Adjusted Interval Digraphs

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Abstract

Interval digraphs were introduced by West et all. They can be recognized in polynomial time and admit a characterization in terms of incidence matrices. Nevertheless, we do not have a forbidden structure characterization nor a low-degree polynomial time algorithm.

We introduce a new class of 'adjusted interval digraphs', obtained by a slight change in the definition. By contrast, these digraphs have a natural forbidden structure characterization, parallel to a characterization for undirected graphs, and admit an easy recognition algorithm.

We relate adjusted interval digraphs to a list homomorphism problem. Each digraph H defines a corresponding list homomorphism problem L-HOM(H). We observe that if H is an adjusted interval digraph, then the problem L-HOM(H) is polynomial time solvable, and conjecture that for all other reflexive digraphs H the problem L-HOM(H) is NP-complete. We present some preliminary evidence for the conjecture.

1 Introduction

An *interval graph* is a graph H which admits an *interval representation*, i.e., a family of intervals $I_v, v \in V(H)$, such that $uv \in E(H)$ if and only if I_u and I_v intersect. An *interval digraph* is a digraph H which admits an *interval pair representation*, which is a family of pairs of intervals $I_v, J_v, v \in V(H)$, such that $uv \in E(H)$ if and only if I_u intersects J_v. Note that an interval graph must be reflexive (each vertex has a loop), but an interval digraph may lack loops. If the intervals $I_v, J_v, v \in V(H)$, can be chosen so that for each v the intervals I_v and J_v have the same left endpoint, we say that H is an *adjusted interval digraph*. It is again clear that an adjusted interval digraph must be reflexive.

In [3] we have studied the special case of adjusted interval digraphs H representable by intervals $I_v, J_v, v \in V(H)$, in which each interval J_v is just one point. These are called *chronological interval digraphs* [3], and we have shown that they can be characterized by the absence of certain special forbidden structures. In [22], a related class of *interval catch digraphs* has been characterized by the absence of certain other forbidden structures.

Here we provide a forbidden structure characterization of adjusted interval digraphs, very similar to a recent forbidden structure characterization of interval graphs, and other structures [16]. The characterization allows a direct polynomial time recognition algorithm of adjusted interval digraphs.

We apply adjusted interval digraphs to the complexity of list homomorphisms. A *homomorphism* f of a digraph G to a digraph H is a mapping $f : V(G) \to V(H)$ in which $f(u)f(v) \in E(H)$ whenever $uv \in E(G)$ [18]. If $L(v), v \in V(G)$, are *lists* (subsets of $V(H)$), then a *list homomorphism* of G to H (with respect to the lists L) is a homomorphism satisfying $f(v) \in L(v)$ for all $v \in V(G)$. The *list homomorphism problem* $L - HOM(H)$ asks whether or not an input digraph G equipped with lists L admits a list homomorphism $f : G \to H$ with respect to L. The complexity of the list homomorphism problem L-HOM(H) for undirected graphs H has been classified in [5, 6, 7]. Of particular interest for this paper is the classification in the special case of reflexive graphs: if H is a reflexive graph, then the problem L-HOM(H) is polynomial time solvable if H is an interval graph, and is NP-complete otherwise [5]. The complexity of $L - HOM(H)$ for any digraph (and more general relational system) has been classified in [1] (see Theorem 4.1). For reflexive digraphs H, we propose a simpler classification. Specifically, we observe that each adjusted interval digraph H has polynomial time solvable list homomorphism problem L-HOM(H), and conjecture that for any other reflexive digraph H the problem L-HOM(H) is NP-complete. We offer some evidence for the conjecture here (and more in a full version of this paper).

2 Invertible Pairs

The *underlying graph* of H has an edge uv whenever $uv \in E(H)$ or $vu \in E(H)$. If u, v are adjacent in the underlying graph of H, the pair uv is a *forward edge* if $uv \in E(H)$, and a *backward edge* if $vu \in E(H)$. Note that a loop is both a forward edge and a backward edge. If $uv \in E(H)$, we say that u *dominates* v in H.

We define two walks $P = x_0, x_1, \ldots, x_n$ and $Q = y_0, y_1, \ldots, y_n$ in H to be *congruent*, if they follow the same pattern of forward and backward edges, i.e., if $x_i x_{i+1}$ is a forward (backward) edge if and only if $y_i y_{i+1}$ is a forward (backward) edge, respectively. If P and Q as above are congruent walks, we say that P *avoids* Q, if there is no edge $x_i y_{i+1}$ in the same direction (forward or backward) as $x_i x_{i+1}$.

An *invertible pair* in H is a pair of vertices u, v such that

- there exist congruent walks P from u to v and Q from v to u, and such that P avoids Q,
- there exist congruent walks P' from v to u and Q' from u to v, such that P' avoids Q'.

It will turn out to be useful to reformulate these definitions in terms of an auxiliary digraph. The *pair-digraph* H^+ associated with H has vertices $V(H^+) = \{(u,v) : u \neq v\}$, and edges $(u,v)(u',v')$, where

$$uu', vv' \in E(H) \text{ and } uv' \notin E(H), \text{ or}$$

$$u'u, v'v \in E(H) \text{ and } v'u \notin E(H).$$

Lemma 2.1 *If H has an invertible pair (u,v), then (u,v) and (v,u) belong to the same strong component C of the pair-digraph H^+; moreover, for any (x,y) in C the reversed pair (y,x) also belongs to C, i.e., each pair in C is invertible.*

If H has no invertible pair, then for each strong component C of H^+ there exists a reversed strong component C' *such that $(x,y) \in C$ if and only if $(y,x) \in C'$.*

Proof. These properties follow from the definition of a strong component and the observation that $(u,v)(u'v') \in E(H^+)$ implies $(v',u')(v,u) \in E(H^+)$. For instance, if $(u,v),(v,u),(x,y) \in C$, then the directed closed walk containing $(u,v),(x,y)$ yields by reversal a directed closed walk containing $(v,u),(y,x)$, and by concatenation with the directed closed walk containing $(u,v),(v,u)$, we obtain a directed closed walk containing $(x,y),(y,x)$. $\square$

An ordering $<$ of the vertices of H is a *min ordering* of H if it satisfies the following property: if $uv \in E(H)$ and $u'v' \in E(H)$, then $\min(u,u')\min(v,v') \in E(H)$. (A min ordering was also called an *X-underbar enumeration* [13, 18]). The following result relates min orderings to adjusted interval digraphs.

Theorem 2.2 *A reflexive digraph is an adjusted interval digraph if and only if it admits a min ordering.*

Proof. Given a min ordering, we can arrange the common starting points of I_v, J_v in the same order as the vertices v appear in the min ordering, and define intervals I_v and J_v as follows. If v has no forward edges towards later vertices, we end the interval I_v at the last vertex w such that vw is a double edge, and end the interval J_v at the last vertex w such that vw is a backward edge. If v has no backward edges towards later vertices, we end the interval J_v at the last vertex w such that vw is a double edge, and end the interval I_v at the last vertex w such that vw is a forward edge. Conversely, given an adjusted interval pair representation $I_v, J_v, v \in V(H)$ we obtain a min ordering of H according to the left to right order of the common left endpoints of the intervals. $\square$

Min orderings also play an important role for list homomorphism problems, cf. [18].

Theorem 2.3 *[13] If H admits a min ordering, then the problem $L - HOM(H)$ is polynomial time solvable.*

Finally, we observe that an invertible pair is an obstruction to the existence of a min ordering.

Lemma 2.4 *If H has an invertible pair, then H does not admit a min ordering.*

Proof. Suppose $(u, v)(u', v')$ is an edge of the pair-digraph H^+. Suppose $<$ is a min ordering of H, and suppose $u < v$. The we must also have $u' < v'$. Following the directed closed walk in H^+ which contains (u, v) and (v, u), we obtain a contradiction. □

3 Adjusted Interval Digraphs

We now strengthen Lemma 2.4.

Theorem 3.1 *A reflexive digraph H admits a min ordering if and only if it has no invertible pair.*

In fact, we shall prove the following stronger result.

Theorem 3.2 *The following statements are equivalent for a reflexive digraph H:*

1. *H is an adjusted interval digraph*
2. *H has a min ordering*
3. *H has no invertible pairs*
4. *The vertices of H^+ can be partitioned into sets D, D' such that*
    - *$(x, y) \in D$ if and only if $(y, x) \in D'$*
    - *$(x, y) \in D$ and (x, y) dominates (x', y') in H^+ implies $(x', y') \in D$*
    - *$(x, y), (y, z) \in D$ implies $(x, z) \in D$.*

Proof. The equivalence of 1 and 2 is proved in Theorem 2.2. Furthermore, Lemma 2.4 shows that 2 implies 3. It is also quite straightforward to see that 4 implies 2; it suffices to define $a < b$ if $(x, y) \in D$. Thus it remains to show that 3 implies 4.

Therefore, we assume that H has no invertible pair. Note that we may assume that H is weakly connected, otherwise we can order each weak component separately. We also note that for each strong component C of H^+, there is a corresponding reversed strong component C' whose pairs are precisely the reversed pairs of the pairs in C; we shall say that C, C' are *coupled* strong components.

The partition of $V(H^+)$ into D, D' will correspond to separating each pair of coupled strong components C, C' of H^+. The vertices of one strong components will be placed in the set D, their reversed pairs will go to D'. We wish to make these choices in such a way as to avoid creating a *circular chain* in D, i.e., a sequence of pairs $(x_0, x_1), (x_1, x_2), \ldots,$ $(x_n, x_0) \in D$.

We shall proceed as follows. Initially the sets D and D' are empty. We say that a strong component C of H^+ is *ripe* when it has no edge *to* another strong component in $H^+ - D$. In the general step, we shall take a ripe component C and place it in D, and simultaneously place C' in D'. (Note that C' need not be ripe, but has no edge *from* another strong component.) We will show that there is always at least one ripe strong component which can be added to D without creating a circular chain.

The sets D, D' will always have the following properties (which are true initially). There is no circular chain in D; each strong component of H^+ belongs entirely to D, D', or to $V(H^+) - D - D'$; the pairs in D' are precisely the reversed pairs of the pairs in D; there is no edge of H^+ from D to a vertex outside of D; and there is no edge of H^+ from a vertex outside of D' to a vertex in D'. At the end of the algorithm each pair (x, y) with $x \neq y$ will belong either to D or to D', and hence the final D will have no circular chain and hence satisfy the transitivity property of 4.

We now prove that the algorithm maintains these properties.

Suppose, for a contradiction, that the current D has no circular chain but the addition of C to D creates a circular chain in $C \cup D$. Suppose $(x_0, x_1), (x_1, x_2), \ldots, (x_n, x_0)$ is a circular chain that has occurred for the first time during the execution of the algorithm, and also suppose that at that time no shorter circular chain has occurred. Since there are no invertible pairs, and since we never place both an edge and its reverse in D, we must have $n \geq 2$. We may assume without loss of generality that $(x_n, x_0) \in C$; note that other pairs of the circular chain could also be in C.

Case 1. Assume that in H, there is at least one edge between the vertices $x_0, x_1, \ldots, x_n$, say an edge $x_a x_b$.

We claim that this implies that H is complete on $x_0, x_1, \ldots, x_n$. We make the following elementary observations, assuming $j \neq i$.

1. If x_j dominates x_i then x_{j-1} dominates x_i in H.
2. If x_j dominates x_i then x_j dominates x_{i-1} in H.

To prove the first observation, we note that if x_j dominates x_i but x_{j-1} not dominate x_i in H, then (x_{j-1}, x_j) dominates (x_{j-1}, x_i) in H^+. Since (x_{j-1}, x_j) is in $C \cup D$, the pair (x_{j-1}, x_i) must belong to $C \cup D$, implying a shorter circular chain in $C \cup D$.

To prove the second observation, we similarly note that if x_j dominates x_i but x_j does not dominate x_{i-1} in H, then (x_{i-1}, x_i) dominates (x_{i-1}, x_j) in H^+, also implying a shorter circular chain.

Consider now the fact that x_a dominates x_b in H. Property 1 implies that x_{a-1}, $x_{a-2}, \ldots, x_{b+1}$ all dominate x_b. Since x_{b+1} dominates x_b, property 2 implies that x_{b+1} dominates x_{b-1}, $x_{b-2}, \ldots, x_{b+2}$, i.e., dominates all other vertices. At this point we use 1 again to derive that x_b dominates x_{b-1}, and repeated application of 2 as before implies that x_b dominates all other vertices. Continuing this way, we see that each x_j dominates all other vertices, i.e., the vertices x_0, $x_1, \ldots, x_n$ induce a complete graph in H.

We conclude the proof of Case 1 by showing that C is a trivial component (with a single vertex). If C has more than one vertex, then so does its corresponding coupled component C', which contains the vertex (x_0, x_n). Hence we assume for contradiction that (x_0, x_n) dominates some (a, b) not in $C \cup D$.

Up to symmetry, we may assume that x_0 dominates a in H, x_n dominates b in H and x_0 does not dominate b in H. Since (a, b) is not in $C \cup D$, the pair (x_0, x_1), which is in C, cannot dominate (a, b), which implies that x_1 does not dominate b in H. If x_2 dominates b in H, then (x_1, x_2) dominates (x_0, b) which dominates (a, b) in H^+; this is impossible, as this is a directed path starting in C and ending outside of $C \cup D$, so some edge would exit from $C \cup D$ against the rules we maintain. Therefore x_2 does not dominate b in H; if x_3 dominates b in H, then (x_2, x_3) dominates (x_1, b) which dominates (x_0, b) which dominates (a, b), yielding the same contradiction. Therefore x_3 does not dominate b in H, and continuing this way we would derive that x_n does not dominate b, which is false.

Thus we have $C = \{(x_n, x_0)\}, C' = \{(x_0, x_n)\}$. The same proof also shows that C' is ripe, as no (a, b) dominated by (x_0, x_n) can exist outside of $C \cup D$. It is now easy to see that if both (x_n, x_0) and (x_0, x_n) complete a circular chain with D, then D already had a circular chain.

Case 2. Assume that vertices $x_0, x_1, \ldots, x_n$ are independent in H.

Lemma 3.3 *Suppose p is a vertex of H, distinct from $x_0, x_1, \ldots, x_n$, which dominates x_{i+1} but not x_i (or which is dominated by x_{i+1} but not by x_i).*

Then $(x_0, x_1), \ldots, (x_i, p), (p, x_{i+2}), \ldots, (x_n, x_0)$ is also a circular chain created at the same time.

Proof. We conclude from the assumption that (x_i, x_{i+1}) dominates (x_i, p) in H^+, and since (x_i, x_{i+1}) is in $C \cup D$, we must also have (x_i, p) in $C \cup D$. Furthermore, since x_{i+1} does not dominate or is dominated by x_{i+2} in H, we also have (x_{i+1}, x_{i+2}) dominating

(p, x_{i+2}), whence (p, x_{i+2}) is in $C \cup D$. In conclusion, we see that any such vertex p can replace x_{i+1} in the circular chain $(x_0, x_1), (x_1, x_2), \ldots, (x_n, x_0)$. $\square$

Lemma 3.4

- *If p is a vertex of H, distinct from $x_0, x_1, \ldots, x_n$, which dominates x_j and x_k with $j \neq k$, then p dominates each x_i.*
- *If p is a vertex of H, distinct from $x_0, x_1, \ldots, x_n$, which is dominated by x_j and x_k with $j \neq k$, then p is dominated by each x_i.*
- *If p, distinct from $x_0, x_1, \ldots, x_n$, dominates x_j and is dominated by x_k with $j \neq k$, then p both dominates and is dominated by each $x_i, i \neq j, k$.*

Proof. If p dominates x_{i+1} but not x_i, then Lemma 3.3 implies that p can replace x_{i+1} in the circular chain; however at least one of x_j, x_k is not equal to x_{i+1}, whence the vertices of the chain are not independent and we conclude by Case 1. The other items are proved similarly. $\square$

We now claim that the circular chain $(x_0, x_1), (x_1, x_2), \ldots, (x_n, x_0)$ has at most one pair, say (x_n, x_0), in C (with all other pairs in D). Otherwise, assume some $(x_i, x_{i+1}), i \neq n$ is also in the strong component C, and let P be a directed path in C from (x_n, x_0) to (x_i, x_{i+1}). Let the penultimate pair on this path be (p, q), and, without loss of generality, assume that $px_i, qx_{i+1} \in E(H), px_{i+1} \notin E(H)$. (In the case $x_i p, x_{i+1} q \in E(H), x_{i+1} p \notin E(H)$, the argument is symmetric.) By Lemma 3.3, p does not dominate any x_j with $j \neq i$. Next we claim that q does not dominate x_i. Indeed, if q dominates x_i, then Lemma 3.4 implies that q dominates all x_j. This is a contradiction, since it would mean that (p, q) dominates (x_i, x_{i+2}) in H^+, implying that (x_i, x_{i+2}) is in $C \cup D$ and thus there is a shorter circular chain in H. Therefore q does not dominate x_i. By a double application of Lemma 3.3, we conclude that we can replace x_i and x_{i+1} by p and q in the circular chain in H. Continuing this way, we replace (p, q) by the previous pair on the path P, until we obtain the pair (p', q') which is the first pair after (x_n, x_0). Since x_0 is adjacent to q', we are back in Case 1.

Thus the circular chain $(x_0, x_1), (x_1, x_2), \ldots, (x_n, x_0)$ has only the pair (x_n, x_0) in C, and any circular chain in $C \cup D$ has exactly one pair in C. We now suppose, in addition to the previous assumptions, that our circular chain minimizes the sum of the lengths of all distances amongst the vertices $x_0, x_1, \ldots, x_n$, in the underlying graph of H .

The digraph H turns out to have a very special structure. We claim that in this situation there exists a non-empty set K of vertices of H such that $H \setminus K$ has weak components $C_1, C_2, \ldots C_m$, where $x_i \in C_i, i = 1, 2, \ldots, n$, and such that if $p \in K$ dominates (respectively is dominated by) a vertex in C_i, then p dominates (respectively is dominated by) all vertices in C_i; moreover, if x'_0, x'_1, $\ldots, x'_n$ are any vertices with $x'_i \in C_i$, then (x'_0, x'_1), $(x'_1, x'_2), \ldots, (x'_n, x'_0)$ is also a circular chain.

Indeed, we let K consist of all vertices of H that dominate each x_i, or are dominated by each x_i. It is easy to see that K must be non-empty, as Lemma 3.4 implies that any p dominated by $x_j, x_k, j \neq k$ belongs to K. Such a p must exist by our new minimality assumption, as otherwise we could replace x_j by its neighbour p on a path joining x_j to x_k by Lemma 3.3.

The same argument shows that two different x_j, x_k cannot lie in the same weak component C_i of $H \setminus K$, as any path joining x_j to x_k was shown to contain a vertex of K. Therefore we can number the components so that C_i contains x_i for $i = 1, 2, \ldots n$. (There may be additional components C_i with $i = n+1, \ldots, m$.) Now Lemma 3.3 implies that each x_i can be replaced by any neighbour in C_i, thus any vertex of C_i can be taken as x_i. Thus each $p \in K$ that dominates a vertex in C_i also dominates all vertices in C_i, and similarly for vertices p dominated by a vertex in C_i.

This creates a situation where any pair (y, y') in the strong component C of H^+ containing (x_n, x_0) must satisfy $y \in C_n, y' \in C_0$. This easily implies that the strong component C does not have any arcs entering it from the outside, and hence the strong component C' coupled with C is also ripe. We claim that C' cannot complete a circular chain with D. Otherwise, the pair (x_0, x_n) would also complete a circular chain by the same argument. Thus both (x_0, x_n) and (x_n, x_0) complete a circular chain with D, whence D must already contain a circular chain, a contradiction.

Of course, if the addition of C' does not create a circular chain, then we add C' to D and C to D'. $\square$

This gives us a polynomially verifiable forbidden subgraph characterization of adjusted interval digraphs. As noted above, checking for invertible pairs amounts to computing the strong components of H^+ and checking for the existence of a pair $(u, v), (v, u)$ in one strong component.

Corollary 3.5 *Let H be a reflexive digraph. Then H is an adjusted interval digraph if and only if it has no invertible pair.* $\diamond$

4 Polymorphisms

The min orderings defined above are a particular case of the following general concept. Let k be a positive integer. The *k-th power* of H is the digraph H^k with vertex set $V(H)^k$ in which $(u_1, u_2, \ldots, u_k)(v_1, v_2, \ldots, v_k)$ is an edge just if each $u_i v_i$ is an edge of H. A *polymorphism of order k* is a homomorphism of H^k to H. A polymorphism f is *conservative* if $f(u_1, u_2, \ldots, u_k)$ always is one of $u_1, u_2, \ldots, u_k$. From now on we shall use the word *polymorphism to mean a conservative polymorphism.*

A polymorphism f of order two is *commutative* if $f(u,v) = f(v,u)$ for any u,v. If H admits a min ordering $<$, then clearly defining $f(u,v) = \min(u,v)$ is a polymorphism, which is commutative.

Two ternary polymorphisms also play a role in the problems L-HOM(H) [1]. A polymorphism $f : H^3 \to H$ is called a *majority polymorphism* if $f(u,u,v) = f(u,v,u) = f(v,u,u) = u$ for any u,v. A ternary polymorphism $f : H^3 \to H$ is called a *Maltsev polymorphism* if $f(u,u,v) = f(v,u,u) = v$ for any u,v. A ternary polymorphism $f : H^3 \to H$ is *majority (respectively Maltsev) over* a,b, if $f(a,a,b) = f(a,b,a) = f(b,a,a) = a, f(b,b,a) = f(b,a,b) = f(a,b,b) = b$ (respectively if $f(a,a,b) = f(b,a,a) = b, f(a,b,b) = f(b,b,a) = a$).

At this point, we can state the classification of L-HOM(H) due to Bulatov. The theorem applies to any relational structure H, but for our purposes we only need to state it for reflexive digraphs. Recall that by our definition each polymorphism is conservative. Also, we formulate the result in a language of binary commutative polymorphisms in place of the more usual semi-lattice operations [1], since it is equivalent and is more convenient in our context.

Theorem 4.1 *[1] Let H be a reflexive digraph.*

If for every pair of vertices a,b of H there exists a polymorphism of H which is either ternary and majority, or Maltsev, over a,b, or is binary and commutative over a,b, then L-HOM(H) is polynomial time solvable.

Otherwise, if some pair of vertices a,b does not admit any of these polymorphisms, then the problem L-HOM(H) is NP-complete.

5 List Homomorphism Problems

The following fact follows directly from Theorems 2.3 (or Theorem 4.1) and 2.2.

Theorem 5.1 *If H is an adjusted interval digraph, then L-HOM(H) is polynomial time solvable.*

Here is an equivalent form of the conjecture from [9, 14].

Conjecture 5.2 *If H is an adjusted interval digraph, then L-HOM(H) is polynomial time solvable.*

(We also had a similar conjecture for irreflexive digraphs [9, 14]. However, that conjecture has turned out to be false [15, 2], and we shall discuss the case of irreflexive digraphs in a companion paper [15].)

We now provide some preliminary evidence to support our conjecture. In the full version of the paper we will offer additional results to this end. A digraph is semi-complete if its underlying graph is complete.

Theorem 5.3 *Suppose H is a reflexive semi-complete digraph. If H contains an invertible pair, then L-HOM(H) is NP-complete.*

Proof. We will appeal to Bulatov's characterization, Theorem 4.1, showing that if there exist invertible pairs in H, then some invertible pair a, b admits no polymorphism as prescribed by Theorem 4.1.

It turns out that some structures in H limit our choices of polymorphisms from the theorem. Let R be the reflexive digraph $V(R) = \{a, b, c\}$ and $E(R) = \{aa, bb, cc, ab, bc, ac, ca\}$.

Lemma 5.4 *There is no polymorphism g on the digraph R which is a majority over a, b.*

Proof. Suppose g is a polymorphism of R which is a majority over a, b, i.e., $g(a, a, b) = g(a, b, a) = g(b, a, a) = a$, and $g(a, b, b) = g(b, a, b) = g(b, b, a) = b$. We claim that g must also be a majority over b, c. Note that $g(c, c, b)g(a, a, b) = g(c, c, b)a \in E(R)$. Hence $g(c, c, b) = c$, as b does not dominate a in R. Similarly, $g(c, b, c) = g(b, c, c) = c$. Also $g(b, b, c)g(b, b, a) = g(b, b, c)b \in E(R)$ thus $g(b, b, c) = b$ and similarly $g(b, c, b) = g(c, b, b) = b$. Now we can conclude that g is also majority over a, c, using the fact that $g(a, a, c)g(b, b, c) \in E(R)$ and $g(b, b, c)g(c, c, a) \in E(R)$.

Now we note that we have $g(a, b, c)g(b, b, c) = g(a, b, c)b \in E(R)$, which implies that $g(a, b, c) \in \{a, b\}$ (since c doesn't dominate b in R); we have $g(a, b, b)g(a, b, c) \in E(R)$, which similarly implies that $g(a, b, c) \in \{b, c\}$; and we have $g(c, a, c)g(a, b, c) \in E(R)$, which similarly implies that $g(a, b, c) \in \{a, c\}$, which is impossible. □

Lemma 5.5 *Suppose H is a reflexive digraph with $ab \in E(H), ba \notin E(H)$. There is no polymorphism h over the digraph H which is a Maltsev operation over a, b.*

Proof. If h is Maltsev over a, b, then $h(a, a, b)h(a, b, b) = ba \in E(H)$, a contradiction. □

Thus let us assume H contains invertible pairs. If H also contains an induced reflexive directed three-cycle $\vec{C}_3$, then L-HOM(H) is known to be NP-complete [9]. Thus we may assume for the proof that H does not contain $\vec{C}_3$. By a similar argument, we may assume that $S(H)$ is an interval graph, and in particular, $S(H)$ does not contain an induced four-cycle [8, 20].

If H contains invertible pairs, then there exist directed closed walks $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n), (x_0, y_0)$ in H^+ which contains both (a, b) and (b, a) for some $a, b \in V(H)$. We say that such a closed walk C is an *inverting walk* for the pair a, b. As noted in Lemma 2.1, each vertex (x_i, y_i) of C is itself invertible.

An inverting walk C in H^+ consists of forward edges only. Recall that, in H, these edges could correspond to pairs of edges x_ix_{i+1}, y_iy_{i+1}, which are either forward or backward.

We first assume that for some C and some i we have the edges $x_ix_{i+1}, x_{i+2}x_{i+1} \in E(H)$ and $y_iy_{i+1}, y_{i+2}y_{i+1} \in E(H)$. Without loss of generality, let us assume $i = 0$, i.e., that x_0x_1, x_2x_1, $y_0y_1, y_2y_1 \in E(H)$ and $x_0y_1, y_2x_1 \notin E(H)$. Therefore, the pair (x_0, y_1) dominates (x_1, y_1) and is dominated by (x_0, y_0), which are consecutive in the cycle C. Thus we may assume that (x_0, y_1) is also in C, and hence is invertible. The same argument shows that (x_1, y_2) is also invertible.

Since H is semi-complete, we must have $y_1x_0, x_1y_2 \in E(H)$. If $y_1x_1 \notin E(H)$, then y_1, x_0, x_1 are all distinct and must induce R, since there is no induced $\vec{C_3}$). Then over y_1, x_0 there is no majority by Lemma 5.4, no Maltsev by Lemma 5.5, and no commutative binary polymorphism by Lemma 2.4. Hence L-HOM(H) is NP-complete by Theorem 4.1.

If $y_1x_1 \in E(H)$, then y_1, x_1, y_2 must be distinct and the same argument as above implies that $x_1y_1 \in E(H)$. Then the same argument again applied to the triple y_1, x_0, x_1 implies that $x_1x_0 \in E(H)$, and applied to the triple y_1, x_1, y_2 implies that $y_1y_2 \in E(H)$. Note that $x_0 \neq y_2$ because $x_0y_1 \notin E(H)$ but $y_2y_1 \in E(H)$. If $y_2x_0 \notin E(H)$ then we have a copy of R over x_0, y_1, y_2; if $x_0y_2 \notin E(H)$ then we have a copy of R over x_0, x_1, y_2. This yields an induced four-cycle x_0, x_1, y_1, y_2, x_0 in $S(H)$, contrary to our assumption.

Thus we may assume that on any inverting walk all edges go in the same direction, forward or backward. Without loss of generality, assume that on C all edges x_ix_{i+1} in H are forward (and similarly for y_iy_{i+1}). If there is a copy of $\vec{C_3}$ or R, the problem L-HOM(H) is NP-complete as above. Otherwise, we claim that all $x_iy_i \in E(H)$ and $y_ix_i \in E(H)$. Indeed if $y_ix_i \notin E(H)$, then a copy of $\vec{C_3}$ or R exists on x_{i-1}, x_i, y_i, unless $x_i = x_{i-1}$. Note that if $x_i = x_{i-1}$ would mean that $x_iy_i \notin E(H)$ also holds, contrary to the fact that H is semi-complete. If $x_iy_i \notin E(H)$, then on some inverting walk involving the invertible pair x_i, y_i, the same argument would show the existence of $\vec{C_3}$ or R. $\square$

Thus the conjecture holds for semi-complete digraphs. In the full version of this paper we also prove the conjecture for oriented trees, and other digraphs.

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