$ respectively, which yield the walk $Z^{*}+Z$ in $C$ from $c_{\ell^{\prime \prime}}$ to $c_{1}$ that has a common pre-image $X^{*}+X$ with $A\left[a_{i}, a_{j}\right]$, as required.

Here is another useful version of the lemma (cf. Figure ??).
Lemma 6.3 Suppose that $n \geq t>1, \ell>0$ are integers, $\left(a_{0}, a_{1}\right),\left(a_{1}, a_{2}\right), \ldots,\left(a_{n}, a_{0}\right)$ is a circuit in $H^{+}$, and, for each $i=0,1, \ldots, t, p_{i}, q_{i}, g_{i}, h_{i}$ are vertices of $H$, and $A_{i}, A_{i}^{\prime}, B_{i}, B_{i}^{\prime}$ are walks in $H$, such that the following statements hold (subscript addition modulo $n+1$ ):


Figure 2: Top Figure : The notation for the walks $A_{i}, B_{i}, A_{i}^{\prime}, B_{i}^{\prime}$ and bottom Figure is an example for such walks

1. $A_{i}$ is a constricted walk from $p_{i}$ to $h_{i}$ of net length $\ell$
2. $B_{i}$ is a constricted walk from $q_{i}$ to $g_{i}$ of net length $\ell$
3. $A_{i}^{\prime}$ is a constricted walk from $h_{i}$ to $a_{i}$ of net length $-\ell$
4. $B_{i}^{\prime}$ is a constricted walk from $g_{i}$ to $a_{i}$ of net length $-\ell$
5. $A_{i}+A_{i}^{\prime}$ is congruent to and avoids $B_{i+1}+B_{i+1}^{\prime}$
6. if $k \neq j+1$, then $\left(p_{i}, q_{i+1}\right) \nLeftarrow \rightarrow\left(a_{j}, a_{k}\right)$

Let $C$ be any one of the walks $A_{i}^{\prime}$ or $B_{i}^{\prime}$ or $A_{i}^{-1}$ or of $B_{i}^{-1}$, and let $D$ be any one of the walks $A_{j}^{\prime}$ or $B_{j}^{\prime}$ or $A_{j}^{-1}$ or of $B_{j}^{-1}$, with $i \neq j$.

Then $C, D$ have common pre-images that avoid each other.
Note that condition 5 implies that $\left(p_{i}, q_{i+1}\right) \rightsquigarrow\left(a_{i}, a_{i+1}\right)$ in $H^{+}$, and therefore condition 6 implies that

7 if $\left(a_{i}, a_{i+1}\right) \rightsquigarrow\left(a_{j}, a_{k}\right)$ in $H^{+}$, then $k=j+1$
Another way to state the conclusion of the lemma is the following.
There are common pre-images of all $A_{i}^{\prime}, B_{i}^{\prime}, A_{i}^{-1}, B_{i}^{-1}, i=0,1, \ldots, t$, such that any two pre-images of walks with different subscripts avoid each other.

The lemma will often be used for walks where $A_{i}^{\prime}=A_{i}^{-1}$ and/or $B_{i}^{\prime}=B_{i}^{-1}$ (or even $A_{i}^{\prime}=A_{i}^{-1}=$ $B_{i}^{\prime}=B_{i}^{-1}$ ).

Proof: We prove the lemma with $t=n$, and it is easy to check that the proof allows any smaller $t, t \geq 2$.

We first prove that any $B_{i}^{\prime}, B_{j}^{\prime}$ with $i \neq j$ have common pre-images that avoid each other. We proceed by induction on $|j-i|$. If $|j-i|=1$, say $j=i+1$, we may apply Corollary 6.2 to the walks $A_{i-1}^{\prime}, B_{i}^{\prime}, A_{i}^{\prime}, B_{i+1}^{\prime}$. Indeed, $A_{i-1}^{\prime}$ avoids $B_{i}^{\prime}$ and $A_{i}^{\prime}$ avoids $B_{i+1}^{\prime}$ by 5 , and since the same condition also implies that $\left(p_{i}, q_{i+1}\right) \rightsquigarrow\left(h_{i}, g_{i+1}\right)$ in $H^{+}$, condition 6 implies that $\left(h_{i}, g_{i+1}\right) \nprec \sim\left(a_{i-1}, a_{i+1}\right)$ and $\left(h_{i}, g_{i+1}\right) \not \nsim\left(a_{i}, a_{i-1}\right)$, and similarly for $\left(h_{i-1}, g_{i}\right)$. (Here we used the fact that $n>1$.)

For the induction step, we again assume that $i<j$ and consider $B_{i}^{\prime}, B_{j}^{\prime}, B_{j+1}^{\prime}$. By the induction hypothesis, there are common pre-images of $B_{i}^{\prime}, B_{j}^{\prime}$ that avoid each other and also common preimages of $B_{j}^{\prime}, B_{j+1}^{\prime}$ that avoid each other. As noted earlier, we may assume that all these pre-images are congruent to each other. Now assume there is a faithful arc from $B_{i}$ to $B_{j+1}$, or from $B_{j+1}$ to $B_{i}$. It is easy to trace walks from $\left(a_{i}, a_{j}\right)$ on reverses of $B_{i}^{\prime}, B_{j}^{\prime}$ (that are known to avoid each other) up to the faithful arc, use the faithful arc, and then follow the walks $B_{j+1}^{\prime}, B_{j}^{\prime}$, also known to avoid each other to $\left(a_{j+1}, a_{j}\right)$. This would imply that $\left(a_{j}, a_{j+1}\right) \rightsquigarrow\left(a_{j}, a_{i}\right)$ which contradicts condition 7 , and completes the induction proof.

A symmetric argument yields that any $A_{i}^{\prime}, A_{j}^{\prime}$ with $i \neq j$ have common pre-images that avoid each other.

It now follows that any $A_{i}^{\prime}, B_{j}^{\prime}, i \neq j$, have common pre-images that avoid each other. Indeed, $A_{i}^{\prime}, B_{i+1}^{\prime}$ are congruent by assumption, so it suffices to take the common pre-images of $B_{i+1}^{\prime}, B_{j}^{\prime}$ that avoid each other, we have just constructed, and also use the same pre-image for $A_{i}^{\prime}$. Then if there was a faithful arc between $A_{i}^{\prime}$ and $B_{j}^{\prime}$ (in either direction), we could use it to reach $\left(a_{j}, a_{i}\right)$ from $\left(p_{i}, q_{i+1}\right)$, using the walks $A_{i}, B_{i+1}, A_{i}^{\prime}$, a portion of $B_{i+1}^{\prime}$, the faithful arc, and a portion of $B_{j}^{\prime}$. This contradicts condition 6 of the lemma. (We note that since $n>2$, we can always choose $i, j$ so that $j+1 \neq i$.)

Next we argue that for each $j \neq i$ there are common pre-images to $A_{i}, B_{i+1}, A_{j}, B_{j+1}$, such that each pair except for the pre-images of $B_{i+1}$ and $A_{j}$ avoid each other. This will in particular imply that the pre-images of $A_{i}$ and $A_{j}$ avoid each other, and the pre-images of $A_{i}$ and $B_{j+1}$ avoid each other. It will also imply that $A_{i}, B_{i+1}$ avoid each other, and thus $A_{i}, B_{j}$ avoid each other for all $j \neq i$. This will imply the corresponding statements also about their reverses. For any $i \neq j$, consider a new digraph $H^{o}$ obtained from $H$ by the addition of three new vertices $u, v, w$ and four new arcs $h_{i} u, g_{i+1} v, h_{j} v, g_{j+1} w$ (see Figure 3). Then in $H^{o}$ we will apply Corollary 6.2 to the walks $A_{i}+h_{i} u, B_{i+1}+g_{i+1} v, A_{j}+h_{j} v, B_{j+1}+g_{j+1} w$, to conclude that $A_{i}, A_{j}$ as well as $A_{i}, B_{j+1}$ have common pre-images that avoid each other. The assumptions of Corollary 6.2 are easy to check using the statements we have already proved. For instance, $\left(p_{i}, q_{i+1}\right) \nLeftarrow(u, w)$ since otherwise we would have $\left(p_{i}, q_{i+1}\right) \rightsquigarrow\left(h_{i}, g_{j+1}\right)$. Since we have already proved that $A_{i}^{\prime}, B_{j+1}^{\prime}$ avoid each other, this implies that $\left(p_{i}, q_{i+1}\right) \rightsquigarrow\left(h_{i}, g_{j+1}\right)$, contradicting assumption 6 of the lemma.

It remains to check the primed walks against the reverses of the unprimed walks. The arguments are symmetric, we focus on finding common pre-images of $A_{i}^{\prime}, B^{\prime} i+1, A_{j}^{-1}, B_{j+1}^{-1}$. We again construct a new digraph $H^{o}$ with added vertices $u, v, w$ and $\operatorname{arcs} a_{i} u, a_{i+1} v, p_{j} v, q_{j+1} w$. It is again easy to check, from the statements already proved, that Corollary 6.2 applies to the walks $A_{i}^{\prime}+a_{i} u, B_{i+1}^{\prime}+a_{i+1} v, A_{j}^{-1}+a_{j} v, B_{j+1}^{-1}+q_{j+1} w$ to imply that $A_{i}^{\prime}$ and $A_{j}^{-1}$ avoid each other, and that $A_{i}^{\prime}$ and $B_{j+1}^{-1}$ as well as $A_{i}^{\prime}$ and $B_{i+1}^{-1}$ also avoid each other.


Figure 3: Induction step in the proof of Lemma 6.3

## 7 Structural properties of a minimal circuit

In this section we analyze a minimal circuit in $H^{+}$under certain conditions and we derive properties of $H^{+}$. We later use them to prove the correctness of the algorithm.

Let $S$ be a set of pairs in $V\left(H^{+}\right)$. Let $\widehat{S}$ denote the set of pairs in $H^{+}$that are reachable from $S$. Note that $\widehat{S}$ includes $S$. We call $\widehat{S}$ the out-section of $S$. Let $\left(a_{0}, a_{1}\right),\left(a_{1}, a_{2}\right), \ldots,\left(a_{n-1}, a_{n}\right),\left(a_{n}, a_{0}\right)$ be a circuit in $\widehat{S}$. We say this circuit is minimal if there is no path from a pair in $S$ that can reach to ( $a_{r}, a_{r+1}$ ) to any of $\left(a_{i}, a_{j}\right), i \neq j-1, r, i, j=0,1,2, \ldots, n$.

Let $S$ be a set of pairs in $H^{+}$. For a pair $(x, y) \in H^{+}$we may or may not have a path from another pair in $S$ to $(x, y)$ with a positive net value. If such a path exists we call $(x, y)$ a $z$-pair with respect to $S$ and associate to each $z$-pair $(x, y)$ a path $Z_{x, y}$ of net value one constricted from below and ending at $(x, y)$.
Lemma 7.1 Let $S$ be a set of pairs in $H^{+}$and suppose $\widehat{S}$ has a minimal circuit $\left(a_{0}, a_{1}\right),\left(a_{1}, a_{2}\right), \ldots$, $\left(a_{n-1}, a_{n}\right),\left(a_{n}, a_{0}\right)(n>1)$ such that each $\left(a_{i}, a_{i+1}\right)$ is a z-pair with respect to $\widehat{S}$. There exists another circuit $\left(a_{0}^{\prime}, a_{1}^{\prime}\right),\left(a_{1}^{\prime}, a_{2}^{\prime}\right), \ldots,\left(a_{n}^{\prime}, a_{0}^{\prime}\right)$ of the pairs in $\widehat{S}$, and walks $P_{i}, Q_{i}, i=0, \ldots, n$, in $H$, such that $P_{i}, Q_{i}$ are walks of net length one, constricted from below, $P_{i}$ from $a_{i}^{\prime}$ to $a_{i}, Q_{i}$ from $a_{i+1}^{\prime}$ to $a_{i+1}$, and such that $P_{i}$ and $Q_{i}$ are congruent and avoid each other.

Proof: Let $Z_{i}=Z_{a_{i}, a_{i+1}}, 1 \leq i \leq n$, be a path from $\left(p_{i}^{\prime}, q_{i+1}^{\prime}\right) \in \widehat{S}$ to $\left(a_{i}, a_{i+1}\right)$. We will find $n$ vertices $a_{i}^{\prime}$ from amongst the $2 n$ vertices $p_{i}^{\prime}, q_{i}^{\prime}$ which satisfy the conclusion. As an intermediate step, we will find $n$ vertices $a_{i}^{*}$ of the $2 n$ vertices $p_{i}, q_{i}$ which also yield a circuit in $\widehat{S}$. For any $i$, if $L_{i-1}<L_{i}$ we let $a_{i}^{*}=q_{i}$ and $a_{i}^{\prime}=q_{i}^{\prime}$ and if $L_{i-1} \geq L_{i}$ we let $a_{i}^{*}=p_{i}$ and $a_{i}^{\prime}=p_{i}^{\prime}$. We first prove that each pair $\left(a_{i}^{*}, a_{i+1}^{*}\right)$ in the circuit $\left(a_{0}^{*}, a_{1}^{*}\right),\left(a_{1}^{*}, a_{2}^{*}\right), \ldots,\left(a_{n}^{*}, a_{0}^{*}\right)$ can be reached from the corresponding pair ( $a_{i}, a_{i+1}$ ).

First consider the case that $L_{i-1}<L_{i}<L_{i+1}$, in which we have $\left(a_{i}^{*}, a_{i+1}^{*}\right)=\left(q_{i}, q_{i+1}\right)$. We refer to Figure 4 to summarize the steps of the proof. First of all, we find corresponding vertices $r, s$ so that the green walks $C$ from $r$ to $a_{i}$ and $D$ from $s$ to $a_{i+1}$ have net length $L_{i}$.

Then Corollary 6.2 is applied to the four green walks $A, B, C, D$, to conclude that $A, B$ avoid each other and $C, D$ also avoid each other. Moreover $A, C$ have a common pre-image that avoid


Figure 4: walks $A, B$ avoid each other and walks $C, D$ avoid each other
each other and $B, D$ have common pre-image that avoid each other. Now we can use Lemma 6.3 on the suggested blue walks. Specifically, the blue walk from $p_{i}$ to $u$, then taking $A$ to $a_{i}$ and then use $A^{-1}+A$ again; the blue walk from $q_{i+1}$ to $v$ followed by $B+C^{-1}+C$; and the blue walk $C^{-1}+C+C^{-1}+C$, and the blue walk $D^{-1}+D+D^{-1}+D$ (see Figure 4). We conclude that the walk $A^{-1}+A$ and the walk $B^{-1}$ concatenated with the walk from $v$ to $q_{i+1}$ have common preimages that avoid each other. Thus $\left(a_{i}, a_{i+1}\right) \rightsquigarrow\left(a_{i}, q_{i+1}\right)$ and $\left(a_{i+1}, a_{i+2}\right) \rightsquigarrow\left(q_{i+1}, a_{i+2}\right)$. Then using similar arguments to the red walks of height $L_{i-1}$ we conclude that $\left(a_{i}, q_{i+1}\right) \rightsquigarrow\left(q_{i}, q_{i+1}\right)$ and from $\left(a_{i-1}, a_{i}\right) \rightsquigarrow\left(a_{i-1}, q_{i}\right)$. This allows us to replace $a_{i}$ by $q_{i}$ and $a_{i+1}$ by $q_{i+1}$ in the circuit $\left(a_{0}, a_{1}\right), \ldots,\left(a_{n}, a_{0}\right)$. (The proof in case $L_{i-1}<L_{i}<L_{i+1}$ is symmetric.)

If $L_{i} \geq L_{i-1}$ and $L_{i} \geq L_{i+1}$, then $\left(a_{i}^{*}, a_{i+1}^{*}\right)=\left(q_{i}, p_{i+1}\right)$ and $\left(a_{i}^{\prime}, a_{i+1}^{\prime}\right)=\left(q_{i}^{\prime}, p_{i+1}^{\prime}\right)$. Lemma 6.3 can be similarly used to conclude that $\left(a_{i}, a_{i+1}\right) \rightsquigarrow\left(q_{i}, p_{i+1}\right)$. In fact, by an argument identical to the one in the proof of Lemma 7.3, as illustrated in the Figure 5, one can show, that $\left(a_{i}, a_{i+1}\right),\left(p_{i}, q_{i+1},\left(p_{i}^{\prime}, q_{i+1}^{\prime}\right)\right.$ are all reachable from each other, in fact that the paths depicted in Figure 5 (of net length zero, constricted from below) avoid each other.

In case $L_{i} \leq L_{i-1}$ and $L_{i} \leq L_{i+1}$, we have $\left(a_{i}^{*}, a_{i+1}^{*}\right)=\left(p_{i}, q_{i+1}\right)$, and by using Lemma 6.3 in a fashion similar to the above proofs, we conclude easily that there are the walks from $a_{i}$ to $a_{i}^{*}$ and from $a_{i+1}$ to $a_{i+1}^{*}$ ) that avoid each other and are of net length zero, constricted from below.

We now show that $a_{i}^{\prime} a_{i+1}^{*}$ and $a_{i+1}^{\prime} a_{i}^{*}$ are not arcs of $H$, completing the proof of the Lemma. In fact, this has already been observed (by appealing to the proof of Lemma 7.3) for the $i$ which have $L_{i} \geq L_{i-1}$ and $L_{i} \geq L_{i+1}$.

If $L_{i-1}<L_{i}<L_{i+1}$. In fact, we assume $L_{i-1}<L_{i}<L_{i+1}<\ldots L_{i+j} \geq L_{i+j+1}$ for some $j \geq 1$. Note that this means that $a_{i}^{*}=q_{i}, a_{i+1}^{*}=q_{i+1}, \ldots, a_{i+j-1}^{*}=q_{i+j-1}, a_{i+j}^{*}=q_{i+j}, a_{i+j+1}^{*}=p_{i+j+1}$. We first note that since $L_{i+j} \geq L_{i+j-1}$ and $L_{i+j} \geq L_{i+j+1}$ we already know that $p_{i+j+1}^{\prime} q_{i+j}$ is not an arc of $H$. By symmetry, $q_{i+j}^{\prime} p_{i+j+1}$ is also not an arc of $H$. Next we argue that $p_{i}^{\prime} q_{i+2}$ is not an arc of $H$ as otherwise $\left(p_{i}^{\prime}, q_{i+1}^{\prime}\right) \rightsquigarrow\left(a_{i+2}^{*}, a_{i+1}^{*}\right)$ contradicting the minimality of $n$. This implies that $q_{i+1}^{\prime} q_{i+2}$ is not an arc of $H$, as otherwise $\left(p_{i}^{\prime}, q_{i+1}^{\prime}\right) \rightsquigarrow\left(p_{i}, q_{i+2}\right)$ and eventually $\left(q_{i}, q_{i+2}\right)=$ $\left(a_{i}^{*}, a_{i+2}^{*}\right)$ (using Lemma 6.3 on suitable portions of the walks). The same arguments imply that $p_{i+1}^{\prime} q_{i+3} \notin A(H)$ and $q_{i+2}^{\prime} q_{i+3} \notin A(H)$, and so on until $p_{i+j-2}^{\prime} q_{i+j} \notin A(H)$ and $q_{i+j-1}^{\prime} q_{i+j} \notin A(H)$. (These will all be used later.) Now we proceed to show that $q_{i+j}^{\prime} q_{i+j-1}$ is not arc of $H$, otherwise


Figure 5: Illustration of the proof of Lemma 7.1
$\left(q_{i+j}^{\prime}, p_{i+j+1}^{\prime}\right)$ (which we have shown to be reachable from $\left.\left(a_{i+j}^{*}, a_{i+j+1}^{*}\right)\right)$ can reach $\left(a_{i+j-1}^{*}, a_{i+j+1}^{*}\right)$ because $q_{i+j}^{\prime} p_{i+j+1}=q_{i+j}^{\prime} a_{i+j+1}^{*}$ is not an arc of $H$. This contradicts the minimality of $n$. Note that now we have both $q_{i+j}^{\prime} q_{i+j-1} \notin A(H)$ and $q_{i+j-1}^{\prime} q_{i+j} \notin A(H)$. The first fact implies that $\left(a_{i+j-1}^{*}, a_{i+j}^{*}\right) \rightsquigarrow\left(a_{i+j-1}^{\prime}, a_{i+j}^{\prime}\right)$. The second fact implies that we can repeat the argument to conclude that $q_{i+j-1}^{\prime} q_{i+j-2} \notin A(H)$, and continue the argument in this way, eventually showing that $a_{i+1}^{\prime} a_{i}^{*}=q_{i+1}^{\prime} q_{i} \notin A(H)$ and $a_{i}^{\prime} a_{i+1}^{*}=q_{i}^{\prime} q_{i+1} \notin A(H)$.

It remains to consider those $i$ that have $L_{i} \leq L_{i-1}$ and $L_{i} \leq L_{i+1}$. In this case, $a_{i}^{*}=p_{i}$ and $a_{i+1}^{*}=q_{i+1}$. If $L_{i+1}<L_{i+2}$, then $a_{i+2}^{*}=q_{i+2}$, and we can use the previous argument to conclude that $q_{i+1}^{\prime} q_{i+2}$ and $q_{i+2}^{\prime} q_{i+1}$ are not arcs of $H$. This implies that $q_{i+1}^{\prime} p_{i}$ is also not an arc of $H$, otherwise the walk $q_{i+1}, q_{i+1}^{\prime}, p_{i}$ avoids the walk $q_{i+2}, q_{i+2}^{\prime}, q_{i+2}$ and hence the pair ( $p_{i}, q_{i+2}$ ) $=\left(a_{i}^{*}, a_{i+2}^{*}\right) \in \widehat{S}$, contradicting the minimality of $n$.

On the other hand, if $L_{i+1} \geq L_{i+2}$ we have $a_{i+2}^{*}=p_{i+2}$ and we use another previous case to conclude that $q_{i+1}^{\prime} p_{i+1}$ and $p_{i+2}^{\prime} q_{i+1}$ are not arcs of $H$, and hence there is walk from $q_{i+1}$ to $p_{i}$ that avoids a walk from $p_{i+2}$ to itself, thereby the pair $\left(p_{i}, p_{i+2}\right)=\left(a_{i}^{*}, a_{i+2}^{*}\right)$ is also in $\widehat{S}$, again yielding a contradiction.

Each unbalanced strong component $C$ of $H^{+}$contains a directed cycle $D$ of non-zero net value $r$, corresponding to two closed walks $D_{1}, D_{2}$ in $H$, with $D_{1}$ avoiding $D_{2}$, each of net length $r$. Note that $r$ could be positive or negative. As observed earlier, if $r>0$, then any extremal vertex (often called extremal pair) $(x, y)$ of $D$ must initiate an infinite directed walk $W=w_{1}, w_{2}, \ldots$ with $w_{1}=(x, y)$, continuously winding around $D$ in the positive direction, such that the net values of any portion $W\left[w_{1}, w_{t}\right]$ are non-negative (the net values are constricted from below) and unbounded (for any $k>0$ some $W\left[w_{1}, w_{t}\right]$ has net value $k$ ).

Theorem 7.2 Let $T$ be a subset of unbalanced pairs such that if $(a, b) \in T$ then $(b, a) \notin \widehat{T}$.
Suppose $\widehat{T}$ contains a circuit and let $\left(a_{0}, a_{1}\right),\left(a_{1}, a_{2}\right), \ldots,\left(a_{n}, a_{0}\right)$ be a minimal circuit in $\widehat{T}$. Then the following statements hold.

1. $n>1$ and there exists a circuit $\left(b_{0}, b_{1}\right),\left(b_{1}, b_{2}\right), \ldots,\left(b_{n-1}, b_{n}\right),\left(b_{n}, b_{0}\right)$ consisting of extremal pairs.
2. Each $\left(a_{i}, a_{i+1}\right)$ can be reached from the corresponding $\left(b_{i}, b_{i+1}\right)$ by a symmetric directed walk in $H^{+}$with positive net value.
3. There are infinite walks $P_{i}, 0 \leq i \leq n$, starting at $b_{i}$ such that $P_{i}, P_{j}, 0 \leq i<j \leq n$, avoid each other.

Proof: For contradiction suppose $n=1$. This means $(a, b) \rightsquigarrow\left(a_{0}, a_{1}\right)$ for some $(a, b) \in T$ and $\left(a^{\prime}, b^{\prime}\right) \rightsquigarrow\left(a_{1}, a_{0}\right)$ for some $\left(a^{\prime}, b^{\prime}\right) \in T$ Now by skew symmetry we have $\left(a^{\prime}, b^{\prime}\right) \rightsquigarrow(b, a)$, a contradiction unless $\left(a_{0}, a_{1}\right)=(a, b)$ and $\left(a_{1}, a_{0}\right)=\left(a^{\prime}, b^{\prime}\right)$ and hence $(a, b),(b, a) \in T$, again a contradiction. Therefore $n>1$.

For each $i$, let $C_{i}$ be the strong component of $H^{+}$containing an extremal pair in $S$ where ( $a_{i}, a_{i+1}$ ) is reached from $C_{i}$, and let $D_{i}$ be a directed cycle in $C_{i}$.

Claim 7.3 We may assume each $\left(a_{i}, a_{i+1}\right)$ is a $z$-pair with respect to $\widehat{T}$.
Proof: Let $Z_{i}=Z_{a_{i}, a_{i+1}}$ be a path from $\left(p_{i}^{\prime}, q_{i+1}^{\prime}\right) \in \widehat{T}$ to $\left(a_{i}, a_{i+1}\right)$ which is constricted from below and has net value one. Consider first a subscript $i$ such that $D_{i}$ has positive net value. Then, as observed above, there is an infinite directed walk $W$ continuously winding around $D_{i}$ in the positive direction, whose net values are constricted from below and unbounded. By following $W$ as far as necessary and then following a path that leads from $C_{i}$ to $\left(a_{i}, a_{i+1}\right)$ (such a path exists both when $\left(a_{i}, a_{i+1}\right)$ is in $C_{i}$ or reachable from $\left.C_{i}\right)$, we obtain a directed walk $W_{i}$ in $H^{+}$that has values constricted from below. We let $\left(p_{i}^{\prime}, q_{i+1}^{\prime}\right)$ be the last vertex on $W_{i}$ such that the net value of $W_{i}\left[\left(p_{i}^{\prime}, q_{i+1}^{\prime}\right),\left(a_{i}, a_{i+1}\right)\right]$ is one, and we set $Z_{i}=W_{i}\left[\left(p_{i}^{\prime}, q_{i+1}^{\prime}\right),\left(a_{i}, a_{i+1}\right)\right]$. Let $L_{i}$ be the maximum net value of a prefix of $Z_{i}$, i.e., of a directed walk $Z_{i}\left[\left(p_{i}^{\prime}, q_{i+1}^{\prime}\right),(x, y)\right]$ for any $(x, y)$. Note that $L_{i}$ could be one, in case the values of $Z_{i}$ are also constricted from above, i.e., $Z_{i}$ is just one arc in $H^{+}$.

We emphasize for future reference that in this case the directed walk $Z_{i}$ arises from $W_{i}$ that started on the cycle $D_{i}$.

A similar argument applies to a subscript $i$ such that $D_{i}$ has negative net value, but following the directed walk $W$ discussed above (unbounded and non-positive) and then a path from $D_{i}$ to $\left(a_{i}, a_{i+1}\right)$, we obtain a directed walk $W_{i}$ in $H^{+}$that has values constricted from above but not constricted from below. Indeed, in such a case we can again let $\left(p_{i}^{\prime}, q_{i+1}^{\prime}\right)$ be the last vertex on $W_{i}$ such that the net value of $W_{i}\left[\left(p_{i}^{\prime}, q_{i+1}^{\prime}\right),\left(a_{i}, a_{i+1}\right)\right]$ is one and set $Z_{i}=W_{i}\left[\left(p_{i}^{\prime}, q_{i+1}^{\prime}\right),\left(a_{i}, a_{i+1}\right)\right]$.

Suppose next that there are two subscripts $i, i+1$ (addition modulo $n$ ) such that both $D_{i}$ and $D_{i+1}$ have negative net values, and both $W_{i}$ and $W_{i+1}$ have constricted values, from some pairs $(u, v),(w, x) \in \widehat{S}$ to $\left(a_{i}, a_{i+1}\right),\left(a_{i+1}, a_{i+2}\right)$ respectively. We may assume that the pairs $(u, v),(w, x)$ are on the cycles $D_{i}, D_{i+1}$, and that the net values of $W_{i}, W_{i+1}$ are the same (by choosing for their starting vertices a suitable extremal vertex on each $D_{i}$ ). In this context, Corollary 6.2 applies to the four walks $A, B, C, D$ in $H$ corresponding to $W_{i}, W_{i+1}$, and we conclude that, in particular, $A, B$ avoid each other and $C, D$ avoid each other. This implies that the reverse traversal of the cycles $D_{i}, D_{i+1}$ is also a cycle in $H^{+}$, of positive net value, and we can proceed as in the case when $D_{i}, D_{i+1}$ had positive net value (see Figure 6 ).


Figure 6: Claim 7.3 when $D_{i}, D_{i+1}$ have positive net value


Figure 7: Claim 7.3 when $D_{i-1}, D_{i+1}$ have positive net value and $D_{i}$ has negative net value

Thus it remains to consider the case when $D_{i}$ has negative net value, and the values of $W_{i}$ are constricted (both from below and from above), and $D_{i-1}, D_{i+1}$ have positive net values. We illustrate this part of the proof in Figure 7. Each directed walk in $H^{+}$is depicted as two walks in $H$ where the first avoids the second. (The avoided arcs are depicted as dotted lines.) We show in the figure 7 the starting vertex $\left(p_{i-1}^{\prime}, q_{i}^{\prime}\right)$ of $Z_{i-1}$, its ending vertex $\left(a_{i-1}, a_{i}\right)$, as well as the starting vertex $\left(p_{i+1}^{\prime}, q_{i+2}^{\prime}\right)$ of $Z_{i+1}$, and its ending vertex $\left.\left(a_{i+1}, a_{i+2}\right)\right)$. In the illustration we assume neither $Z_{i-1}$ nor $Z_{i+1}$ is constricted from above. (The proof in the cases where one or both are constricted is similar and easier.) Without loss of generality, we assume that $L_{i-1} \geq L_{i+1}$ as depicted. As shown, we let $\left(p_{i-1}, q_{i}\right)$ be the second vertex of $Z_{i-1}$ and $\left(p_{i+1}, q_{i+2}\right)$ the second vertex of $Z_{i+1}$. We also show a constricted directed walk $W_{i}$ ending in $\left(a_{i}, a_{i+1}\right)$. We assume $(g, h)$ is the last vertex on $Z_{i-1}$ that maximizes the net value of the prefix $Z_{i-1}\left[\left(p_{i-1}^{\prime}, q_{i-1}^{\prime}\right),(g, h)\right]$, and $\left(g^{\prime}, h^{\prime}\right)$ the last vertex of $W_{i}$ so that $Z_{i-1}\left[(g, h),\left(a_{i-1}, a_{i}\right)\right]$ and $W_{i}\left[\left(g^{\prime}, h^{\prime}\right),\left(a_{i}, a_{i+1}\right)\right]$ have the same net values. (Note that both these directed walks have constricted values.) The directed walk $Z_{i-1}$ in $H^{+}$corresponds to two walks $p_{i-1}^{\prime}+A+A^{\prime}, q_{i}^{\prime}+B+B^{\prime}$ in $H$, as depicted, where the first avoids the second, in particular $A$ avoids $B$ and $A^{\prime}$ avoids $B^{\prime}$. ( $A$ is the portion from $p_{i-1}$ to $g$ and $A^{\prime}$ the portion from $g$ to $a_{i-1}$, and similarly for $B$ ). Now we claim the following facts.

1 The walks $A^{\prime}, B^{\prime}, C^{\prime}, D^{\prime}$ have common pre-images that avoid each other, except for the pre-
images of $B^{\prime}$ and $C^{\prime}$
2 The walks $A^{\prime-1}+A^{\prime}$ and $B^{\prime-1}+B^{-1}$ have common pre-images that avoid each other
3 The pairs $\left(a_{i-1}, a_{i}\right)$ and $\left(a_{i-1}, q_{i}\right)$ are reachable from each other
4 The walks $B^{\prime-1}+B^{-1}$ and $D+D^{\prime}$ have common pre-images that avoid each other
5 The pairs $\left(a_{i}, a_{i+1}\right)$ and $\left(q_{i}, a_{i+1}\right)$ are reachable from each other
Item 1 follows directly by Lemma 6.1 because the minimality of the circuit $\left(a_{0}, a_{1}\right)$, $\left(a_{1}, a_{2}\right), \ldots,\left(a_{n}, a_{0}\right)$ implies that $\left(a_{i-1}, a_{i+1}\right)$ or $\left(a_{i+1}, a_{i}\right)$ or $\left(a_{i}, a_{i-1}\right)$ can not be in $\widehat{T}$.

Therefore Lemma 6.3 applies to the walks $A+A^{\prime}+A^{\prime-1}+A^{\prime}, B+B^{\prime}+C+C^{\prime}, D+D^{\prime}+D+D^{\prime}$, using the same minimality arguments, verifying items 2 and 4 . Items 3 and 5 follow from 2 and 4.

Items 3 and 5 imply that we can replace $a_{i}$ by $a_{i}^{\prime}=q_{i}$ and obtain another circuit of pairs

$$
\left(a_{0}, a_{1}\right),\left(a_{1}, a_{2}\right), \ldots,\left(a_{i-1}, a_{i}^{\prime}\right),\left(a_{i}^{\prime}, a_{i+1}\right), \ldots,\left(a_{n}, a_{0}\right)
$$

in $\widehat{T}$. A similar argument shows that we may replace $a_{i+1}$ by $a_{i+1}^{\prime}=p_{i+1}$. In the rest of the proof we assume that we have made the replacement, i.e., that $a_{i}=q_{i}, a_{i+1}=p_{i+1}$.

We now show that, in the new circuit, the path $Z_{i}$ actually exists, namely, that the (single-arc) walks $q_{i}^{\prime} q_{i}$ and $p_{i+1}^{\prime} p_{i+1}$ avoid each other, and hence the new $\left(a_{i}, a_{i+1}\right)$ and $\left(q_{i}^{\prime}, p_{i+1}^{\prime}\right)$ are reachable from each other. First, we observe that $p_{i-1}^{\prime} p_{i+1}$ is not an arc, otherwise ( $\left.p_{i-1}^{\prime}, q_{i}^{\prime}\right) \rightsquigarrow\left(a_{i+1}, a_{i}\right)$, contradicting the minimality of the circuit. Then $q_{i}^{\prime} p_{i+1}$ is not an arc, otherwise $\left(p_{i-1}^{\prime}, q_{i}^{\prime}\right) \rightsquigarrow$ ( $a_{i+1}, a_{i+1}$ ), yielding the same kind of contradiction. Finally, $p_{i+1}^{\prime} q_{i}$ is not an arc, otherwise ( $a_{i}, q_{i+2}$ ) is reachable from $\left(p_{i+1}^{\prime}, q_{i+2}^{\prime}\right)$, and hence $\left(p_{i+1}^{\prime}, q_{i+2}^{\prime}\right) \rightsquigarrow\left(a_{i}, a_{i+2}\right)$, by one more application of Lemma 6.3.

Having the vertices $p_{i}^{\prime}, q_{i}^{\prime}, i=0,1, \ldots, n$, allows us to modify the circuit $\left(a_{0}, a_{1}\right),\left(a_{1}, a_{2}\right), \ldots$, $\left(a_{n}, a_{0}\right)$ one step towards satisfying the Theorem 7.2.

Lemma 7.1 has a stronger version as follows:
Claim 7.4 For every integer $k>0$, there exists circuit $\left(a_{0}^{k}, a_{1}^{k}\right),\left(a_{1}^{k}, a_{2}^{k}\right), \ldots,\left(a_{n}^{k}, a_{0}^{k}\right)$ of pairs in $\widehat{T}$, and walks $P_{i}^{k}, Q_{i}^{k}, i=0, \ldots, n$, in $H$ such that each pair $P_{i}^{k}, Q_{i}^{k}$ are walks of net length $k$ constricted from below, $P_{i}^{k}$ is from $a_{i}^{k}$ to $a_{i}, Q_{i}^{k}$ from $a_{i+1}^{k}$ to $a_{i+1}$, such that $P_{i}^{k}$ and $Q_{i}^{k}$ are congruent and avoid each other. Moreover, the walk $P_{i}^{k+1}$ is a suffix of the walk $P_{i}^{k}$ for each $k$, and similarly for $Q^{k+1}$.

By reversing the walks $P_{i}^{k}, Q_{i}^{k}$ and increasing $k$ arbitrarily high, we prove the following fact.
Claim 7.5 There exist infinite walks $S_{i}, T_{i}, i=0,1, \ldots, n$ in $H$, where $S_{i}$ starts in $a_{i}$ and $T_{i}$ starts in $a_{i+1}$ such that $S_{i}, T_{i}$ are congruent and avoid each other. Moreover, the net lengths of these walks are unbounded from below.

Consider the set of pairs $(s, t)$ of corresponding vertices in $S_{i}, T_{i}$; since $S_{i}, T_{i}$ avoid each other, these pairs $(s, t)$ are all from the same strong component of $H^{+}$as $\left(a_{i}, a_{i+1}\right)$ (which we called $C_{i}$ ). Therefore, eventually the same pair $(s, t)$ is repeated. This means that each $C_{i}$ contains a circuit of
positive net value. Hence, we may assume that each cycle $D_{i}$ (our unbalanced cycle of choice in the component $C_{i}$ ) is in fact of positive net value. Thus we can avoid the complications in the proof of Lemma 7.3: the first alternative, that $D_{i}$ has positive net value, applies in all cases. This has effect on the walks $P_{i}, Q_{i}, P_{i}^{k}, Q_{i}^{k}$ and also $S_{i}, T_{i}$ above. In particular, the corresponding pairs ( $s, t$ ) of vertices on walks $S_{i}, T_{i}$ may be assumed to eventually reach the cycle $D_{i}$, and then remain on that cycle. This uses Lemma 7.3 in conjunction with Lemma 7.1, continually replacing the circuit until we get a circuit $\left(b_{0}, b_{1}\right),\left(b_{1}, b_{2}\right), \ldots,\left(b_{n-1}, b_{n}\right),\left(b_{n}, b_{0}\right)$ until each $\left(b_{i}, b_{i+1}\right)$ is on the cycle $D_{i}$ in $H^{+}$. Moreover, we can continue doing this until $\left(b_{i}, b_{i+1}\right)$ is an extremal vertex of $D_{i}$. (Recall that there exist $r$ extremal pairs, where $r$ is the net length of $D_{i}$.)

For future reference we note that we can reverse the walks $S_{i}, T_{i}$.
Claim 7.6 There exist infinite walks $Q_{i}, R_{i}, i=0,1, \ldots, n$ in $H$, where $Q_{i}$ starts in $b_{i}$ and $R_{i}$ starts in $b_{i+1}$ such that $Q_{i}, R_{i}$ are congruent and avoid each other. Moreover, these walks are constricted from below and their net lengths are unbounded from above.

It is important to note that the walks $Q_{i}, R_{i}$ are obtained by continuously following the cycle $D_{i}$ in the positive direction.

By applying Corollary 6.3 to suitable (repeated) increasing prefixes of the walks $Q_{i}, R_{i}$ we can easily conclude that they have pre-images, say $P_{i}$, that all avoid each other, except for the corresponding (intersecting) walks $Q_{i}, R_{i-1}$. In particular we obtain the following useful conclusion that complete the proof of Theorem 7.2 (3) :

There are infinite walks $P_{i}, i=0,1, \ldots, n$, in $D_{i}$, starting in $b_{i}$, that all avoid each other.

Corollary 7.7 Let $T$ be a subset of unbalanced pairs such that if $(a, b) \in T$ then $(b, a) \notin \widehat{T}$.
Suppose $\widehat{T}$ contains a circuit and let $\left(a_{0}, a_{1}\right),\left(a_{1}, a_{2}\right), \ldots,\left(a_{n}, a_{0}\right)(n>1)$ be a minimal circuit in $\widehat{T}$. Then there is no path in $H^{+}$from $\left(a_{i}, a_{i+1}\right)$ to any of $\left(a_{j}, a_{j+1}\right),\left(a_{j+1}, a_{j}\right), i \neq j$.

Proof: By Theorem 7.2 both pairs $\left(b_{i}, b_{i+1}\right),\left(b_{j}, b_{j+1}\right)$ are extremal pairs on cycles $D_{i}$ and $D_{j}$ respectively. Let $W$ be the directed walk from $\left(b_{i}, b_{i+1}\right)$ to $\left(b_{j}, b_{j+1}\right)$ we may assume that $D_{i}$ is constricted from above. Otherwise we add to the beginning of $W$ a directed walk $W^{\prime}$ obtained by going around the cycle $D_{i}$ (in negative direction) sufficiently many times and adding to the end of $W$ directed walk $W^{\prime \prime}$ obtained by going around the cycle $D_{j}$ (in negative direction) sufficiently many times (this is possible since $P_{i}, P_{i+1}$ avoid each other, and $P_{j}, P_{j+1}$ avoid each other). Let $W=$ $\left(X_{1}, X_{2}\right)$. Consider two vertices $p^{\prime} \in P_{j+1}$ and $q^{\prime} \in P_{j+2}$. Where the portion of $A^{\prime}=P_{j+1}^{-1}\left[p^{\prime}, b_{j+1}\right]$ and $B^{\prime}=P_{j+2}^{-1}\left[q^{\prime}, b_{j+2}\right]$ are congruent (avoid each other) and have the same net value as $W$. Now by Corollary 6.2 we conclude that $X_{1}$ and $X_{2}$ avoid each other. Therefore $\left(b_{i}, b_{i+1}\right)$ and $\left(b_{j}, b_{j+1}\right)$ are in the same strong component. Thus by We may assume that $D_{i}$ is the cycle that contains both $\left(b_{i}, b_{i+1}\right),\left(b_{j}, b_{j+1}\right)$ (we replace $D_{i}$ by a cycle that goes around $D_{i}$ (in positive direction) sufficiently many times and then it follows $W$ to $\left(b_{j}, b_{j+1}\right)$ and then going around $D_{j}$ once and then back to $\left(b_{i}, b_{i+1}\right)$ on $W^{-1}$ ). Observe that $D_{i}$ now has net value greater than 1 .

By Theorem 7.2 (2) the walks $P_{i}$ and $P_{j}$ around $D_{i}$ avoid each other and we can apply Corollary 4.4 to conclude that some component of $H^{+}$contains a circuit, a contradiction. By similar argument from from the previous proposition we have the following.

There is no path in $H^{+}$from $\left(a_{i+1}, a_{i}\right)$ to $\left(a_{i}, a_{i+1}\right)$, and in particular there is no path from $\left(a_{0}, a_{n}\right)$ to $\left(a_{n}, a_{0}\right)$.

The following proposition will be used in the last section. Recall that we have denoted by $C_{i}$ the (strong) component of $H^{+}$containing the pair $\left(a_{i}, a_{i+1}\right)$. A digraph is symmetric if for each arc $u v$ the arc $v u$ is also present.

Proposition 7.8 Let $T$ be a subset of unbalanced pairs such that if $(a, b) \in T$ then $(b, a) \notin \widehat{T}$.
Suppose $\widehat{T}$ contains a circuit and let $\left(a_{0}, a_{1}\right),\left(a_{1}, a_{2}\right), \ldots,\left(a_{n}, a_{0}\right)(n>1)$ be a minimal circuit in $\widehat{T}$. Then each component $C_{i}$ is a symmetric digraph.

Proof: We will show that every directed walk $W^{\prime}$ in $C_{i}$ from some $(c, d)$ to $\left(a_{i}, a_{i+1}\right)$ consists of two walks, $X$ from $c$ to $a_{i}$ and $Y$ from $d$ to $a_{i+1}$, that avoid each other. Since every arc of $C_{i}$ lies on a such a walk, this proves the Proposition. By Theorem 7.2 (2), there exists a circuit $\left(b_{0}, b_{1}\right),\left(b_{1}, b_{2}\right), \ldots,\left(b_{n}, b_{0}\right)$ of extremal pairs where $\left(b_{i}, b_{i+1}\right),\left(a_{i}, a_{i+1}\right)$ are in the same component $C_{i}$.

Recall that $\left(b_{i}, b_{i+1}\right)$ lies on $D_{i}$ (which is a closed walk of positive net value). We also note by Theorem 7.2 (3) that $P_{i}, P_{i+1}$ are obtained by repeatedly following cycle $D_{i}$ in positive direction (see also Claim 7.6).

Consider a directed walk $W$ from $\left(b_{i}, b_{i+1}\right)$ to $(c, d)$ in $C_{i}$ that has a negative net value and it is constricted from above. Such a directed walk is obtained by starting at $\left(b_{i}, b_{i+1}\right)$ and going around the cycle $D_{i}$ in negative direction sufficiently many times and then going to $(c, d)$. Now consider the directed walk $W^{\prime}$ going from $(c, d)$ to $\left(b_{i}, b_{i+1}\right)$ and then following the directed walk $W^{\prime \prime}$ around the cycle $D_{i}$ in the negative direction, so that $W W^{\prime} W^{\prime \prime}$ is constricted and has negative net value, (Again this can be obtained by going around $D_{i}$ in negative direction sufficiently many times). The directed walk $W W^{\prime} W^{\prime \prime}$ gives two constricted walks $A, B$ from some $b_{i}, b_{i+1}$ to $b_{i}, b_{i+1}$ respectively, where $A$ avoids $B$. Now let $C, D$ be two walks from $p, q$ to $b_{i+1}, b_{i+2}$ respectively, that avoid each other and have the same negative net value as $A$. (We may assume $p \in P_{i+1}$ and $p \in P_{i+2}$ ). Now by Corollary 6.2 we conclude that $A, B$ avoid each other and hence the walks $X$ and $Y$ constituting $W^{\prime}$ avoid each other. This implies that $C_{i}$ is symmetric.

## 8 Balanced pairs and minimal circuits

A strong component of $H^{+}$is balanced if every closed directed walk in that component has net value zero, i.e., the same number of positive and negative arcs. A pair of $H^{+}$is called balanced if it lies in a balanced strong component.

Let $H^{*}$ be the sub-digraph of $H^{+}$induced by balanced pairs. The pairs in $H^{*}$ can be assigned levels so that if $(a, b)(c, d)$ is a positive arc in $H^{+}$, then the level of $(c, d)$ is one more than the level of $(a, b)$, and if $(a, b)(c, d)$ is a negative arc in $H^{+}$, then the level of $(c, d)$ is one less than the level of $(a, b)$.

Definition 8.1 We say a set $T$ of the pairs in $H^{+}$is good with respect to level $\ell$ if it has at least one pair of $H^{*}$ on level $\ell$ and for all pairs $(x, y)$ of $H^{*}$ with level $\ell^{\prime}<\ell$ exactly one of the $(x, y),(y, x)$ belongs to $T$.

Recall that $\widehat{T}$ is the set of pairs that are reachable from $T$.
Theorem 8.2 Let $T$ be a good set with respect to level $\ell$.
Suppose $\widehat{T}$ contains a circuit, and let $\left(a_{0}, a_{1}\right),\left(a_{1}, a_{2}\right), \ldots,\left(a_{n}, a_{0}\right)$ be a minimal circuit in $\widehat{T}$. Then one of the following statements holds.

1. $n=1$ and there exists $(a, b) \in T$ such that $(b, a) \in \widehat{T}$
2. $n>1$ and if $\left(a_{i}, a_{i+1}\right),\left(a_{i+1}, a_{i+2}\right)$ are $z$-pairs with respect to $\widehat{T}$ then there is no path in $H^{+}$ from $\left(a_{i}, a_{i+2}\right)$ to any of $\left(a_{i+1}, a_{i}\right),\left(a_{i+2}, a_{i+1}\right)$, and $\left(a_{i+2}, a_{i}\right)$.

Proof: If $n=1$ then there exists $(a, b),\left(a^{\prime}, b^{\prime}\right) \in T$ such that $(a, b) \rightsquigarrow\left(a_{0}, a_{1}\right)$ and $\left(a^{\prime}, b^{\prime}\right) \rightsquigarrow$ $\left(a_{1}, a_{0}\right)$. Now $\left(a^{\prime}, b^{\prime}\right) \rightsquigarrow(b, a)$. Therefore $\left(a^{\prime}, b^{\prime}\right) \rightsquigarrow(b, a)$ and hence $(a, b) \in T$ and $(b, a) \in \widehat{(T)}$.

It is easy to see that if $n>1$ then the circuit is on level $\ell$ of $H^{*}$, i.e. the pairs of the circuit that belong to $H^{*}$ are on level $\ell$. First we note that ( $a_{i}, a_{i+2}$ ) is not a $z$-pair. Otherwise according to assumptions for $T,\left(a_{i}, a_{i+2}\right) \in \widehat{T}$, and hence we get a shorter circuit.

We may assume $L_{i} \leq L_{i+1}$. Now we show that $L<L_{i+1}-1$. Let ( $p_{i}^{\prime}, q_{i+1}^{\prime}$ ) be the first vertex on $Z_{a_{i}, a_{i+1}}$ and $\left(p_{i}, q_{i+1}\right)$ be the second vertex and we may assume that $Z_{a_{i}, a_{i+1}}\left[\left(p_{i}, q_{i+1}\right),\left(a_{i}, a_{i+1}\right)\right]$ has net value zero and constricted from below (Note that we use the notation used in the Lemma 7.1 and most of the argument for this case is similar to argument in Lemma 7.1). By the argument similar to the one in Lemma 7.1 it follows that $p_{i+1}^{\prime} p_{i}$ is not an arc of $H$ (as otherwise $\left(p_{i+1}^{\prime}, q_{i+2}^{\prime}\right)\left(p_{i}, q_{i+2}\right) \in$ $A\left(H^{+}\right)$and also $\left(p_{i}, q_{i+2}\right),\left(a_{i}, a_{i+2}\right)$ are in the same strong component and hence $Z_{a_{i}, a_{i+2}}$ exists, a contradiction). Moreover, $p_{i}^{\prime} q_{i+2} \notin A(H)$ as otherwise $\left(p_{i}^{\prime}, q_{i+1}^{\prime}\right)\left(q_{i+2}, q_{i+1}\right) \in A\left(H^{+}\right)$. Observe that as it shown in Lemma $7.1\left(a_{i+1}, a_{i+2}\right),\left(q_{i+1}, q_{i+2}\right)$ are in the same strong component of $H^{+}$. Now $\left(p_{i}^{\prime}, q_{i+1}^{\prime}\right) \rightsquigarrow\left(a_{i+2}, a_{i+1}\right)$, and $\left(p_{i}^{\prime}, q_{i+1}^{\prime}\right) \rightsquigarrow\left(a_{i+1}, a_{i+2}\right)$. Therefore for some $\left(x^{\prime}, y^{\prime}\right) \in S$ we have $\left(x^{\prime}, y^{\prime}\right) \rightsquigarrow\left(y^{\prime}, x^{\prime}\right)$, a contradiction.

Note that $\left(p_{i}, q_{i+1}\right)\left(p_{i}^{\prime}, q_{i+1}^{\prime}\right) \in A\left(H^{+}\right)\left(\right.$since $p_{i+1}^{\prime} p_{i}$ is not an arc). Now ( $\left.p_{i}^{\prime}, q_{i+1}^{\prime}\right)$ and $\left(p_{i+1}^{\prime}, q_{i+2}^{\prime}\right)$ are in $\widehat{S}$ since they are on a lower level than $\ell$ and therefore $\left.\left(p_{i}^{\prime}, q_{i+2}^{\prime}\right) \in \widehat{S}\right)$ or $\left(q_{i+2}^{\prime}, p_{i}^{\prime}\right) \in \widehat{S}$ (the former is not possible as it would imply a circuit of level $\ell^{\prime}<\ell$ in $T$, a contradiction to $T$ being a good set). Thus we have $\left(p_{i}^{\prime}, q_{i+2}^{\prime}\right) \in \widehat{S}$. Now $\left(p_{i}^{\prime}, q_{i+2}^{\prime}\right)\left(p_{i}, q_{i+2}\right) \in A\left(H^{+}\right)$and (as shown in Lemma 7.1, $\left(p_{i}, q_{i+2}\right),\left(a_{i}, a_{i+2}\right)$ are in the same strong component of $\left.H^{+}\right)$. Therefore $\left(p_{i}^{\prime}, q_{i+2}^{\prime}\right) \rightsquigarrow\left(a_{i}, a_{i+2}\right)$. Thus $\left(a_{i}, a_{i+2}\right)$ is a $z$-pair, a contradiction.

We note that if $\left(a_{i+2}, a_{i}\right)$ is a $z$-pair then we according to Lemma 7.1 there is a circuit on a lower level of $H^{*}$. Therefore $\left(a_{i+2}, a_{i}\right)$ is not a $z$-pair. We also note $\left(a_{i}, a_{i+2}\right) \nprec\left(a_{i+1}, a_{i}\right)$ as otherwise $\left(a_{i}, a_{i+1}\right) \rightsquigarrow\left(a_{i+2}, a_{i}\right)$ and hence $\left(a_{i+2}, a_{i}\right)$ is a $z$-pair, a contradiction. Similarly $\left(a_{i}, a_{i+2}\right) \nLeftarrow\left(a_{i+2}, a_{i+1}\right)$. Now suppose $\left(a_{i}, a_{i+2}\right) \in T$ and there exits a path $W$ from $\left(a_{i}, a_{i+2}\right)$ to $\left(a_{i+2}, a_{i}\right)$. Since neither of $\left(a_{i+2}, a_{i}\right),\left(a_{i}, a_{i+2}\right)$ is a $z$-pair, $W$ is constricted from below.

Let $L_{0}$ be the height of $Z_{a_{i}, a_{i+1}}$ and $L_{1}$ be the height of $Z_{a_{i+1}, a_{i+2}}$ and assume $L_{0} \leq L_{1}$ (the argument for $L_{0}>L_{1}$ is symmetric).

As we showed we have $L<L_{1}-1$. Now by applying the Lemma 6.2 on suitable portion of $Z_{a_{i+1}, a_{i+2}}$ and $W$ we conclude that $A_{0}, A_{2}$ avoid each other where $W=\left(A_{0}, A_{2}\right)$. This would mean $\left(a_{i}, a_{i+2}\right)$ and $\left(a_{i+2}, a_{i}\right)$ are in the same strong component of $H^{+}$, a contradiction to the choice of $T$.

The stronger version of Theorem 8.2 is the following Corollary.

Corollary 8.3 Let $T$ be a good set with respect to level $\ell$.
Suppose $\left(a_{1}, a_{2}\right),\left(a_{2}, a_{3}\right), \ldots$,
$\left(a_{j-1}, a_{j}\right), 1<j-1$ are $z$-pairs in $T$. Then there is no path in $H^{+}$from $\left(a_{i}, a_{t}\right), 1 \leq i<t \leq j$, to $\left(a_{r}, a_{i}\right), 1 \leq i<r \leq j$.

Proof: By comparing the height of the path from $\left(a_{i}, a_{t}\right)$ to $\left(a_{r}, a_{i}\right)$ and the height of $Z_{a_{i}, a_{i+1}}, Z_{a_{t-1}, a_{t}}$, $Z_{a_{r-1}, a_{r}}$, and applying similar argument as in the proof of Theorem 8.2, we get a contradiction. $\diamond$

Following the proof of the Lemma 7.1 one can obtain the following corollary.
Corollary 8.4 Let $S$ be a set of pairs. Suppose $\widehat{S}$ contains a circuit and let $\left(a_{0}, a_{1}\right),\left(a_{1}, a_{2}\right), \ldots$, $\left(a_{n}, a_{0}\right)(n>1)$ be a minimal circuit in $\widehat{S}$.

If $\left(a_{i}, a_{i+1}\right)$ is a z-pair with respect to $\widehat{S}$ then let $\left(p_{i}, q_{i+1}\right)$ be the second vertex on $Z_{a_{i}, a_{i+1}}$ otherwise let $X_{i}$ be a constricted directed walk from below of net value zero from $\left(p_{i}, q_{i+1}\right) \in S$ to $\left(a_{i}, a_{i+1}\right)$ ( $X_{i}$ could be just a path in strong component containing $\left(a_{i}, a_{i+1}\right)$.

Then there exists another circuit $\left(a_{0}^{\prime \prime}, a_{1}^{\prime \prime}\right),\left(a_{1}^{\prime \prime}, a_{2}^{\prime \prime}\right), \ldots,\left(a_{n}^{\prime \prime}, a_{0}^{\prime \prime}\right)$ of pairs, and walks $P_{i}^{\prime}, Q_{i}^{\prime}, i=$ $0, \ldots, n$, in $H$, such that $P_{i}^{\prime}, Q_{i}^{\prime}$ are walks of net length zero, constricted from below, $P_{i}^{\prime}$ from $a_{i}^{\prime \prime}$ to $a_{i}, Q_{i}$ from $a_{i+1}^{\prime \prime}$ to $a_{i+1}$, and such that $P_{i}^{\prime}$ and $Q_{i}^{\prime}$ are congruent and avoid each other. Here each $\left(a_{i}^{\prime \prime}, a_{i+1}^{\prime \prime}\right)$ is either $\left(p_{i}, q_{i+1}\right)$ or $\left(q_{i}, p_{i+1}\right)$ or $\left(q_{i}, q_{i+1}\right)$ or $\left(p_{i}, p_{i+1}\right)$.

Corollary 8.5 Let $S$ be a set of pairs. Suppose $\widehat{S}$ contains a circuit and let $\left(a_{0}, a_{1}\right),\left(a_{1}, a_{2}\right), \ldots$, $\left(a_{n}, a_{0}\right),(n>1)$ be a minimal circuit in $\widehat{S}$ Let $W_{1}$ be a directed walk from $(p, q) \in S$ to $\left(a_{i}, a_{i+1}\right)$ of net value zero and let $W_{2}$ be a directed walk from $(p, q)$ to $\left(a_{i+1}, a_{i+2}\right)$ of net value zero. Then at least one of the $W_{1}, W_{2}$ is not constricted from below.

Proof: For contradiction $W_{1}$ and $W_{2}$ both are constricted from below. We may assume $L_{1}$ the height of $W_{1}$ is at most $L_{2}$; the height of $W_{2}$. Now by Corollary 8.4 and the proof of the Lemma $7.1\left(a_{i}, a_{i+1}\right),(q, q)$ are in the same strong component, a contradiction. Similarly if $L_{2}<L_{1}$ then $\left(a_{i}, a_{i+1}\right)$ and $(p, p)$ are in the same strong component of $H^{+}$, a contradiction.

Lemma 8.6 Let $x, y, z$ be three vertices such that there exist a path $W_{x}$ in $H^{+}$from $(y, x)$ to $(x, y)$ and there exists a path $W_{y}$ in $H^{+}$from $(z, y)$ to $(y, z)$ where both $W_{x}, W_{y}$ are constricted from below and have net value zero.

Then there is no path $W_{z}$ in $H^{+}$from $(x, z)$ to $(z, x)$ which is contracted from below and has net value zero.

Proof: Suppose this is not the case and $W_{z}$ exists. Up to symmetry suppose $L_{z}$ the height of $W_{z}$ is bigger than $L_{y}$ the height of $W_{y}$ respectively.

We observe that $(x, z) \nVdash(x, y)$ and $(x, z) \nsim(y, z)$. For contradiction suppose $(x, z) \rightsquigarrow(y, z)$. Let $T=\{(x, y),(y, z),(x, z)\}$ and now there is a circuit $(x, y),(y, z),(z, x)$ in $\widehat{T}$. Since $(x, z) \rightsquigarrow(z, x)$ and $(x, z) \rightsquigarrow(y, z)$, we get a contradiction by Corollary 8.5. Similarly $(x, z) \nprec(x, y)$. These would also imply that $(x, z) \nLeftarrow \rightarrow(y, x)$ and $(x, z) \nprec \rightarrow(z, y)$. By the symmetry $(y, x) \nsim \rightarrow(y, z),(y, x) \nsim \rightarrow(z, x)$ and $(z, y) \nsim(x, y),(z, y) \nsim(z, x)$.

Let $W_{z}=\left(Z, Z^{\prime}\right)$ and $W_{y}=\left(Y, Y^{\prime}\right)$ and let $(g, h)$ be a vertex on $W_{z}$ where $W_{z}[(g, h),(z, x)]$ is constricted from above and has height $-L_{y}$. Let $W_{z}[(g, h),(z, x)]=\left(A_{2}, A_{0}\right)$. Now by applying Corollary 6.2 on $Y, Y^{\prime}, A_{2}, A_{0}$ we conclude that $Y^{\prime}, Y^{\prime}$ avoid each other. This would imply that $(y, z)$ and $(z, y)$ are in the same strong component of $H^{+}$, contradiction.

## 9 Choosing pairs of unbalanced components

We say a pair $(x, y)$ in an unbalanced strong component of $H^{+}$is half extremal if at least one of the $(x, y),(y, x)$ is extremal.

Lemma 9.1 Let $U=\left\{u_{1}, u_{2}, \ldots, u_{m}\right\}$ and let $R$ be a set of ordered pairs $\left(u_{i}, u_{j}\right), i<j$ each being half extremal pair in $H^{+}$. Let $R_{k}$ be a subset of $R$ consisting of all pairs $\left(u_{i}, u_{j}\right) \in R$ with $k \leq i<j$. Suppose that for each $R_{k}, u_{k}$ is a source. Then $\widehat{R}$ has no circuit.

Proof: For contradiction suppose $\widehat{R}$ contains a circuit and let $\left(a_{0}, a_{1}\right),\left(a_{1}, a_{2}\right), \ldots,\left(a_{n}, a_{0}\right)$ be a minimal circuit in $\widehat{R}$.

Since each $R_{i}$ has a source, it is easy to see that $n>1$ (c.f. the proof of Theorem 7.2). According to the Theorem 7.2 there exists another circuit $\left(b_{0}, b_{1}\right),\left(b_{1}, b_{2}\right), \ldots,\left(b_{n}, b_{0}\right)$ where each $\left(b_{i}, b_{i+1}\right)$ is extremal and lies in the directed cycle $D_{i}$ in a strong component of $H^{+}$. By Theorem 7.2, $\left(a_{i}, a_{i+1}\right),\left(b_{i}, b_{i+1}\right)$ are in the same strong component of $H^{+}$.

By Theorem $7.2(2)$ we observe that for every $i, j,\left(b_{i}, b_{j}\right)$ is in an unbalanced component. Therefore if one of the $b_{\ell}$ 's is a source for some $R_{i}$ then it would mean we have $\left(b_{\ell}, b_{\ell+2}\right) \in \widehat{R}$ as well, yielding a shorter circuit. Thus it remain to prove that we may assume one of the $b_{\ell}$ is a source.

Claim 9.2 At least one of the $b_{\ell}$ 's is a source.
Proof: For contradiction none of the $b_{i}$ 's is a source. Now each $\left(b_{i}, b_{i+1}\right)$ is in some strong component $C_{i}$ where it has an extremal pair $(p, q)$ and $p$ was a source for some $R_{j}$ where $j$ is the minimum subscript. Now by Proposition 7.8 we may assume that $(p, q)$ and $\left(b_{i}, b_{i+1}\right)$ lies on the same cycle $D_{i}$. This can be achieved by starting from $\left(b_{i}, b_{i+1}\right)$ and going around $D_{i}$ and then to $(p, q)$ on the path in $C_{i}$ and then back on the same path (the arcs in $C_{i}$ are symmetric by Corollary 7.8 ) to $\left.\left(b_{i}, b_{i+1}\right)\right)$.

Therefore by applying the Corollary 7.4 sufficiently many times (the net value between $(p, q)$ and $\left(b_{i}, b_{i+1}\right)$ in $\left.D_{i}\right)$ we may assume that there exist another circuit such that $(p, r)$ is one of the pair of the circuit and $\left(b_{i}, b_{i+1}\right)$ and $(p, r)$ are in the same strong component. Since $j$ is the minimum subscript, there does not exist $(q, p) \in R$ such that $(q, p),(p, r)$ are the pairs of this new circuit. This means that $n=1$. In this case we have $(p, r) \rightsquigarrow\left(a_{0}, a_{1}\right)$ and $(p, s) \rightsquigarrow\left(a_{1}, a_{0}\right)$. Now by skew symmetry we have $(p, r) \rightsquigarrow(s, p)$ contradiction to $p$ being a source for $R_{j}$.

This completes the proof of the lemma.
Lemma 9.3 Any set $S$ of half extremal pairs such that $(a, b) \in S$ implies that $(b, a) \in S$ has a source.

Proof: Let $U$ be the set of vertices of $H$ that appear, as a first or second coordinate, in pairs of $S$. We use induction on $|U|$. We first observe that if $|U|=2$, then $S$ has a source. Indeed, suppose $U=\left\{u_{1}, u_{2}\right\}$. If $u_{1}$ is not a source for $S$, we must have both $\left(u_{1}, u_{2}\right),\left(u_{2}, u_{1}\right)$ in $S$ and $\left(u_{2}, u_{1}\right)$ reachable from $\left(u_{1}, u_{2}\right)$ in $H^{+}$; and if $u_{2}$ is not a source for $S$ we also must have ( $u_{1}, u_{2}$ ) reachable from $\left(u_{2}, u_{1}\right)$ in $H^{+}$. This would mean that $u_{1}, u_{2}$ is an invertible pair, i.e., a circuit in a strong component of $H^{+}$, a contradiction. Thus there is a source for $S$.

Assume the statement for any set of size $m$ and let $U$ have $m+1$ elements say $U=\left\{u, u_{1}, u_{2}, \ldots, u_{m}\right\}$. For simplicity when we say $x \in V(H)$ is a source for $X \subseteq V(H)$, we mean $x$ is a source for all the pairs $(a, b) \in S$ where $a, b \in X$. By induction hypothesis we may assume that $u_{1}$ is a source for $U \backslash\{u\}$. Observe that there exists $2 \leq t \leq m$ such that $u_{i}, 2 \leq i \leq t$ is a source for $\left\{u_{i}, u_{i+1}, \ldots, u_{m}, u\right\}$ and $u_{i}, t+1 \leq i \leq m$ is a source for $\left\{u_{i}, u_{i+1}, \ldots, u_{m}\right\}$ and $u$ is a source for $\left\{u, u_{t+1}, u_{t+2}, \ldots, u_{m}\right\}$.

We say $u_{i}, u_{i+1}$ are not exchangeable if $\left(u_{i}, u_{i+1}\right) \in S$ and $\left(u_{i+1}, u_{i^{\prime}}\right) \rightsquigarrow\left(u_{i}, u_{i+1}\right), 2 \leq i+$ $1<i^{\prime} \leq m$. If $u_{i}, u_{i+1}$ are exchangeable then we can assume that $u_{i+1}$ is also source for $u_{i+1}, u_{i}, \ldots, u_{t}, u, u_{t+1}, \ldots, u_{m}$. Otherwise we must have $\left(u_{i+1}, u_{i}\right) \rightsquigarrow\left(u_{i^{\prime}}, u_{i+1}\right)$ and hence $u_{i}, u_{i+1}$ are not exchangeable.

Now suppose $u_{1}$ is not a source for $U$ as otherwise we are done. This would mean $\left(u_{1}, y\right) \rightsquigarrow\left(x, u_{1}\right)$ where $x, y \in U$. Since $u_{1}$ is a source for $U \backslash\{u\}$, either $x=u$ or $y=u$. If $y \neq u$ then we may assume $y=u_{\ell}$ where $\ell$ is the smallest subscript.

We also assume $u$ is not a source for $U$ as otherwise we are done. Therefore there is a path in $H^{+}$from $\left(u, u_{r}\right)$ to $\left(u_{j}, u\right)$. Note either $r=1$ or $j=1$ as otherwise $U \backslash\left\{u_{1}\right\}$ satisfies the condition of the Lemma 9.1. Without loss of generality assume that $\left(u_{j}, u\right)$ is reachable from $\left(u, u_{1}\right)$ in $H^{+}$ and $j$ is the smallest subscript.

To summarize, one of the following happens :

1. $\left(u, u_{1}\right) \rightsquigarrow\left(u_{j}, u\right)$ and $\left(u_{1}, u_{\ell}\right) \rightsquigarrow\left(u, u_{1}\right)$.
2. $\left(u, u_{1}\right) \rightsquigarrow\left(u_{j}, u\right)$ and $\left(u_{1}, u\right) \rightsquigarrow\left(u, u_{1}\right)$.

We prove (1) and the proof of (2) is similar. We may assume that $j$ is a minimum subscript. Let $i_{1}<i_{2}<\cdots<i_{k-1}<i_{k}=j$ where $u_{i_{r}}$ and $u_{i_{r+1}}, 1 \leq r \leq k$ are not exchangeable and $u_{i_{k-1}}$, $u_{j}$ are not exchangeable and $i_{1}$ is the smallest subscript. We may assume that $i_{1}=1$ as otherwise we can exchange $u_{i_{1}-1}, u_{i_{1}}$.

No we may assume that $u_{i_{2}}$ is the minimum subscript. This would mean we assume that $u_{i_{2}}=u_{2}$ as otherwise we would exchange $u_{i_{2}}$ with $u_{i_{2}-1}$ and continuing exchanging $u_{i_{2}}$ with the previous element. By continuing this argument we may assume that $u_{i_{3}}$ is $u_{3}$ and so on. Therefore we may assume that $i_{1}=1, i_{2}=2, \ldots, i_{k}=j$ and $j=k$. This means $\left(u_{1}, u_{2}\right),\left(u_{2}, u_{3}\right), \ldots,\left(u_{j-1}, u_{j}\right) \in S$.

Case 1. $\quad \ell \leq j$. Now there exists a circuit $C_{1}=\left(u, u_{1}\right),\left(u_{1}, u_{2}\right), \ldots,\left(u_{j-1}, u_{j}\right),\left(u_{j}, u\right)$ in $T=\left\{\left(u, u_{1}\right),\left(u_{1}, u_{2}\right), \ldots,\left(u_{j-1}, u_{j}\right),\left(u_{j}, u\right)\right\}$. Either $C_{1}$ is a minimal circuit and has length more than two or $\left(u_{r}, u_{r+s}\right) \rightsquigarrow\left(u_{1}, u\right), 1 \leq r \leq j-1$. In the former case we have $\left(u_{r}, u_{r+s}\right) \rightsquigarrow\left(u_{1}, u\right)$ and we also have $\left(u_{1}, u_{\ell}\right) \rightsquigarrow\left(u, u_{1}\right)$. These would imply that $\left(u_{r}, u_{r+s}\right) \rightsquigarrow\left(u_{\ell}, u_{1}\right)$, a contradiction to $\widehat{R}$ does not have a circuit according to Lemma 9.1. Here $R=\left\{\left(u_{i}, u_{j}\right) \in S \mid i<j\right\}$.

We continue by assuming $C_{1}$ is minimal. This means we may assume $\left(u_{r}, u_{r+1}\right) \nLeftarrow \rightarrow\left(u_{i}, u_{i^{\prime}}\right)$, $i<i^{\prime}-1$. Moreover, $\left(u_{r}, u_{r+1}\right) \not \nsim\left(u_{i^{\prime}}, u_{i}\right)$ as otherwise we have a circuit in $\widehat{R}$. However, there exists a path from $\left(u_{1}, u_{\ell}\right) \rightsquigarrow\left(u, u_{1}\right)$. This is a contradiction by Lemma 7.7.

Case 2. $\quad \ell>j$. First suppose $t \leq \ell$. We may assume that $u$ is on the its latest position. By that we mean $u$ can not be exchanged with $u_{t+1}$. This means that $u, u_{t+1}$ are not exchangeable and hence similar to the previous case there exists a circuit $\left(u_{1}, u\right),\left(u, u_{t+1}\right),\left(u_{t+1}, u_{t+2}\right), \ldots,\left(u_{\ell-1}, u_{\ell}\right),\left(u_{\ell}, u_{1}\right)$
in $T=\left\{\left(u_{1}, u\right),\left(u, u_{t+1}\right),\left(u_{t+1}, u_{t+2}\right), \ldots,\left(u_{\ell-1}, u_{\ell}\right),\left(u_{\ell}, u_{1}\right)\right\}$. However, since $\left(u_{1}, u\right) \rightsquigarrow\left(u_{\ell}, u_{1}\right)$, we get a contradiction by Lemma 7.7. Similar argument would apply when $t>\ell$.

Proposition 9.4 In every execution of line 5 of the algorithm, the set $S$ has a source. Thus we can always execute line 8 of the algorithm. Moreover, after executing the loop in lines 9-11, the set $V_{c}$ will not contain a circuit.

Proof: Note that at each stage of the algorithm the set $S$ in lines 5,8 has all the remaining extremal pairs and hence set $T=S \cup\{(y, x) \mid(x, y) \in S\}$ is the set of all remaining half extremal pairs. By Lemma 9.3, $T$ has a source and hence $S$ has a source. Now it is easy to see that $V_{c}$ is $\widehat{R}$ where $R$ satisfies the conditions of Lemma 9.1. Therefore $V_{c}$ will not contain a circuit.

## 10 Choosing pairs of balanced components

Proposition 10.1 Any set of pairs on a level $\ell$ of $H^{*}$ has a source that respects transitivity in $V_{c}$. Thus we can always execute line 16 of the algorithm. Moreover, after executing the loop in lines 17-19, the set $V_{c}$ will not contain a circuit.

Proof: We first show the following fact. Let $\left(u_{1}, u_{2}\right),\left(u_{2}, u_{3}\right), \ldots,\left(u_{m-1}, u_{m}\right)$ be a chain of pairs in $V_{c}$, each being a $z$-pair with respect to $V_{c}$. Then each $u_{i}, 1 \leq i \leq m$, is a source for $R_{i}$ consisting of all pairs $\left(u_{i}, u_{j}\right), i<j \leq m$. Otherwise suppose $u_{i}$ is not a source for $R_{i}$. Therefore $\left(u_{i}, u_{j}\right) \rightsquigarrow\left(u_{r}, u_{i}\right), i<j \leq r$. But this is a contradiction according to Corollary 8.3.

Let $U=\left\{u_{1}, u_{2}, \ldots, u_{m}\right\} \subseteq V(H)$ and let $S_{1}$ be the set of all pairs $\left(u_{i}, u_{j}\right), 1 \leq i<j \leq m$, of $H^{+}$ such that $\left(u_{i}, u_{j}\right)$ is either a $z$-pair with respect to $V_{c}$ or $\left(u_{i}, u_{j}\right)$ is on level $\ell$ of $H^{*}$. Suppose $u_{i} \in U$, $1 \leq i \leq m$, is a source for $S_{i}$ consisting of all pairs in $S_{1}$ that involves only vertices $\left\{u_{i}, u_{i+1}, \ldots, u_{m}\right\}$. We also assume $S_{1}$ has the following property. $S_{1}$ contains all the pairs $(x, y)$ that are $z$-pairs with respect to $V_{c}$. It contains the pairs $(x, y)$ where there is a chain $\left(x, x_{1}\right),\left(x_{1}, x_{2}\right), \ldots,\left(x_{t-1}, x_{t}\right),\left(x_{t}, y\right)$ of all $z$-pairs with respect to $V_{c}$ and lastly it contains some pairs $(x, y)$ on level $\ell$ of $H^{*}$ with priority given to $(x, y)$ such that $(y, x) \rightsquigarrow(y, x)$ if one exists.

Claim 10.2 Let $u_{0}$ be any vertex in $V(H) \backslash U$ where $\left(u_{0}, u_{i}\right),\left(u_{j}, u_{0}\right) \notin \widehat{S_{1}}$ for any $1 \leq i \leq j \leq m$, $\left(u_{0}, u_{1}\right)$ is on level $\ell$ of $H^{*}$. Moreover if there are several such $u_{0}$ we chose $u_{0}$ such that $\left(u_{1}, u_{0}\right) \rightsquigarrow$ $\left(u_{0}, u_{1}\right)$ if one exists. Let $S_{2}$ be the set of all pairs $\left(u_{0}, u_{i}\right),\left(u_{j}, u_{0}\right), 1 \leq i, j \leq m$, each on level $\ell$ of $H^{*}$. Then $S_{2}$ has a source that respects transitivity in $S_{1}$.

Proof: For contradiction suppose there is no source for $S_{2}$. Since $u_{0}$ is not a source, we have $\left(u_{0}, u_{r}\right) \rightsquigarrow\left(u_{j}, u_{0}\right), 1 \leq j \leq r \leq m$. Suppose also $u_{1}$ is not a source for $S_{1} \cup S_{2}$. Therefore we have $\left(u_{1}, y\right) \rightsquigarrow\left(x, u_{1}\right)$ where $x, y \in\left\{u_{0}, u_{1}, \ldots, u_{m}\right\}$. Moreover since $\left(u_{1}, u_{t}\right) \in S_{1}, 1 \leq t \leq m, y=u_{0}$ as otherwise $\left(x, u_{1}\right) \in S_{1}$. Similarly $x=u_{1}$ as otherwise when $x=u_{t^{\prime}}$ we have $\left(u_{1}, u_{t^{\prime}}\right) \rightsquigarrow\left(u_{0}, u_{1}\right)$ and hence $\left(u_{0}, u_{1}\right) \in S_{1}$. To summarize we have $\left(u_{1}, u_{0}\right) \rightsquigarrow\left(u_{0}, u_{1}\right)$ and $\left(u_{0}, u_{r}\right) \rightsquigarrow\left(u_{j}, u_{0}\right)$.

Case 1. $j=1$. We have $\left(u_{0}, u_{r}\right) \rightsquigarrow\left(u_{1}, u_{0}\right)$. Now we have $\left(u_{0}, u_{r}\right) \rightsquigarrow\left(u_{1}, u_{0}\right) \rightsquigarrow\left(u_{0}, u_{1}\right) \rightsquigarrow$ $\left(u_{r}, u_{0}\right)$.

Now there is a circuit $\left(u_{0}, u_{1}\right),\left(u_{1}, u_{2}\right), \ldots,\left(u_{r-1}, u_{r}\right),\left(u_{r}, u_{0}\right)$ in $S_{1} \cup S_{2}$. By definition $\left(u_{1}, u_{3}\right) \in$ $S_{1}$ and hence we may assume $r=2$ as otherwise we get a shorter circuit. To summarize, we have
$\left(u_{0}, u_{2}\right) \rightsquigarrow\left(u_{1}, u_{0}\right) \rightsquigarrow\left(u_{0}, u_{1}\right) \rightsquigarrow\left(u_{2}, u_{0}\right)$. Now there exists a circuit $C=\left(u_{0}, u_{1}\right),\left(u_{1}, u_{2}\right),\left(u_{2}, u_{0}\right)$ in $S_{1} \cup S_{2}$. Note that we may assume at least one of the $\left(u_{0}, u_{1}\right),\left(u_{1}, u_{2}\right),\left(u_{2}, u_{0}\right)$ is not a $z$-pair with respect to $V_{c}$. Otherwise we get a circuit on a lower level of $V_{c}$ by Lemma 7.1. Moreover we may assume at least one of the $\left(u_{2}, u_{0}\right),\left(u_{0}, u_{1}\right)$ is not a $z$-pair with respect to $V_{c}$. First suppose two of the three pairs in $C$ are $z$-pairs with respect to $V_{c}$, say $\left(u_{1}, u_{2}\right),\left(u_{2}, u_{0}\right)$. Now all the pairs of $V_{c}$ together with three pairs $\left(u_{1}, u_{2}\right),\left(u_{2}, u_{0}\right),\left(u_{0}, u_{1}\right)$ satisfies the conduction of Theorem 8.2. Since $\left(u_{1}, u_{0}\right) \rightsquigarrow\left(u_{0}, u_{1}\right)$, we get a contradiction by Theorem 8.2. Now we may assume that at most one of the pairs in $C$ is a $z$-pair with respect with $V_{c}$. Note that this means none of $\left(u_{2}, u_{0}\right),\left(u_{0}, u_{1}\right)$ is a $z$-pair. Therefore the path from $\left(u_{1}, u_{0}\right)$ to $\left(u_{2}, u_{0}\right)$ and the path from $\left(u_{1}, u_{0}\right)$ to $\left(u_{0}, u_{1}\right)$ are constricted from below and have net value zero. Now let $T$ be the set of $\left(u_{1}, u_{2}\right),\left(u_{2}, u_{0}\right),\left(u_{1}, u_{0}\right)$. It is easy to see that circuit $C$ is minimal in $\widehat{T}$ and since $\left(u_{1}, u_{0}\right) \rightsquigarrow\left(u_{2}, u_{0}\right)$, and $\left(u_{1}, u_{0}\right) \rightsquigarrow\left(u_{0}, u_{1}\right)$, we get a contradiction by Corollary 8.5.

Case 2. $1<j \leq r$. First assume that $r=j$. Note that $\left(u_{1}, u_{j}\right) \in S_{1}$ so in this case we may assume $j=2$. This means we have $\left(u_{0}, u_{2}\right) \rightsquigarrow\left(u_{2}, u\right)$. If $\left(u_{1}, u_{2}\right)$ is not a $z$-pair with respect to $V_{c}$ then $\left(u_{2}, u_{1}\right) \rightsquigarrow\left(u_{1}, u_{2}\right)$ according to the choice of $S_{1}$ (we have $\left.\left(u_{1}, u_{0}\right) \rightsquigarrow\left(u_{0}, u_{1}\right)\right)$ and we get a contradiction by Lemma 8.6. So we may assume that $\left(u_{1}, u_{2}\right)$ is a $z$-pair with respect to $V_{c}$. Now again by similar argument as in the proof of Lemma 8.6 (considering the height of the path from $\left(u_{2}, u_{1}\right)$ to $\left(u_{1}, u_{2}\right)$ and the height of the path from $\left(u_{1}, u_{0}\right)$ to $\left(u_{0}, u_{1}\right)$ and the height of $\left.Z_{u_{1}, u_{2}}\right)$ we get a contradiction.

Therefore $j<r$. Note that $\left(u_{j}, u_{r}\right) \in S_{1}$. Now we have the circuit $\left(u_{0}, u_{1}\right),\left(u_{1}, u_{2}\right), \ldots,\left(u_{j-1}, u_{j}\right)$, $\left(u_{j}, u_{r}\right),\left(u_{r}, u_{0}\right)$ in $S_{1} \cup S_{2}$. Since $\left(u_{1}, u_{j}\right) \in S$, we may assume that $\left(u_{0}, u_{1}\right),\left(u_{1}, u_{j}\right),\left(u_{j}, u_{r}\right),\left(u_{r}, u_{0}\right)$ is a circuit in $S_{1} \cup S_{2}$. For simplicity we may assume $j=2$. We note that $W_{1}$ which is a path in $H^{+}$from $\left(u_{1}, u_{0}\right)$ to $\left(u_{0}, u_{1}\right)$ is constricted from below and has net value zero. Also $W_{r}$ which a the path in $H^{+}$from $\left(u_{0}, u_{2}\right)$ to $\left(u_{r}, u_{0}\right)$ is constricted from below and has net value zero.

First assume that the height of $W_{1}$ is smaller than the height of $W_{r}$. Now consider the set $T=\left\{\left(u_{1}, u_{r}\right),\left(u_{r}, u_{0}\right),\left(u_{0}, u_{2}\right),\left(u_{1}, u_{0}\right)\right\}$ and observe that there is a circuit $\left(u_{1}, u_{r}\right),\left(u_{r}, u_{0}\right),\left(u_{0}, u_{1}\right)$ in $\widehat{T}$. Similar to the proof of Lemma 8.6 (by applying Lemma 6.3 ) we conclude that $\left(u_{0}, u_{1}\right),\left(u_{0}, u_{1}\right)$ are in the same strong component of $H^{+}$, a contradiction. Therefore we may assume that the height of $W_{1}$ is greater than the height of $W_{r}$. Now consider the circuit $\left(u_{0}, u_{1}\right),\left(u_{1}, u_{2}\right),\left(u_{2}, u_{r}\right),\left(u_{r}, u_{0}\right)$ is $\widehat{T}$ where $T=\left\{\left(u_{1}, u_{2}\right),\left(u_{0}, u_{2}\right),\left(u_{1}, u_{0}\right)\right\}$. Therefore again by similar argument as in the proof of Lemma 8.6, we conclude that the two walks of $X, Y$ in $H$ where $W_{r}=(X, Y)$ avoid each other. By Lemma 6.3 on suitable portion of $X, Y$ and the walks in $H$ that give rise to $W_{1}$ ( and the fact that $X, Y$ avoid each other), we conclude that $\left(u_{2}, u_{r}\right) \rightsquigarrow\left(u_{2}, u_{0}\right)$, implying that $\left(u_{2}, u_{0}\right) \in V_{c}$, a contradiction.

The above Claim allows us to view remaining balanced pairs on level $\ell$ step by step and at each step we show that there is a source for a subsets of unbalanced pairs on level $\ell$.

Claim 10.3 After executing the loop in lines 17-19, the set $V_{c}$ will not contain a circuit.
Proof: Suppose by selecting a pair $(x, y)$ on level $\ell$ at some stage of the algorithm (from line 17 to 19) we close a circuit $\left(a_{0}, a_{1}\right),\left(a_{1}, a_{2}\right), \ldots,\left(a_{n}, a_{0}\right)$. We again suppose that up to this point no circuits were created, and that $n$ is as small as possible.

If $\left(a_{i}, a_{i+1}\right)$ is not a $z$-pair then let $\left(p_{i}, q_{i+1}\right)$ be a selected vertex on level $\ell$ where $\left(p_{i}, q_{i}\right) \rightsquigarrow$ $\left(a_{i}, a_{i+1}\right)$ and let $W_{i}$ be a directed walk from $\left(p_{i}, q_{i+1}\right)$ to ( $a_{i}, a_{i+1}$ ) and we may assume that $W_{i}$ reaches to the maximum height.

If $\left(a_{j}, a_{j+1}\right)$ is a $z$-pair then let $\left(p_{j}, q_{j+1}\right)$ be a vertex on the level $\ell$ such that the portion of $Z_{a_{j}, a_{j+1}}$ from $\left(p_{j}^{\prime}, q_{j+1}^{\prime}\right)$ to $\left(a_{j}, a_{j+1}\right)$ is constricted from below and has net value zero. Let $L_{i}$ be the height of $W_{i}$ and $L_{j}$ be the height of $Z_{j_{j}, j_{j+1}}$ minus one.

Let $\left(p_{r}, q_{r}\right)$ be a pair where $p_{r}$ is the source. This means for every other $\left(p_{j}, q_{j}\right)$ the algorithm selects $C_{r}$ before $C_{j}$ which contains ( $p_{j}, q_{j}$ ). Now as we argued in the unbalanced case and according to Corollary 8.4 there exists another circuit $\left(a_{0}^{*}, a_{1}^{*}\right),\left(a_{1}^{*}, a_{2}^{*}\right), \ldots,\left(a_{n-1}^{*}, a_{n}^{*}\right),\left(a_{n}^{*}, a_{0}^{*}\right)$ where $\left(a_{i}^{*}, a_{i+1}^{*}\right)$ is already chosen or is selected (belongs to $V_{c}$ ). Note that $\left\{a_{0}^{*}, a_{1}^{*}, \ldots, a_{n}^{*}\right\}$ is a subset of $T=$ $\left\{p_{0}, q_{0}, p_{1}, q_{1}, \ldots, p_{n}, q_{n}\right\}$. However, according to the choice of $p_{0}, p_{1}, \ldots, p_{n}, q_{0}, q_{1}, \ldots, q_{n}$ there must be a source in $T$ but this is a contradiction to having a source.

## $11 \mathbf{k}$-arc digraphs and $\mathbf{k}$-min ordering

A $k$-min ordering of a digraph $H$ is a partition of $V(H)$ into $k$ subsets $V_{0}, V_{1}, \ldots, V_{k-1}$, and a linear ordering $<$ of each of these subsets $V_{i}$, such that each arc of $H$ belongs to some $V_{i} \times V_{i+1}$, $0 \leq i \leq k-1$, and $u<w, z<v$ and $u v, w z \in A(H)$ imply that $u z \in A(H)$ for any $u, w \in V_{i}$, $v, z \in V_{i+1}$, with all subscript addition modulo $k$. Theorem 4.1 can be extended to $k$-min orderings as follows. A $k$-arc representation of a digraph $H$ on a circle $C$ with $2 k$ special points (poles $N_{0}, N_{1}, \ldots, N_{k-1}, S_{0}, S_{1}, \ldots, S_{k-1}$ (in this clockwise order) consists of intervals $I_{v}, J_{v}, v \in V(H)$ consistent as before, now each $I_{v}$ containing $S_{i+1}, S_{i+2}, \ldots, S_{k-1}, N_{0}, N_{1}, \ldots, N_{i}$, for some $0 \leq i \leq$ $k-1$, and no other poles, and each $J_{v}$ containing $N_{i+1}, N_{i+2}, \ldots, N_{k-1}, S_{0}, S_{1}, \ldots, S_{i}$, for some $0 \leq i \leq k-1$, and no other poles, such that $u v \in A(H)$ if and only if $I_{u}$ and $J_{v}$ are disjoint.

Theorem 11.1 $A$ digraph $H=(V, A)$ is a $k$-arc digraph if and only if it admits a $k$-min ordering. $\diamond$

In some cases when min orderings do not exist, there may still exist extended min orderings, which is sufficient for the polynomial solvability of $\operatorname{LHOM}(H)[25]$. We denote by $\vec{C}_{k}$ the directed cycle on vertices $0,1, \ldots, k-1$. We shall assume in this section that $H$ is weakly connected. This assumption allows us to conclude that any two homomorphisms $\ell, \ell^{\prime}$ of $H$ to $\vec{C}_{k}$ define the same partition of $V(H)$ into the sets $V_{i}=\ell^{-1}(i)$, and we will refer to these sets without explicitly defining a homomorphism $\ell$. Thus suppose $H$ is homomorphic to $\vec{C}_{k}$, and let $V_{i}$ be the partition of $V(H)$ corresponding to all such homomorphisms.

Note that any $H$ is homomorphic to the one-vertex digraph with a loop $\vec{C}_{k}$, and a 1-min ordering of $H$ is just the usual min ordering. Also note that a min ordering of a digraph $H$ becomes a $k$-min ordering of $H$ for any $\vec{C}_{k}$ that $H$ is homomorphic to. However, there are digraphs homomorphic to $\vec{C}_{k}$ which have a $k$-min ordering but do not have a min ordering - for instance $\vec{C}_{k}$ (with $k>1$ ).

We observe for future reference that an unbalanced digraph $H$ has only a limited range of possible values of $k$ for which it could be homomorphic to $\vec{C}_{k}$, and hence a limited range of possible values of $k$ for which it could have a $k$-min orderings. It is easy to see that a cycle $C$ admits a homomorphism to $\vec{C}_{k}$ only if the net length of $C$ is divisible by $k$ [24]. Thus any cycle of net length $q>0$ in $H$ limits the possible values of $k$ to the divisors of $q$. If $H$ is balanced, it is easy to see that $H$ has a $k$-min ordering for some $k$ if and only if it has a min ordering.

For a digraph $H$ homomorphic to $\vec{C}_{k}$ we shall consider the following version of the pair digraph. The digraph $H^{(k)}$ is the subgraph of $H^{+}$induced by all ordered pairs $(x, y)$ belonging to the same set $V_{i}$. We say that $(u, v)$ is a symmetrically $k$-invertible pair in $H$ if $H^{(k)}$ contains a directed walk joining $(u, v)$ and $(v, u)$. Thus a symmetrically $k$-invertible pair is a symmetrically invertible pair in $H$ in which $u$ and $v$ belong to the same set $V_{i}$. Note that $H$ may contain symmetrically invertible pairs, but no symmetrically $k$-invertible pair. Consider, for instance the directed hexagon $\vec{C}_{6}$. The pair 0,3 is symmetrically invertible and symmetrically 3 -invertible, but not symmetrically 6 -invertible.

The extended version of our main theorem follows.
Theorem 11.2 The following statements are equivalent for a weakly connected digraph $H$.

1. $H$ admits a $k$-min ordering
2. there exists a positive integer $k$ such that $H$ is homomorphic to $\vec{C}_{k}$ and no component of $H^{(k)}$ contains a circuit

Proof: We shall in fact prove that the following statements are equivalent for a positive integer $k$ such that $H$ is homomorphic to $\vec{C}_{k}$ :

1. $H$ admits a $k$-min ordering
2. no component of $H^{(k)}$ contains a circuit

Suppose $H$ admits linear orderings < of sets $V_{i}$ satisfying the Min property between consecutive sets $V_{i}, V_{i+1}$. Any circuit $\left(x_{0}, x_{1}\right),\left(x_{1}, x_{2}\right), \ldots,\left(x_{n}, x_{0}\right)$ in $H^{(k)}$ must have all vertices $x_{0}, x_{1}, \ldots, x_{n}$ in the same set $V_{i}$, and hence if all the pairs $\left(x_{i}, x_{i+1}\right)$ were in the same component of $H^{(k)}$ we would obtain the same contradiction with transitivity of $<$ as above the statement of Theorem 4.2. This proves that 1 implies 2 .

Now we prove that 2 implies 1. Thus assume that $H$ is homomorphic to $\vec{C}_{k}$ and no strong component of $H^{(k)}$ contains a circuit. We shall construct a $k$-min ordering of $H$. We have again the components of $H^{(k)}$ in dual pairs $C, C^{\prime}$, where $C^{\prime}$ consists of the reverses of the pairs in $C$, and we can proceed with a similar algorithm as before. At each stage of the algorithm, some component of $H^{(k)}$ is chosen and its dual component discarded. We again choose a component $X$ according to the rules in Algorithm 1.

The proof of correctness is analogous to the proof of Propositions 9.4, 10.1.
We again note that the theorem implies a polynomial time algorithm to test whether an input digraph $H$ has a $k$-min min ordering. As noted above, it suffices to check for each component of $H$ separately, so we may assume that $H$ is weakly connected. If $H$ is balanced, we have already observed this is only possible if $H$ has a min ordering, which we can check in polynomial time. Otherwise we find any unbalanced cycle in $H$, say, of net length $q$, and then test for circuits in components $H^{(k)}$ for all $k$ that divide $q$.

## 12 Conclusions

We have provided polynomial time algorithms, obstruction characterizations, and geometric representations, for digraphs admitting a min ordering, i.e., a CSL polymorphism. We believe they are a
useful generalization of interval graphs, encompassing adjusted interval digraphs, monotone proper interval digraphs, complements of circular arcs of clique covering number two, two-dimensional ray graphs, and other well known classes.

We have also similarly characterized digraphs admitting a CC polymorphism.
We now point out that the class of digraphs admitting a set polymorphism, i.e., CTS polymorphisms of all orders, coincides with the the class of digraphs with a min ordering, and so is equal to the class of bi-arc digraphs.

Theorem 12.1 If there exists a circuit in a strong component of $H^{+}$then either $H^{+}$contains an invertible pair or there exists a closed walk $W$ in $H$ composed of walks $W\left[v_{0}, v_{1}\right], W\left[v_{1}, v_{2}\right], \ldots, W\left[v_{r}, v_{0}\right]$ with the following properties: each $W\left[v_{i}, v_{i+1}\right]$ is constricted from below, has a positive net length $r$ and $W\left[v_{i}, v_{i+1}\right]$ and $W\left[v_{j}, v_{j+1}\right]$ avoid each other for every $0 \leq i<j \leq r\left(v_{r+1}=v_{0}\right)$

Proof: Suppose $C:\left(a_{0}, a_{1}\right),\left(a_{1}, a_{2}\right), \ldots,\left(a_{n}, a_{0}\right)$ is a circuit in a strong component $S$ of $H^{+}$.
Claim 12.2 We may assume that $C$ is minimal and $n>1$. Moreover $S$ is an unbalanced component.

Proof: For contradiction suppose $C$ is not minimal. Thus there is a minimal circuit which is implied by $S$. Now by the properties shown in Section 7 for a minimal circuit when $S$ is unbalance then there is also a minimal circuit which all belong to the same strong component $S$.

If $S$ is balanced then one can show there is a pair $(x, y)$ on the lowest level of $S$ where for each pair in the circuit there exists a path constricted from below from $(x, y)$ to that pair. This means we can apply the Lemma 7.1 several times as necessary and conclude that there is a circuit on the lowest level of $S$. Now we may assume that for every pair of the circuit there exists a path (with net value zero and constricted from below) from $(x, y)$ to that pair. But this is a contradiction according to 8.5 as it would imply that there exists an invertible pair in a strong component of $H^{+}$.

Suppose $S$ is unbalanced and observe that by Theorem 7.2 we may assume that $\left(b_{i}, b_{i+1}\right)=$ $\left(a_{i}, a_{i+1}\right), 0 \leq i \leq n$, and $\left(a_{i}, a_{i+1}\right)$ is extremal and all lie on one cycle $(X, Y)=D_{i}$ in $S$. This can be done because all the pairs lie on the same strong component $S$. We also assume that the net value of such $D_{i}$ is minimum. We may assume that the net value of a directed path $W$ from $\left(a_{i}, a_{i+1}\right)$ to $\left(a_{i+1}, a_{i+2}\right)$ is not zero. Otherwise $W$ is constricted from below since $\left(a_{i}, a_{i+1}\right)$ is an extremal pair and hence $W=\left(A_{i}, B_{i}\right)$ where $A_{i}$ and $B_{i}$ avoid each other (part of $P_{i}, P_{i+1}$ ). Now it is easy to see that there is a path from $\left(a_{i}, a_{i+1}\right)$ to $\left(a_{i}, a_{i+2}\right)$ (a walk from $a_{i}$ to a vertex with the maximum height on $A_{i}$ and then back to $a_{i}$ and a walk on $B_{i}$ from $a_{i+1}$ to $a_{i+2}$ would give to a path in $H^{+}$from $\left(a_{i}, a_{i+1}\right)$ to $\left.\left(a_{i}, a_{i+2}\right)\right)$ and hence we get a shorter circuit.

Consider the walks $P_{i}$ from Theorem 7.2 (2). Each walk $P_{i}$ starts at $a_{i}$ and it is constricted and has unbounded positive net length. Every $P_{i}$ and $P_{j}$ avoid each other. Moreover $P_{i}$ is obtained by walking around the closed walk $X$. Thus without loss of generality let the net value of portion of $D_{i}$ from $\left(a_{0}, a_{1}\right)$ to ( $a_{1}, a_{2}$ ) has the smallest positive net value $\ell$ and let the net value of $D_{i}$ be $m$ where $m$ is minimum. Note that $(n+1) \ell \leq m$.

Note that $P_{0}$ and $P_{1}$ avoid each other. Let $X^{\prime}$ be the closed walk starting at $a_{0}$ corresponding to $P_{0}$. Note that $X^{\prime}$ is also a closed walk starting at $a_{1}$ corresponding to $P_{1}$.

Now consider the following walks : $W_{0}=X^{\prime}\left[a_{0}^{\prime}, a_{1}^{\prime}\right]$, where $a_{0}^{\prime}=a_{0}$ and $a_{1}^{\prime}=a_{1}$ and $W_{j}=$ $X^{\prime}\left[a_{j}^{\prime}, a_{j+1}^{\prime}\right] ; j=1,2, \ldots$ where $a_{j+1}^{\prime}$ is a extremal vertex on $X^{\prime}$ and $W_{j}$ has net length $\ell$. Observe that $W_{i}, W_{i+1}, 1 \leq i$ avoid each other since $P_{0}, P_{1}$ avoid each other. Now at some point we must have $a_{r}^{\prime}=a_{k}^{\prime}$. Without loss of generality we may assume that $a_{0}^{\prime}=a_{k}^{\prime}$. Note that $r \leq m$ because $X^{\prime}$ has at most $m$ extremal pairs. Note that the net value of $D_{i}$ is $(r+1) \ell$.

Now $\left(a_{0}^{\prime}, a_{1}^{\prime}\right),\left(a_{1}^{\prime}, a_{2}^{\prime}\right), \ldots,\left(a_{r}^{\prime}, a_{0}^{\prime}\right)$ is a circuit in $S$. We show that $W_{i}$ and $W_{j}$ avoid each other for every $0 \leq i<j \leq r$. Note that if there is a faithful arc from the $q$-th vertex of $W_{i}^{-1}$ to the $(q+1)$-th vertex of $W_{i+2}^{-1}$ then we would have $\left(a_{i}^{\prime}, a_{i+1}^{\prime}\right) \rightsquigarrow\left(a_{i+2}^{\prime}, a_{i+1}^{\prime}\right)$ (using the faithful arc at index $q$ ) and because $W_{i}, W_{i+1}$ avoid each other and $W_{i+1}, W_{i+2}$ avoid each other we conclude that $\left(a_{i}^{\prime}, a_{i+1}^{\prime}\right)$ and $\left(a_{i+2}^{\prime}, a_{i+1}^{\prime}\right)$ are both in $S$. On the other hand both $\left(a_{i}^{\prime}, a_{i+1}^{\prime}\right),\left(a_{i+1}^{\prime}, a_{i+2}^{\prime}\right)$ are in $S$ and hence $\left(a_{i+1}^{\prime}, a_{i+2}^{\prime}\right),\left(a_{i+2}^{\prime}, a_{i+1}^{\prime}\right)$ are in $S$, a contradiction. Therefore for every $i$ we have $W_{i}, W_{i+2}$ avoid each other. Now if $r=2$ then we are done so we may assume that $r \geq 3$.

Let $d$ be the smallest integer such that $W_{j}$ and $W_{j+d}$ do not avoid each other $(d \geq 2)$. Note without loss of generality we may assume that $d \leq r / 2$. Without loss of generality assume that $W_{0}, W_{d}$ do not avoid each other and $d \leq r / 2$. By considering the last faithful arc on $W_{0}^{-1}$ to $W_{d}^{-1}$ we conclude that $X_{1}:\left(a_{d-1}^{\prime}, a_{d}^{\prime}\right) \rightsquigarrow\left(a_{d}^{\prime}, a_{1}^{\prime}\right)$ has net value $\ell$ and since $W_{d-1}, W_{d}$ avoid each other and $W_{0}, W_{d-1}$ avoid each other, $\left(a_{d-1}^{\prime}, a_{d}^{\prime}\right),\left(a_{d}^{\prime}, a_{1}^{\prime}\right)$ are in $S$. We also note that using the faithful arc from $W_{0}^{-1}$ to $W_{d}^{-1}$ we have $\left(a_{0}^{\prime}, a_{1}^{\prime}\right) \rightsquigarrow\left(a_{d}^{\prime}, a_{1}^{\prime}\right)$ which is a directed walk of net value zero and hence $X_{2}:\left(a_{d}^{\prime}, a_{1}^{\prime}\right) \rightsquigarrow\left(a_{0}^{\prime}, a_{1}^{\prime}\right) \rightsquigarrow\left(a_{1}^{\prime}, a_{2}^{\prime}\right)$ has net value $\ell$. Now we have a circuit $\left(a_{1}^{\prime}, a_{2}^{\prime}\right),\left(a_{2}^{\prime}, a_{2}^{\prime}\right), \ldots,\left(a_{d-1}^{\prime}, a_{d}^{\prime}\right),\left(a_{d}^{\prime}, a_{1}^{\prime}\right)$, and a closed walk that consists of $a_{1}^{\prime}, a_{2}^{\prime}, \ldots, a_{d}^{\prime}, a_{1}^{\prime}$ and has net length $(d-1) \ell+\ell+\ell=(d+1) \ell<(r+1) \ell$. This is a contradiction to the assumption about the net value of $D_{i}$.

Theorem 12.3 $A$ digraph $H$ admits a conservative CSL polymorphism if and only if it admits a conservative set polymorphism.

Proof: Since a min ordering allows to define a conservative set polymorphism as the minimum, it suffices to show that a digraph that does not have a min ordering also cannot have a conservative set polymorphism. We show this by showing that a circuit in one component of $H^{+}$means that $H$ does not have a conservative set polymorphism.

So suppose $\left(a_{0}, a_{1}\right),\left(a_{1}, a_{2}\right), \ldots,\left(a_{n}, a_{0}\right)$ is a circuit in a strong component $C$ of $H^{+}$. By Theorem 12.1 either there exists an invertible pair and hence there is no CLS polymorphism or there exists a closed walk $W$ composed of walks $W\left[v_{0}, v_{1}\right], W\left[v_{1}, v_{2}\right], \ldots, W\left[v_{r}, v_{0}\right]$ with the following properties: Each $W\left[v_{i}, v_{i+1}\right]$ is constricted from below, and has a positive net value $r$ and $W\left[v_{i}, v_{i+1}\right]$ and $W\left[v_{j}, v_{j+1}\right]$ avoid each other for every $0 \leq i<j \leq r\left(v_{r+1}=v_{0}\right)$.

Now any conservative set function must assign $f\left(v_{0}, v_{1}, \ldots, v_{r}\right)=f\left(v_{1}, v_{2}, \ldots, v_{r}, v_{0}\right)$ but since the walks $W\left[v_{i}, v_{i+1}\right], W\left[v_{j}, v_{j+1}\right], 0 \leq i \neq r$ avoid each other, this is not possible.

We remark that in the proof we have only used the fact that $H$ does not have a conservative cyclic polymorphism. Thus we have actually proved that the class of bi-arc digraphs coincides with each of the following classes of digraphs:

1. digraphs with conservative semi-lattice polymorphisms
2. digraphs with a conservative set polymorphism
3. digraphs with conservative cyclic polymorphisms of all arities.

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