THE DICHOTOMY OF MINIMUM COST HOMOMORPHISM PROBLEMS FOR DIGRAPHS

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Abstract. The minimum cost homomorphism problem has arisen as a natural and useful optimization problem in the study of graph (and digraph) coloring and homomorphisms: it unifies a number of other well studied optimization problems. It was shown by Gutin, Rafiey, and Yeo that the minimum cost problem for homomorphisms to a digraph \( H \) that admits a so-called extended Min-Max ordering is polynomial time solvable, and these authors conjectured that for all other digraphs \( H \) the problem is NP-complete. In a companion paper, we gave a forbidden structure characterization of digraphs that admit extended Min-Max orderings. In this paper, we apply this characterization to prove Gutin’s conjecture.

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1. Introduction. A homomorphism of a digraph \( G \) to a digraph \( H \) is a mapping \( f : V(G) \to V(H) \) such that \( xy \in A(G) \) implies \( f(x)f(y) \in A(H) \). The minimum cost homomorphism problem for \( H \), denoted Min\(\text{HOM}(H) \), asks whether or not an input digraph \( G \), with integer costs \( c_i(u), u \in V(G), i \in V(H), \) and an integer \( k \), admits a homomorphism to \( H \) of total cost \( \sum_{u \in V(G)} c_{f(u)}(u) \) not exceeding \( k \). The problem Min\(\text{HOM}(H) \) was first formulated in [19]; it unifies and generalizes several other problems [21, 28, 30, 31, 33], including two other well studied homomorphism problems, the problem HOM\(\text{}(H) \) asking for just the existence of homomorphisms [22], and the problem List\(\text{HOM}(H) \) asking for the existence of homomorphisms in which vertices of \( G \) map to vertices of \( H \) from given allowed lists [10].

For undirected graphs \( H \), the complexity of HOM\(\text{}(H) \), List\(\text{HOM}(H) \), and Min\(\text{HOM}(H) \) was classified in [22, 10, 15]. Namely, HOM\(\text{}(H) \) is polynomial time solvable if \( H \) is bipartite or has a loop, List\(\text{HOM}(H) \) is polynomial time solvable if \( H \) is a bi-arc graph, and Min\(\text{HOM}(H) \) is polynomial time solvable if each component of \( H \) is a reflexive proper interval graph or an irreflexive proper interval bigraph. In all other cases the problems are NP-complete. Thus in all three cases, the classification is a dichotomy, in the sense that each problem HOM\(\text{}(H) \), List\(\text{HOM}(H) \), or Min\(\text{HOM}(H) \) is polynomial time solvable or NP-complete. Moreover, given a graph \( H \), deciding whether \( H \) is bipartite or has a loop, whether \( H \) is a bi-arc graph, and whether each component of \( H \) is a reflexive proper interval graph or an irreflexive proper interval bigraph is polynomial in terms of the size of the graph \( H \) [22, 10, 15]. Thus these dichotomies are polynomial time classifications, in terms of \( H \).
For digraphs, the dichotomy of $\text{HOM}(H)$ is an important unproved conjecture, equivalent to the so-called dichotomy conjecture [13, 5]. Recent progress specifically on classifying the complexity of $\text{HOM}(H)$ for classes of digraphs $H$ was reported in [2, 3]; cf. [1].

A dichotomy of List$\text{HOM}(H)$ for general structures is given in [4]. It implies dichotomy for digraphs; however, this general dichotomy is not a polynomial classification. A polynomial dichotomy classification of List$\text{HOM}(H)$ for digraphs is reported in [27]; cf. also [11, 12].

A dichotomy of Min$\text{HOM}(H)$ for general structures (cf. [9]) is given in [34]. It again implies dichotomy for digraphs but is not a polynomial classification. A polynomial dichotomy classification of Min$\text{HOM}(H)$ for reflexive digraphs was proved in [14]. Other special cases were treated in [25]. In this paper we provide a polynomial dichotomy classification of Min$\text{HOM}(H)$ for general digraphs. (A preliminary version was posted in [24].)

At the heart of our minimum cost homomorphism algorithms is the following concept.

Let $H$ be any digraph. A linear ordering $<$ of $V(H)$ is a Min-Max ordering of $H$ if it satisfies the following Min-Max property:

if $u < w$ and $z < v$ and $uw, wz \in A(H)$, then $uz \in A(H)$ and $wv \in A(H)$.

An undirected graph (viewed as a symmetric digraph) admits a Min-Max ordering if and only if each component is either a reflexive proper interval graph or an irreflexive proper interval bigraph [15]. Thus digraphs with Min-Max orderings can be viewed as digraph analogues of proper interval graphs and bigraphs. It turns out that this is not a coincidence—we have shown in a companion paper that the digraphs that admit a Min-Max ordering also have an equivalent characterization using an interval representation akin to that for proper interval graphs and bigraphs [26].

Proper interval graphs (and bigraphs) are characterized by simple forbidden structures and recognized in polynomial time [32]; cf. [15]. In our companion paper, we have given a forbidden structure characterization of digraphs admitting a Min-Max ordering.

It follows from [15, 14] that both for symmetric digraphs (undirected graphs) and for reflexive digraphs, Min$\text{HOM}(H)$ is polynomial time solvable if $H$ admits a Min-Max ordering, and is NP-complete otherwise. This is not the case for general digraphs, as certain extended Min-Max orderings (defined in a later section) also imply a polynomial time algorithm [17]. However, it was conjectured by Gutin, Rafiey, and Yeo [17] that Min$\text{HOM}(H)$ is NP-complete unless $H$ admits an extended Min-Max ordering. Several special cases of the conjecture have been verified [14, 15, 16, 17, 18]. We apply our characterization of digraphs with extended Min-Max ordering to prove this conjecture, obtaining a polynomial dichotomy classification of the minimum cost homomorphism problems in digraphs. The problem Min$\text{HOM}(H)$ is polynomial time solvable if $H$ has an extended Min-Max ordering, and is NP-complete otherwise. We have shown in [26] that there is a polynomial time algorithm to test whether $H$ has an extended Min-Max ordering.

2. Background. If $uv \in A(H)$, we say that $uv$ is an arc of $H$, or that $uv$ is a forward arc of $H$; we also say that $vu$ is a backward arc of $H$. In any event, we say that $u, v$ are adjacent in $H$ if $uv$ is a forward or a backward arc of $H$. A walk in $H$ is a sequence $P = x_0, x_1, \ldots, x_n$ of consecutively adjacent vertices of $H$; note that a walk has a designated first and last vertex. A path is a walk in which all $x_i$ are distinct. A walk is closed if $x_0 = x_n$ and is a cycle if all other $x_i$ are distinct. A walk is directed
if all arcs are forward. The net length of a walk is the number of forward arcs minus the number of backward arcs. A closed walk is balanced if it has net length zero; otherwise it is unbalanced. Note that in an unbalanced closed walk we may always choose a direction in which the net length is positive (or negative). A digraph is balanced if it does not contain an unbalanced closed walk (or equivalently an unbalanced cycle); otherwise it is unbalanced. It is easy to see that a digraph is balanced if and only if it admits a labeling of vertices by nonnegative integers so that each arc goes from some level \( i \) to the level \( i + 1 \). The height of \( H \) is the maximum net length of a walk in \( H \). Note that an unbalanced digraph has infinite height, and the height of a balanced digraph is the greatest label in a nonnegative labeling in which some vertex has label zero.

For any walk \( P = x_0, x_1, \ldots, x_n \) in \( H \), we consider the minimum height of \( P \) to be the smallest net length of an initial subwalk \( x_0, x_1, \ldots, x_i \), and the maximum height of \( P \) to be the greatest net length of an initial subwalk \( x_0, x_1, \ldots, x_i \). Note that when \( i = 0 \), we obtain the trivial subwalk \( x_0 \) of net length zero, and when \( i = n \), we obtain the entire walk \( P \). We shall say that \( P \) is constricted from below if the minimum height of \( P \) is zero (no initial subwalk \( x_0, x_1, \ldots, x_i \) has negative net length), and constricted if moreover the maximum height is the net length of \( P \) (no initial subwalk \( x_0, x_1, \ldots, x_i \) has greater net length than \( x_0, x_1, \ldots, x_n \)). It is easy to see that a walk which is constricted from below can be partitioned into two constricted pieces by dividing it at any vertex achieving the maximum height.

For walks \( P \) from \( a \) to \( b \), and \( Q \) from \( b \) to \( c \), we denote by \( PQ \) the walk from \( a \) to \( c \) which is the concatenation of \( P \) and \( Q \), and by \( P^{-1} \) the walk \( P \) traversed in the opposite direction, from \( b \) to \( a \). We call \( P^{-1} \) the reverse of \( P \). For a closed walk \( C \), we denote by \( C^a \) the concatenation of \( C \) with itself \( a \) times.

The following lemma is well known. (For a proof, see [20, 35] or Lemma 2.36 in [23].)

**Lemma 2.1.** Let \( P_1 \) and \( P_2 \) be two constricted walks of net length \( r \). Then there is a constricted path \( P \) of net length \( r \) that admits a homomorphism \( f_1 \) to \( P_1 \) and a homomorphism \( f_2 \) to \( P_2 \), such that each \( f_i \) takes the starting vertex of \( P \) to the starting vertex of \( P_i \), and the ending vertex of \( P \) to the ending vertex of \( P_i \).

We shall call \( P \) a common preimage of \( P_1 \) and \( P_2 \). In [26] we have proved the following corollary of Lemma 2.1.

**Corollary 2.2.** Let \( P_1 \) and \( P_2 \) be two walks of infinite height, constricted from below. Assume that \( P_i \) starts in \( p_i \), \( i = 1, 2 \), and let \( q_i \) be a vertex on \( P_i \), such that the infinite portion of \( P_i \) starting from \( q_i \) is also constricted from below, and the portions of \( P_i \) from \( p_i \) to \( q_i \) have the same net length for \( i = 1, 2 \).

Then there is a path \( P \) that admits homomorphisms \( f_i \) to \( P_i \) taking the starting vertex of \( P \) to \( p_i \) and the ending vertex of \( P \) to \( q_i \), for \( i = 1, 2 \).

We define two walks \( P = x_0, x_1, \ldots, x_n \) and \( Q = y_0, y_1, \ldots, y_n \) in \( H \) to be congruent if they follow the same pattern of forward and backward arcs; i.e., \( x_i x_{i+1} \) is a forward (backward) arc if and only if \( y_i y_{i+1} \) is a forward (backward) arc (respectively). Suppose the walks \( P, Q \) as above are congruent. We say an arc \( x_i y_{i+1} \) is a faithful arc from \( P \) to \( Q \) if it is a forward (backward) arc when \( x_i x_{i+1} \) is a forward (backward) arc (respectively), and we say an arc \( y_i x_{i+1} \) is a faithful arc from \( Q \) to \( P \) if it is a forward (backward) arc when \( x_i x_{i+1} \) is a forward (backward) arc (respectively). We say that \( P, Q \) avoid each other if there is no pair of faithful arcs \( x_i y_{i+1} \) from \( P \) to \( Q \), and \( y_i x_{i+1} \) from \( Q \) to \( P \), for some \( i = 0, 1, \ldots, n \).

We observe that if \( < \) is a Min-Max ordering of \( H \) and \( P = x_0, x_1, \ldots, x_n \) and \( Q = y_0, y_1, \ldots, y_n \) are two congruent walks in \( H \) that avoid each other, then \( x_0 < y_0 \)
In particular, an induced cycle with more than one vertex does not contain a loop.

A **symmetrically invertible pair** in \( H \) is a pair of distinct vertices \( u, v \) such that there exist congruent walks \( P \) from \( u \) to \( v \) and \( Q \) from \( v \) to \( u \) that avoid each other. It follows from the above observation that if \( H \) has a symmetrically invertible pair, then it cannot have a Min-Max ordering. It can also be shown that a digraph \( H \) that contains an induced cycle of net length greater than one cannot have a Min-Max ordering [26]. In fact, we have proved the following theorem.

**Theorem 2.3** (see [26]). A digraph \( H \) admits a Min-Max ordering if and only if \( H \) has no induced cycle of net length greater than one and no symmetrically invertible pair.

A cycle of \( H \) is **induced** if \( H \) contains no other arcs on the vertices of the cycle. In particular, an induced cycle with more than one vertex does not contain a loop.

We denote by \( \bar{C}_k \) the directed cycle on vertices \( 0, 1, \ldots, k - 1 \). We shall assume in this section that \( H \) is strongly connected. Indeed, the minimum cost homomorphism problem to \( H \) can be easily separated into subproblems corresponding to the weak components of \( H \); moreover, any version of the Min-Max property also applies to each individual weak component of \( H \) separately. This assumption allows us to conclude that any two homomorphisms \( \ell, \ell' \) of \( H \) to \( \bar{C}_k \) define the same partition of \( V(H) \) into the sets \( V_i = \ell^{-1}(i) \), and we will refer to these sets without explicitly defining a homomorphism \( \ell \).

A **k-Min-Max ordering** of a digraph \( H \) homomorphic to \( \bar{C}_k \) is a linear ordering < of each set \( V_i \), so that the Min-Max property \((u < w, z < v \text{ and } uw, wz \in A(H)) \) imply that \( uz \in A(H), vw \in A(H) \) is satisfied for \( u, w \) and \( v, z \) in any two circularly consecutive sets \( V_i \) and \( V_{i+1} \), respectively (script addition modulo \( k \)). Any digraph \( H \) is homomorphic to the one-vertex digraph with a loop \( \bar{C}_k \), and a 1-Min-Max ordering of \( H \) is just the usual Min-Max ordering. A Min-Max ordering of a digraph \( H \) becomes a k-Min-Max ordering of \( H \) for any \( \bar{C}_k \) to which \( H \) is homomorphic. There are digraphs \( H \) with a k-Min-Max ordering that do not have a Min-Max ordering, say \( H = \bar{C}_k \) (with \( k > 1 \)). An extended Min-Max ordering of \( H \) is a k-Min-Max ordering of \( H \) for some positive integer \( k \).

We observe for future reference that an unbalanced digraph \( H \) has only a limited range of possible values of \( k \) for which it could have a homomorphism to \( \bar{C}_k \), and hence a limited range of possible values of \( k \) for which it could have a k-Min-Max ordering. It is easy to see that a cycle \( C \) admits a homomorphism to \( \bar{C}_k \) only if the net length of \( C \) is divisible by \( k \) [23]. Thus any cycle of net length \( q > 0 \) in \( H \) limits the possible values of \( k \) to the divisors of \( q \). If \( H \) is balanced, it is easy to see that \( H \) has a k-Min-Max ordering for some \( k \) if and only if it has a Min-Max ordering.

In [26] we have also proved the following theorem.

**Theorem 2.4** (see [26]). Let \( H \) be a weakly connected digraph homomorphic to \( \bar{C}_k \) for some positive integer \( k \).

Then \( H \) admits a k-Min-Max ordering if and only if it contains no induced cycle of positive net length other than \( k \), and no symmetrically invertible pair such that \( u \) and \( v \) belong to the same set \( V_i \).

We have also proved in [26] that the conditions in each of the two theorems can be tested in polynomial time. This implies that we can decide if \( H \) has a Min-Max ordering or an extended Min-Max ordering in polynomial time. Suppose we want to test whether or not a digraph \( H \) has an extended Min-Max ordering. As noted above,
it suffices to check for each component of $H$ separately, so we may assume that $H$ is weakly connected. If $H$ is balanced, it is easy to see that $H$ has an extended Min-Max ordering if and only if it has a Min-Max ordering. Otherwise $H$ has an unbalanced cycle, say, of net length $q$. Then $H$ has an extended Min-Max ordering if and only if it has a $k$-Min-Max ordering for some $k$ that divides $q$.

We apply Theorem 2.4 to prove the following result. The first statement (that the existence of a $k$-Min-Max ordering implies a polynomial time algorithm) is proved in [17]. The second statement (the NP-completeness claim) was a conjecture of Gutin; cf. [17]. The third statement is justified above. (Note that the third statement refers to polynomiality in terms of the size of $H$.)

**THEOREM 2.5.** Let $H$ be any digraph.

If $H$ has an extended Min-Max ordering, then $\text{MinHOM}(H)$ is polynomial time solvable.

Otherwise, $\text{MinHOM}(H)$ is NP-complete.

There is a polynomial time algorithm for deciding whether $H$ has an extended Min-Max ordering.

We prove Theorem 2.5 using our characterization in Theorem 2.4 by showing that $\text{MinHOM}(H)$ is NP-complete if $H$ contains an induced unbalanced cycle of net length other than $k$, or a symmetrically invertible pair $u, v$ with $u, v$ in the same set $V_i$; this will be done in the next section.

**3. The NP-completeness claims.** Our basic NP-completeness tool is summarized in the next lemma.

**LEMMA 3.1.** Let $H$ be a digraph with two vertices $x, y$, and let $S$ be a digraph with two vertices $s, t$. Suppose we have costs $c_j(i)$ of mapping vertices $i$ of $S$ to vertices $j$ of $H$ where $c_x(s) = c_x(t) = 1, c_y(s) = c_y(t) = 0$, and such that there exists

- a homomorphism $f : S \rightarrow H$ mapping $s$ to $x$ and $t$ to $y$ of total cost 1 (i.e., in which all other vertices of $S$, different from $s, t$, map to vertices of $H$ with cost 0);
- a homomorphism $g : S \rightarrow H$ mapping $s$ to $x$ and $t$ to $x$ of total cost 2 (other vertices map with cost 0);
- a homomorphism $h : S \rightarrow H$ mapping $s$ to $y$ and $t$ to $x$ of total cost 1 (other vertices map with cost 0);
- no homomorphism $S \rightarrow H$ mapping $s$ to $y$ and $t$ to $y$ of total cost smaller than 2.

Then $\text{MinHOM}(H)$ is NP-complete.

**Proof.** Let $G$ be an arbitrary graph, an instance of the maximum independent set problem. We construct a corresponding instance $D$ of $\text{MinHOM}(H)$ by replacing every edge of $G$ by a copy of $S$. Note that $D$ contains all old vertices of $G$, as well as the new vertices, each lying in a separate copy of $S$. The costs $c_i(j), i \in V(H), j \in V(D)$, are defined as follows:

- If $v$ is an old vertex of $G$, then $c_x(v) = 1, c_y(v) = 0$, and $c_z(v) = |V(G)|$ for all other $z \in V(H)$;
- if $v$ is a new vertex of $D$ lying in a copy of $S$ and corresponding to the vertex $v'$ in $S$, then its costs are determined by the costs in $S$, namely $c_i(v) = c_i(v')$ for all $i \in V(H)$.

Note that since we have $c_x(s) = c_x(t) = 1, c_y(s) = c_y(t) = 0$, the two parts of the definition do not conflict. We now claim that $G$ has an independent set of size $k$ if and only if there exists a homomorphism of $D$ to $H$ of cost $|V(G)| - k$. Indeed, if $I$ is an independent set in $G$, we define a homomorphism $\phi : D \rightarrow H$ by setting
\( \phi(j) = y \) if \( j \in I \), setting \( \phi(j) = x \) if \( j \in V(G) \setminus I \), and extending this mapping to a homomorphism of \( D \) to \( H \), using the mappings \( f, g, h \). It is clear that the cost of \( \phi \) is exactly \( |V(G)| - |I| \). Conversely, let \( f \) be any homomorphism of \( D \) to \( H \) of total cost less than \( |V(G)| \). Thus the old vertices of \( G \) must map to either \( x \) or \( y \). If two adjacent vertices of \( G \) are mapped to \( y \), we incur a cost of at least 2. By mapping one of the two vertices instead to \( x \) we decrease the cost of the mapping by at least 2 and increase it by 1, giving a net decrease of at least 1. Thus we may assume that those vertices that map to \( y \) are independent. Since the old vertices of \( G \) that map to \( x \) contribute a cost of 1 each, we conclude that if there is a homomorphism of cost \( |V(G)| - k \), then there is an independent set of size \( k \) in \( G \). 

One example in which we can easily use this lemma deals with a special case of symmetrically invertible pairs.

**Corollary 3.2.** Suppose \( u, v \) is a symmetrically invertible pair in \( H \) with corresponding walks \( P, Q \), such that there exists at least one faithful arc from \( P \) to \( Q \), but there exist no faithful arcs from \( Q \) to \( P \).

Then the problem \( \text{MinHOM}(H) \) is NP-complete.

**Proof.** We assume \( P = u = a_1 \ldots a_n = v \) and \( Q = v = b_1 \ldots b_m = u \), and let \( S = s_1 \ldots s_n \) be a path (all vertices are distinct) congruent to \( P \) (and \( Q \)). Define the cost of mapping vertices of \( S \) to \( H \) as follows. Set \( c_{a_i}(s_1) = c_{a_i}(s_n) = 1 \), \( c_{v_i}(s_1) = c_{v_i}(s_n) = 0 \), and \( c_{a_i}(s_i) = c_{b_i}(s_i) = 0 \) for \( 1 < i < n \). In any other case the cost is 1.

Clearly there are obvious homomorphisms \( \phi : S \to P \) and \( \psi : S \to Q \). Let \( a_1 b_{t+1} \) be a faithful arc from \( P \) to \( Q \). Define also \( \zeta : S \to H \) to be the homomorphism defined by \( \zeta(s_i) = a_i \) for \( 1 \leq i \leq t \) and by \( \zeta(s_i) = b_i \) for \( t + 1 \leq i \leq n \). Suppose there is a homomorphism \( g : V(S) \to V(P) \cup V(Q) \) such that \( g(s_1) = g(s_n) = v \). Then the cost of \( g \) is at least \( n \) unless \( g(r_i) \) is \( a_i \) or \( b_i \). Since \( g(s_1) = g(s_n) = v \), there has to be a faithful arc from \( Q \) to \( P \) in \( H \), which is a contradiction. Now by Lemma 3.1 the problem \( \text{MinHOM}(H) \) is NP-complete.

We next consider the case where some symmetrically invertible pair has faithful arcs both from \( P \) to \( Q \) and from \( Q \) to \( P \).

It was noted in [15] that the following problem \( \Pi_3 \) is NP-complete. Given a three-colored graph \( G \) and an integer \( k \), decide if there exists an independent set of \( k \) vertices. It is easy to see that this fact can be generalized to the following problem \( \Pi_{2m+1} \): Given a graph \( G \) with a homomorphism \( f : G \to C_{2m+1} \), decide if there exists an independent set of \( k \) vertices.

**Lemma 3.3.** Each problem \( \Pi_{2m+1} \) is NP-complete.

**Proof.** Modify every instance \( G \) of \( \Pi_{2m-1} \) to an instance \( G' \) of \( \Pi_{2m+1} \) by replacing each edge of \( G \) between classes \( f^{-1}(1) \) and \( f^{-1}(2) \) by a path of length three.

We apply this result as follows.

**Lemma 3.4.** Suppose \( u, v \) is a symmetrically invertible pair in \( H \) with corresponding walks \( P, Q \), such that there exists at least one faithful arc from \( P \) to \( Q \) as well as at least one faithful arc from \( Q \) to \( P \).

Then \( \text{MinHOM}(H) \) is NP-complete.

**Proof.** The walks \( P = x_0, x_1, \ldots, x_n \) and \( Q = y_0, y_1, \ldots, y_m \) can be organized into segments \( P_1, \ldots, P_k, Q_1, \ldots, Q_k \), where for each \( i \) all faithful arcs between \( P \) and \( Q \) go from \( P \) to \( Q \) or from \( Q \) to \( P \). Assume \( P_i = x_{r_i-1}, x_{r_i-1+1}, \ldots, x_{r_i} \) and \( Q_i = y_{r_i-1}, y_{r_i-1+1}, \ldots, y_{r_i} \), with \( r_0 = 0, r_k = n \), and assume, without loss of generality, that there are faithful arcs from \( P_1 \) to \( Q_1 \) but no faithful arcs from \( Q_1 \) to \( P_1 \). There are faithful arcs from \( Q_2 \) to \( P_2 \) but no faithful arcs from \( P_2 \) to \( Q_2 \); etc. Note that if \( k \) is odd, the faithful arcs of the last segment \( P_k \) and \( Q_k \) are independent, and if \( k \) is even, they go from \( P \) to \( Q \). Let \( R_i \) be a path congruent to \( P_i \) (and \( Q_i \)), and for simplicity assume that \( R_i = r_{i-1}, \ldots, r_i \).
Case 1. Assume $k$ is odd.

We reduce $\Pi_k$ to MinHom($H$) as follows. Consider an instance of $\Pi_k$, namely, a graph $G$ with a homomorphism $f$ to $C_k$. Suppose the vertices of $C_k$ are $1, 2, \ldots, k$ (consecutively, and viewed modulo $k$). Replace each edge $uv$ of $G$ having $u \in f^{-1}(i)$ and $v \in f^{-1}(i+1)$ (modulo $k$) by a copy $R_i(u, v)$ of $R_i$, identifying $r_{i-1}$ with $u$ and $r_i$ with $v$, obtaining a digraph $D$. The costs of mapping an old vertex (from $G$) $u$ in $f^{-1}(i)$ with $i$ odd will be $c_{x_r}(u) = 1, c_{y_r}(u) = 0$, while the costs of mapping an old vertex $u$ in $f^{-1}(i)$ with $i$ even will be $c_{x_r}(u) = 0, c_{y_r}(u) = 1$. For vertices inside the substituted copies of $R$, we proceed as above, defining their costs to be zero only for the corresponding vertices in $R(u, v)$. All other costs are $|V(G)|$.

Suppose $i$ is odd. Each homomorphism of $R_i$ to $D$ taking $r_{i-1}$ to $x_{r_{i-1}}$ and $r_i$ to $y_r$, has a very high cost, but all other possibilities ($r_{i-1}$ to $x_{r_{i-1}}$ and $r_i$ to $r_i$; $r_{i-1}$ to $y_{r_{i-1}}$ and $r_i$ to $r_i$; and $r_{i-1}$ to $y_{r_{i-1}}$ and $r_i$ to $x_{r_i}$) have cost 1. A similar analysis applies to $i$ even. A special consideration is needed for the last segment $R_k$, where we use the fact that $x_{r_k} = y_0$ and $y_{r_k} = y_n = x_0$.

As in the proof of Corollary 3.2, these facts imply that $G$ has an independent set of size $\ell$ if and only if $D$ has a homomorphism to $H$ of cost $|V(G)| - \ell$.

Case 2. Assume $k$ is even.

In this case instead of the symmetrically invertible pair $u, v$ with walks $P, Q$ we consider the symmetrically invertible pair $y_{r_1}, x_{r_1}$ with walks $P', Q'$ where $P' = y_{r_1}, \ldots, y_{r_{i-1}}, y_{r_i} = y_n = x_0, \ldots, x_{r_1}$, and $Q' = x_{r_1}, \ldots, x_{r_{i-1}}, x_{r_i} = x_n = y_0, \ldots, y_{r_1}$. Note that there are no faithful arcs from $x_{r_{i-1}}, \ldots, x_{r_k}$ to $y_{r_{i-1}}, \ldots, y_{r_k} = y_n = x_0, \ldots, x_{r_1}$. Thus we obtain an odd number of segments and can proceed as above, unless $k = 2$, in which case we have only one segment and Corollary 3.2 applies.

We can now handle the case when $H$ is balanced. Recall that this means that the vertices of $H$ have levels $0, 1, \ldots, h$ so that each arc goes from some level $i$ to level $i + 1$. It is easy to see that in a balanced digraph a symmetrically invertible pair $u, v$ must have $u$ and $v$ on the same level. Thus all symmetrically invertible pairs $u, v$ must have $\ell(u) = \ell(v)$ in any homomorphism $\ell : H \rightarrow \bar{C}_k$. Therefore, the NP-completeness part of Theorem 2.5 in the balanced case reduces to the following statement.

**Theorem 3.5.** If a balanced digraph $H$ contains a symmetrically invertible pair, then MinHom($H$) is NP-complete.

**Proof.** By Corollary 3.2 and Lemma 3.4, we may assume that we have a symmetrically invertible pair $u, v$ and corresponding walks $P, Q$ with no faithful arcs between $P$ and $Q$. Consider the walk $W$ in $H^*$ from $(u, v)$ to $(v, u)$ corresponding to $P$ and $Q$. If some $(a, b)$ lies on $W$, then there is a walk in $H^*$ from $(a, b)$ to $(b, a)$ (because $H^*$ has an arc from $(x, y)$ to $(x', y')$ if and only if it has an arc from $(y, x)$ to $(y', x')$). Thus we may assume that $u, v$ are on the lowest level of $P$ and $Q$.

Let $z$ be a vertex on the highest level of $P$, and let $w$ be the corresponding vertex on $Q$. Let $R$ be the walk obtained by following $Q$ from $v$ to $u$ and then following $Q^{-1}$ back from $w$ to $v$. Let the path $S$ be the common preimage of $P, Q$, and $R$, obtained by applying Lemma 2.1 twice, since $P, Q, R$ consist of two constricted pieces. Let $f$ be the corresponding homomorphism of $S$ to $P$, let $g$ be the corresponding homomorphism of $S$ to $Q$, and let $h$ be the corresponding homomorphism of $S$ to $R$. We define the cost of mapping an internal vertex of $S$ to a vertex $i$ of $H$ as 0 if $i \in \{f(j), g(j), h(j)\}$; the cost of mapping the first and the last vertex of $S$ to $v$ is 1 and to $u$ is 0. In all other cases the cost is $|V(S)|$. Note that there is no homomorphism from $S$ to $H$ which maps both the beginning and the end of $S$ to $u$ of total cost smaller than $|V(S)|$, as otherwise there would be a faithful arc from $P$ to $Q$. Now by applying Lemma 3.1 to $S$ and $f, g, h$ we conclude that MinHom($H$) is NP-complete.
Corollary 3.6. Theorem 2.5 holds for balanced digraphs $H$.

Specifically, for a balanced digraph $H$ the problem MinHOM($H$) is polynomial time solvable if $H$ has a Min-Max ordering, and is NP-complete otherwise.

We observe that the same proof also applies for an unbalanced digraph $H$ as long as $P$ (and hence $Q$) has net length zero. Specifically, if any digraph $H$ has a symmetrically invertible pair $u,v$ with corresponding walks $P,Q$ which have net length zero, then MinHOM($H$) is NP-complete.

We now focus on unbalanced digraphs $H$.

Theorem 3.7. Suppose $H$ is weakly connected and contains two induced cycles $C_1, C_2$, with net lengths $k, n > 0, k \neq n$.

Then MinHOM($H$) is NP-complete.

Proof. Suppose $k > n$, so $k$ does not divide $n$. We may assume that $H$ is minimal, in the sense that no weakly connected subgraph $H'$ of $H$ with fewer vertices contains two induced cycles with different nonzero net lengths. Indeed, if $H'$ were such a subgraph, then MinHOM($H'$) would be polynomially reduced to MinHOM($H$) by setting the cost of mapping to vertices of $H$ not in $H'$ to be very high.

Each cycle $C_i$, $i = 1, 2$, contains a vertex $u_i$ such that the walk starting in $u_i$ and following $C_i$ (in the positive direction) is constricted from below. Let $U$ be a walk in $H$ from $u_1$ to $u_2$, and let $u$ be a vertex on $U$ of minimum height. By minimality, we may assume $V(H) = V(C_1) \cup V(C_2) \cup U$. Let $P_i, i = 1, 2$, be the walk from $u$ to $u_i$ following $U$ (or $U^{-1}$), then once around $C_i$ (in the positive direction), and then back from $u$ following $U^{-1}$ (or $U$). It follows that each $P_i$ is constricted from below. The net length of $P_1$ is $k$ and the net length of $P_2$ is $n$. Let $Q_i, i = 1, 2$, be the infinite walk starting at $u$ obtained by repeatedly concatenating $P_i$, and let $Q_i'$ be the two-way infinite walk obtained by expanding $Q_i$ in the opposite direction by repeatedly concatenating $P_i^{-1}$.

Let $d$ be greatest common divisor of $n$ and $k$, and let $a = k/d - 2$. Thus $(a + 2)n$ is the smallest positive common multiple of $n$ and $k$. We now define the following three walks $W_1, W_2, W_3$ in $H$ of net length $(a + 1)n$:

1. The walk $W_1$ starts at $u$ and follows $Q_1$ going around $P_1$ until the last vertex $v$ such that the net length of the resulting walk is $(a + 1)n$.
2. $W_2$ also starts at $u$ and follows $Q_2$ going around $P_2$ fully $(a + 1)$ times, ending at $u$.
3. $W_3$ starts at $v$ and follows $P_1$ until the first occurrence of $u$, and then continues $a$ times around $P_2$, ending again at $u$.

Now we define, in analogy with $Q_1, Q_2$, also the infinite walk $Q_3$, obtained from $W_3$ by continuing to go around $P_2$. Because we chose $v$ to be the last vertex on $Q_1$ with the right net length, the walk $W_3$ is constricted from below; of course $W_1, W_2$ are also constricted from below. Hence $Q_1, Q_2, Q_3$ are also constricted from below; they have infinite heights because $C_1, C_2$ have positive net length. Thus we can apply Corollary 2.2 to $Q_1, Q_2, Q_3$, obtaining a common preimage which is a path $S$, say, $s = s_0, s_1, \ldots, s_q = t$, with homomorphisms $f, g, h$ of $S$ to $Q_1, Q_2, Q_3$, respectively, such that

1. $f(s) = u, f(t) = v$,
2. $g(s) = g(t) = u$,
3. $h(s) = v, h(t) = u$.

Note that the walk $W'_1$ equal to $u = f(s_0), f(s_1), \ldots, f(s_q) = v$, the walk $W'_2$ equal to $v = g(s_0), g(s_1), \ldots, g(s_q) = u$, and the walk $W'_3$ equal to $v = h(s_0), h(s_1), \ldots, h(s_q) = u$ are congruent.
Assume first that $W_1', W_2'$ do not avoid each other; i.e., for some $i$ we have both the {faithful} arcs (forward or backward) $f(s_i)h(s_{i+1})$, $h(s_i)f(s_{i+1})$. Note that $W_1'\cup W_2'\cup W_3'$ contains all the vertices of $H$, so the minimality of $H$ easily implies that all four vertices $f(s_i), h(s_i), f(s_{i+1}), h(s_{i+1})$ must belong to $C_1 \cup C_2$. Since the cycles are induced, we must have two vertices in each cycle. Up to symmetry, we may assume we have forward arcs $ab \in C_1$ and $cd \in C_2$, as well as forward arcs $ad, cb$ in $H$. Then, say, $a = f(s_i), b = f(s_{i+1}), c = h(s_i), d = h(s_{i+1})$.

We first claim that $C_1, C_2$ do not have common vertices, or arcs joining them other than $ad, cb$. Otherwise, let $x$ on $C_1$ be the first vertex following $b$ in the direction opposite to $a$, equal to or adjacent with some $y$ on $C_2$, and assume that $y$ is the first vertex of $C_2$ following $d$, in the direction opposite to $c$, adjacent to $x$. Consider the cycle $D_1$ with arcs $ab, ad, xy$: the portion of $C_1$ between $b$ and $x$ not containing $a$; and the portion of $C_2$ between $d$ and $y$ not containing $c$. Also consider the cycle $D_2$ with arcs $cb, cd, xy$, and the same portions of $C_1, C_2$. The cycles $D_1, D_2$ have the same net length $m$. If $m$ is not zero and not $k$, we could delete $c$ and obtain a smaller weakly connected $H'$ with two different nonzero net lengths. If $m$ is not zero and not $n$, we could likewise delete $a$. Thus $m = 0$. If $x$ has no neighbors on $C_2$ other than $y$, then consider instead of $D_2$ the cycle $D_2'$ obtained from $D_2$ by replacing the portion of $C_2$ between $c$ and $y$ containing $d$ by the portion of $C_2$ between $c$ and $y$ not containing $d$. Since $m = 0$, the net length of $D_2'$ is $n$, so we can delete $d$ and obtain a smaller weakly connected $H'$ with two different nonzero net lengths. Otherwise, let $y_1, y_2, \ldots, y_\beta$ be all the neighbors of $x$ on $C_2$ after $y = y_0$, numbered consecutively in the direction from $y$ to $c$, away from $d$. Consider the cycles $Y_i$ containing $x, y_i, y_{i+1}$ and the segment of $C_2$ between $y_i$ and $y_{i+1}$ not containing $d$. Each $Y_i$ is an induced cycle in $H$, and the sum of their net lengths is $n$. Hence at least one $Y_i$ has a nonzero net length and we similarly obtain a contradiction with the minimality of $H$.

Thus $H$ consists of $C_1, C_2$, and the two extra arcs (forward or backward) $ad, cb$; in particular $u \in C_1 \cup C_2$, and the path $U$ uses $ad$ or $bc$. Without loss of generality, we may assume that it uses $bc$, since we can replace $ad$ by $ab, bc, cd$. Suppose first that $u \in C_1$, whence we also have $v \in C_1$. Consider the initial portion of $W_1'$ from $v$ to $b = f(s_{i+1})$: it has net length equal to a multiple of $k$ (corresponding to going full rounds around the cycle $C_1$) plus the net length of the portion $X_1$ of $C_1$ (in the positive direction) from $u$ to $b$. Consider next the initial portion of $W_2'$ from $v$ to $c$ followed by the arc joining $c$ and $b$: it has net length equal to $n$ (corresponding to going from $v$ to $u$, which must precede $c \in C_2$) plus a multiple of $n$ (corresponding to going full rounds around the closed walk $P_2$ from $u$ to $u$) plus the net length of the portion $X_2$ of $P_2$ (in the positive direction) from $u$ to $c$ concatenated with the arc joining $c$ and $b$. However, from $u$ to $c$ we must use the arc joining $b$ and $c$. Thus $X_2$ uses the arc joining $b$ and $c$ first in one direction and then in the opposite direction, whence the net lengths of $X_1, X_2$ are the same. This means that a multiple of $n$, smaller than $(a + 2)n$ is also a multiple of $k$, which is impossible, by our choice of $a$.

It remains to consider the case when $W_1', W_2'$ avoid each other. We now assume that of all homomorphisms $f, g, h$ of $S$ to $Q_1, Q_2, Q_3$ satisfying properties (1, 2, 3) and such that the resulting walks $W_1', W_3'$ avoid each other, we have chosen ones that maximize the number of vertices with $f(s_i) = g(s_i)$ or $g(s_i) = h(s_i)$.

If $W_1', W_2'$ have at least some faithful arcs, then Corollary 3.2 and Lemma 3.4 imply MinHOM($H$) is NP-complete. Thus we may assume that there are no faithful arcs between $W_1'$ and $W_3'$. 


We now define the costs of mapping vertices \( x \) of \( S \) to vertices \( j \) of \( H \) as follows: \( c_1(x) = 2 \) except for \( c_u(s) = c_u(t) = 1 \), \( c_v(s) = c_v(t) = 0 \), and \( c_j(s_i) = 0 \) when \( j \in \{f(s_i), g(s_i), h(s_i)\} \), \( j \neq u \).

By properties 1, 2, 3, we see that to apply Lemma 3.1 it remains to show that there is no homomorphism of \( S \) to \( H \) of cost less than 2, taking both \( s \) and \( t \) to \( v \). Suppose, for a contradiction, that there is such a homomorphism \( \phi \). Then we must have \( \phi(s_0) = h(s_0) \), \( \phi(s_q) = f(s_q) \), and each \( \phi(s_i) \in \{f(s_i), g(s_i), h(s_i)\} \). Since there are no faithful arcs between \( W'_1 \) and \( W'_3 \), we cannot have \( h(s_i) \) and \( f(s_{i+1}) \) adjacent. Thus, because of the costs, some \( h(s_i) \) and \( g(s_{i+1}) \) must be adjacent, and also some \( g(s_j) \) and \( f(s_{j+1}) \) must be adjacent, with \( i < j \). We now claim that this contradicts the maximality of \( f, g, h \). Indeed, we could redefine \( f \) to equal \( g \) up to \( s_j \) (and then, continuing as before, taking advantage of the arc joining \( g(s_j) \) and \( f(s_{j+1}) \)), obtaining a new \( W'_1 \) with at least one more vertex (namely \( s_{i+1} \)) having equality of \( f \) and \( g \). (We need to observe that the new \( W'_1 \) still avoids \( W'_3 \), which also follows by maximality of \( f, g, h \): there cannot be an arc between \( g(s_p) \neq h(s_p) \) and \( h(s_{p+1}) \).) \( \Box \)

From the theorem we also derive the following corollary that will complete the proof of Theorem 2.5.

**Theorem 3.8.** Suppose \( H \) is a digraph containing an induced cycle of net length \( k > 0 \). If \( H \) is homomorphic to \( C_k \) and contains a symmetrically invertible pair \( u, v \) with \( u, v \) in the same set \( V_i \), then MinHOM(\( H \)) is NP-complete.

**Proof.** Recall that \( P \) is a walk from \( u \) to \( v \) and that \( Q \) is a congruent walk with \( \ell(P) = \ell(Q) \), such that the walk \( \ell(P) \) is divisible by \( k \). If there are faithful arcs from \( P \) to \( Q \) or from \( Q \) to \( P \), then by Corollary 3.2 or Lemma 3.4, MinHOM(\( H \)) is NP-complete. So we may assume that there are no such faithful arcs. We may also assume that the net length of \( P \) is greater than zero, as otherwise the remark following Corollary 3.6 implies that MinHOM(\( H \)) is NP-complete. We now proceed to find congruent walks from \( u \) to \( v \) and from \( v \) to \( u \) that avoid each other, and another congruent walk from \( u \) to \( u \), so that we can apply Lemma 3.1 in a fashion similar to what was done in the proof of Theorem 3.7.

We may assume that \( P \) is constricted from below, as otherwise we replace \( u, v \) by vertices \( u', v' \in P \), where \( u' \) is a vertex of \( P \) with the minimum height, and \( v' \) is the corresponding vertex of \( v' \) in \( Q \). We have observed that \( u', v' \) is also a symmetrically invertible pair; thus there are walks \( P' \) from \( u' \) to \( v' \) and \( Q' \) from \( v' \) to \( Q' \) that avoid each other. It is easy to see that the minimality of \( u' \) implies that this new \( P' \) is constricted from below. Let \( C \) be a walk in \( H \) from \( u \) to a cycle of net length \( k \), followed by going around the cycle once in the positive direction and then returning back on the same walk to \( u \). Note that the net length of this walk is \( k \). We may again assume that \( C \) is constricted from below, as otherwise instead of \( P, Q \) we could use \( P_1, Q_1 \), where \( P_1 \) is obtained by concatenating \( P \) with \( (QP)^a \) and \( Q_1 \) is obtained by concatenating \( Q \) with \( (PQ)^a \) for some positive \( a \), such that the walk from \( u \) to \( Q_1 \) (at the beginning of \( P_1 \)) to the \( (a - 1) \)th appearance of \( u \) in \( P_1 \) followed by \( C \) is a walk constricted from below.

Let the net length of \( P \) be \( \ell_k \), with \( \ell > 0 \). Let \( W \) be the infinite walk obtained by repeatedly concatenating \( C \); note that \( W \) is constricted from below. Let \( P' \) be the infinite walk obtained by concatenating \( P \) with infinitely many repetitions of \( QP \). Let \( Q' \) be the infinite walk congruent to \( P' \) obtained by similarly concatenating \( Q \) with repetitions of \( PQ \). Let \( C' \) be the walk in \( W \), from \( u \) to a vertex \( u' \) that is the \( \ell \)th occurrence of \( u \) in \( W \). Now we apply Corollary 2.2 to obtain a path \( S = s_0, s_1, \ldots, s_t \) which is the common preimage of \( P, C', Q \). In this application, we use \( P', W, Q' \) as the
infinite walks and use the ends of $P, C', Q$ as the vertices $q_i$. (Note that $P, C', Q$ all have net length $k$. ) Corollary 2.2 also yields homomorphisms $f, g, h$ of $S$ to $P', W, Q'$ taking $s_0$ to the beginnings of $P', W, Q'$ (also to the beginnings of $P, C', Q$), and taking $s_t$ to the ends of $P, C', Q$. Let $P''$ be the walk $f(s_0), f(s_1), \ldots, f(s_t)$, let $Q''$ be the walk $h(s_0), h(s_1), \ldots, h(s_t)$, and let $C''$ be the walk $g(s_0), g(s_1), \ldots, g(s_t)$. Observe that $P'', Q''$ avoid each other, and between the walks $P'', Q''$ there are no faithful arcs, because that was the case for $P, Q$.

Note that $f(s_0) = u$ and $f(s_t) = v$, $g(s_0) = g(s_t) = u$, and $h(s_0) = v, h(s_t) = u$. We define the costs as follows: $c_u(s_0) = c_u(s_t) = 1$, and $c_v(s_0) = c_v(s_t) = 0$, and $c_x(x) = 0$ when $x \in \{f(x), g(x), h(y)\}$, $x \neq u$. For any other case the cost is $|V(S)|$.

We now conclude the proof as in Theorem 3.7, assuming that the homomorphisms $f, g, h$ of $S$ to $V(P'') \cup V(C'') \cup V(Q'')$ satisfy properties 1, 2, 3, and maximize the number of vertices with $f(s_i) = g(s_i)$ or $g(s_i) = h(s_i)$.

We are finally ready to conclude the Proof of Theorem 2.5, i.e., to prove Gutin’s conjecture [17].

Recall that the polynomial case of the theorem has been established in [17]. For the NP-completeness claim, the case when $H$ is balanced is handled by Corollary 3.6. Thus we may assume that $H$ has an induced cycle of some positive net length $k$. It is a well-known fact (e.g., Corollary 1.17 in [23]) that $H$ has a homomorphism to $\tilde{C}_k$ if and only if it does not contain a closed walk of net length not divisible by $k$. Suppose first that $H$ does not admit a homomorphism to $\tilde{C}_k$. Then the above fact implies that $H$ contains an induced cycle of net length not divisible by $k$. Hence the problem MinHOM($H$) is NP-complete by Theorem 3.7. If, on the other hand, $H$ does admit a homomorphism to $\tilde{C}_k$, with a symmetrically invertible pair $u, v$ from the same set $V_i$, then MinHOM($H$) is NP-complete by Theorem 3.8. This completes the proof. □

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