SIAM J. DISCRETE MATH.
Vol. 26, No. 4, pp. 1597–1608

THE DICHOTOMY OF MINIMUM COST HOMOMORPHISM PROBLEMS FOR DIGRAPHS*

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Abstract. The minimum cost homomorphism problem has arisen as a natural and useful optimization problem in the study of graph (and digraph) coloring and homomorphisms: it unifies a number of other well studied optimization problems. It was shown by Gutin, Rafiey, and Yeo that the minimum cost problem for homomorphisms to a digraph H that admits a so-called extended Min-Max ordering is polynomial time solvable, and these authors conjectured that for all other digraphs H the problem is NP-complete. In a companion paper, we gave a forbidden structure characterization of digraphs that admit extended Min-Max orderings. In this paper, we apply this characterization to prove Gutin's conjecture.

Key words. minimum cost homomorphisms, Min-Max orderings, dichotomy

AMS subject classifications. 05C75, 05C85

DOI. 10.1137/100783856

1. Introduction.

A *homomorphism* of a digraph G to a digraph H is a mapping $f : V(G) \to V(H)$ such that $xy \in A(G)$ implies $f(x)f(y) \in A(H)$. The *minimum cost homomorphism problem* for H, denoted MinHOM(H), asks whether or not an input digraph G, with integer costs $c_i(u)$, $u \in V(G)$, $i \in V(H)$, and an integer k, admits a homomorphism to H of total cost $\sum_{u \in V(G)} c_{f(u)}(u)$ not exceeding k. The problem MinHOM(H) was first formulated in [19]; it unifies and generalizes several other problems [21, 28, 30, 31, 33], including two other well studied homomorphism problems, the problem HOM(H) asking for just the existence of homomorphisms [22], and the problem ListHOM(H) asking for the existence of homomorphisms in which vertices of G map to vertices of H from given allowed lists [10].

For undirected graphs H, the complexity of HOM(H), ListHOM(H), and MinHOM(H) was classified in [22, 10, 15]. Namely, HOM(H) is polynomial time solvable if H is bipartite or has a loop, ListHOM(H) is polynomial time solvable if H is a bi-arc graph, and MinHOM(H) is polynomial time solvable if each component of H is a reflexive proper interval graph or an irreflexive proper interval bigraph. In all other cases the problems are NP-complete. Thus in all three cases, the classification is a dichotomy, in the sense that each problem HOM(H), ListHOM(H), or MinHOM(H) is polynomial time solvable or NP-complete. Moreover, given a graph H, deciding whether H is bipartite or has a loop, whether H is a bi-arc graph, and whether each component of H is a reflexive proper interval graph or an irreflexive proper interval bigraph is polynomial in terms of the size of the graph H [22, 10, 15]. Thus these dichotomies are *polynomial time classifications*, in terms of H.

*Received by the editors January 25, 2010; accepted for publication (in revised form) July 30, 2012; published electronically October 25, 2012.
http://www.siam.org/journals/sidma/26-4/78385.html

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For digraphs, the dichotomy of HOM(H) is an important unproved conjecture, equivalent to the so-called dichotomy conjecture [13, 5]. Recent progress specifically on classifying the complexity of HOM(H) for classes of digraphs H was reported in [2, 3]; cf. [1].

A dichotomy of ListHOM(H) for general structures is given in [4]. It implies dichotomy for digraphs; however, this general dichotomy is not a polynomial classification. A polynomial dichotomy classification of ListHOM(H) for digraphs is reported in [27]; cf. also [11, 12].

A dichotomy of MinHOM(H) for general structures (cf. [9]) is given in [34]. It again implies dichotomy for digraphs but is not a polynomial classification. A polynomial dichotomy classification of MinHOM(H) for reflexive digraphs was proved in [14]. Other special cases were treated in [25]. In this paper we provide a polynomial dichotomy classification of MinHOM(H) for general digraphs. (A preliminary version was posted in [24].)

At the heart of our minimum cost homomorphism algorithms is the following concept.

Let H be any digraph. A linear ordering $<$ of $V(H)$ is a *Min-Max ordering* of H if it satisfies the following *Min-Max property*:

if $u < w$ and $z < v$ and $uv, wz \in A(H)$, then $uz \in A(H)$ and $wv \in A(H)$.

An undirected graph (viewed as a symmetric digraph) admits a Min-Max ordering if and only if each component is either a reflexive proper interval graph or an irreflexive proper interval bigraph [15]. Thus digraphs with Min-Max orderings can be viewed as digraph analogues of proper interval graphs and bigraphs. It turns out that this is not a coincidence—we have shown in a companion paper that the digraphs that admit a Min-Max ordering also have an equivalent characterization using an interval representation akin to that for proper interval graphs and bigraphs [26].

Proper interval graphs (and bigraphs) are characterized by simple forbidden structures and recognized in polynomial time [32]; cf. [15]. In our companion paper, we have given a forbidden structure characterization of digraphs admitting a Min-Max ordering.

It follows from [15, 14] that both for symmetric digraphs (undirected graphs) and for reflexive digraphs, MinHOM(H) is polynomial time solvable if H admits a Min-Max ordering, and is NP-complete otherwise. This is not the case for general digraphs, as certain extended Min-Max orderings (defined in a later section) also imply a polynomial time algorithm [17]. However, it was conjectured by Gutin, Rafiey, and Yeo [17] that MinHOM(H) is NP-complete unless H admits an extended Min-Max ordering. Several special cases of the conjecture have been verified [14, 15, 16, 17, 18]. We apply our characterization of digraphs with extended Min-Max ordering to prove this conjecture, obtaining a polynomial dichotomy classification of the minimum cost homomorphism problems in digraphs. The problem MinHOM(H) is polynomial time solvable if H has an extended Min-Max ordering, and is NP-complete otherwise. We have shown in [26] that there is a polynomial time algorithm to test whether H has an extended Min-Max ordering.

2. Background.

If $uv \in A(H)$, we say that uv is an arc of H, or that uv is a *forward arc* of H; we also say that vu is a *backward arc* of H. In any event, we say that u, v are *adjacent* in H if uv is a forward or a backward arc of H. A *walk* in H is a sequence $P = x_0, x_1, \ldots, x_n$ of consecutively adjacent vertices of H; note that a walk has a designated first and last vertex. A *path* is a walk in which all x_i are distinct. A walk is *closed* if $x_0 = x_n$ and is a *cycle* if all other x_i are distinct. A walk is *directed* if all arcs are forward. The *net length* of a walk is the number of forward arcs minus the number of backward arcs. A closed walk is *balanced* if it has net length zero; otherwise it is *unbalanced*. Note that in an unbalanced closed walk we may always choose a direction in which the net length is positive (or negative). A digraph is *balanced* if it does not contain an unbalanced closed walk (or equivalently an unbalanced cycle); otherwise it is *unbalanced*. It is easy to see that a digraph is balanced if and only if it admits a *labeling* of vertices by nonnegative integers so that each arc goes from some level i to the level $i+1$. The *height* of H is the maximum net length of a walk in H. Note that an unbalanced digraph has infinite height, and the height of a balanced digraph is the greatest label in a nonnegative labeling in which some vertex has label zero.

For any walk $P = x_0, x_1, \ldots, x_n$ in H, we consider the *minimum height* of P to be the smallest net length of an initial subwalk $x_0, x_1, \ldots, x_i$, and the *maximum height* of P to be the greatest net length of an initial subwalk $x_0, x_1, \ldots, x_i$. Note that when $i = 0$, we obtain the trivial subwalk x_0 of net length zero, and when $i = n$, we obtain the entire walk P. We shall say that P is *constricted from below* if the minimum height of P is zero (no initial subwalk $x_0, x_1, \ldots, x_i$ has negative net length), and *constricted* if moreover the maximum height is the net length of P (no initial subwalk $x_0, x_1, \ldots, x_i$ has greater net length than $x_0, x_1, \ldots, x_n$). It is easy to see that a walk which is constricted from below can be partitioned into two constricted pieces by dividing it at any vertex achieving the maximum height.

For walks P from a to b, and Q from b to c, we denote by PQ the walk from a to c which is the concatenation of P and Q, and by P^{-1} the walk P traversed in the opposite direction, from b to a. We call P^{-1} the *reverse* of P. For a closed walk C, we denote by C^a the concatenation of C with itself a times.

The following lemma is well known. (For a proof, see [20, 35] or Lemma 2.36 in [23].)

LEMMA 2.1. *Let P_1 and P_2 be two constricted walks of net length r. Then there is a constricted path P of net length r that admits a homomorphism f_1 to P_1 and a homomorphism f_2 to P_2, such that each f_i takes the starting vertex of P to the starting vertex of P_i and the ending vertex of P to the ending vertex of P_i.*

We shall call P a *common preimage* of P_1 and P_2. In [26] we have proved the following corollary of Lemma 2.1.

COROLLARY 2.2. *Let P_1 and P_2 be two walks of infinite height, constricted from below. Assume that P_i starts in p_i, $i = 1, 2$, and let q_i be a vertex on P_i, such that the infinite portion of P_i starting from q_i is also constricted from below, and the portions of P_i from p_i to q_i have the same net length for $i = 1, 2$.*

Then there is a path P that admits homomorphisms f_i to P_i taking the starting vertex of P to p_i and the ending vertex of P to q_i for $i = 1, 2$.

We define two walks $P = x_0, x_1, \ldots, x_n$ and $Q = y_0, y_1, \ldots, y_n$ in H to be *congruent* if they follow the same pattern of forward and backward arcs; i.e., $x_i x_{i+1}$ is a forward (backward) arc if and only if $y_i y_{i+1}$ is a forward (backward) arc (respectively). Suppose the walks P, Q as above are congruent. We say an arc $x_i y_{i+1}$ is *a faithful arc from P to Q* if it is a forward (backward) arc when $x_i x_{i+1}$ is a forward (backward) arc (respectively), and we say an arc $y_i x_{i+1}$ is *a faithful arc from Q to P* if it is a forward (backward) arc when $x_i x_{i+1}$ is a forward (backward) arc (respectively). We say that P, Q *avoid each other* if there is no pair of faithful arcs $x_i y_{i+1}$ from P to Q, and $y_i x_{i+1}$ from Q to P, for some $i = 0, 1, \ldots, n$.

We observe that if $<$ is a Min-Max ordering of H and $P = x_0, x_1, \ldots, x_n$ and $Q = y_0, y_1, \ldots, y_n$ are two congruent walks in H that avoid each other, then $x_0 < y_0$

if and only if $x_n < y_n$. Indeed, if $x_i < y_i$ and $y_{i+1} < x_{i+1}$ then the Min-Max property is not satisfied for $x_i, y_i, x_{i+1}, y_{i+1}$; a similar contradiction arises if $y_i < x_i$ and $x_{i+1} < y_{i+1}$.

A *symmetrically invertible pair* in H is a pair of distinct vertices u, v such that there exist congruent walks P from u to v and Q from v to u that avoid each other. It follows from the above observation that if H has a symmetrically invertible pair, then it cannot have a Min-Max ordering. It can also be shown that a digraph H that contains an induced cycle of net length greater than one cannot have a Min-Max ordering [26]. In fact, we have proved the following theorem.

THEOREM 2.3 (see [26]). *A digraph H admits a Min-Max ordering if and only if H has no induced cycle of net length greater than one and no symmetrically invertible pair.*

A cycle of H is *induced* if H contains no other arcs on the vertices of the cycle. In particular, an induced cycle with more than one vertex does not contain a loop.

We denote by $\vec{C}_k$ the directed cycle on vertices $0, 1, \ldots, k-1$. We shall assume in this section that H is weakly connected. Indeed, the minimum cost homomorphism problem to H can be easily separated into subproblems corresponding to the weak components of H; moreover, any version of the Min-Max property also applies to each individual weak component of H separately. This assumption allows us to conclude that any two homomorphisms ℓ, ℓ' of H to $\vec{C}_k$ define the same partition of $V(H)$ into the sets $V_i = \ell^{-1}(i)$, and we will refer to these sets without explicitly defining a homomorphism ℓ.

A *k-Min-Max ordering* of a digraph H homomorphic to $\vec{C}_k$ is a linear ordering $<$ of each set V_i, so that the Min-Max property ($u < w, z < v$ and $uv, wz \in A(H)$ imply that $uz \in A(H), wv \in A(H)$) is satisfied for u, w and v, z in any two circularly consecutive sets V_i and V_{i+1}, respectively (subscript addition modulo k). Any digraph H is homomorphic to the one-vertex digraph with a loop $\vec{C}_k$, and a 1-Min-Max ordering of H is just the usual Min-Max ordering. A Min-Max ordering of a digraph H becomes a k-Min-Max ordering of H for any $\vec{C}_k$ to which H is homomorphic. There are digraphs H with a k-Min-Max ordering that do not have a Min-Max ordering, say $H = \vec{C}_k$ (with $k > 1$). An *extended Min-Max ordering* of H is a k-Min-Max ordering of H for some positive integer k.

We observe for future reference that an unbalanced digraph H has only a limited range of possible values of k for which it could have a homomorphism to $\vec{C}_k$, and hence a limited range of possible values of k for which it could have a k-Min-Max ordering. It is easy to see that a cycle C admits a homomorphism to $\vec{C}_k$ only if the net length of C is divisible by k [23]. Thus any cycle of net length $q > 0$ in H limits the possible values of k to the divisors of q. If H is balanced, it is easy to see that H has a k-Min-Max ordering for some k if and only if it has a Min-Max ordering.

In [26] we have also proved the following theorem.

THEOREM 2.4 (see [26]). *Let H be a weakly connected digraph homomorphic to $\vec{C}_k$ for some positive integer k.*

Then H admits a k-Min-Max ordering if and only if it contains no induced cycle of positive net length other than k, and no symmetrically invertible pair such that u and v belong to the same set V_i.

We have also proved in [26] that the conditions in each of the two theorems can be tested in polynomial time. This implies that we can decide if H has a Min-Max ordering or an extended Min-Max ordering in polynomial time. Suppose we want to test whether or not a digraph H has an extended Min-Max ordering. As noted above, it suffices to check for each component of H separately, so we may assume that H is weakly connected. If H is balanced, it is easy to see that H has an extended Min-Max ordering if and only if it has a Min-Max ordering. Otherwise H has an unbalanced cycle, say, of net length q. Then H has an extended Min-Max ordering if and only if it has a k-Min-Max ordering for some k that divides q.

We apply Theorem 2.4 to prove the following result. The first statement (that the existence of a k-Min-Max ordering implies a polynomial time algorithm) is proved in [17]. The second statement (the NP-completeness claim) was a conjecture of Gutin; cf. [17]. The third statement is justified above. (Note that the third statement refers to polynomiality in terms of the size of H.)

THEOREM 2.5. *Let H be any digraph.*

If H has an extended Min-Max ordering, then MinHOM(H) is polynomial time solvable.

Otherwise, MinHOM(H) is NP-complete.

There is a polynomial time algorithm for deciding whether H has an extended Min-Max ordering.

We prove Theorem 2.5 using our characterization in Theorem 2.4 by showing that MinHOM(H) is NP-complete if H contains an induced unbalanced cycle of net length other than k, or a symmetrically invertible pair u, v with u, v in the same set V_i; this will be done in the next section.

3. The NP-completeness claims.

Our basic NP-completeness tool is summarized in the next lemma.

LEMMA 3.1. *Let H be a digraph with two vertices x, y, and let S be a digraph with two vertices s, t. Suppose we have costs $c_j(i)$ of mapping vertices i of S to vertices j of H where $c_x(s) = c_x(t) = 1, c_y(s) = c_y(t) = 0$, and such that there exists*

- *a homomorphism $f : S \to H$ mapping s to x and t to y of total cost 1 (i.e., in which all other vertices of S, different from s, t, map to vertices of H with cost 0);*
- *a homomorphism $g : S \to H$ mapping s to x and t to x of total cost 2 (other vertices map with cost 0);*
- *a homomorphism $h : S \to H$ mapping s to y and t to x of total cost 1 (other vertices map with cost 0);*
- *no homomorphism $S \to H$ mapping s to y and t to y of total cost smaller than 2.*

Then MinHOM(H) is NP-complete.

Proof. Let G be an arbitrary graph, an instance of the maximum independent set problem. We construct a corresponding instance D of MinHOM(H) by replacing every edge of G by a copy of S. Note that D contains all *old* vertices of G, as well as the *new* vertices, each lying in a separate copy of S. The costs $c_i(j), i \in V(H), j \in V(D)$, are defined as follows:

- If v is an old vertex of G, then $c_x(v) = 1, c_y(v) = 0$, and $c_z(v) = |V(G)|$ for all other $z \in V(H)$;
- if v is a new vertex of D lying in a copy of S and corresponding to the vertex v' in S, then its costs are determined by the costs in S, namely $c_i(v) = c_i(v')$ for all $i \in V(H)$.

Note that since we have $c_x(s) = c_x(t) = 1, c_y(s) = c_y(t) = 0$, the two parts of the definition do not conflict. We now claim that G has an independent set of size k if and only if there exists a homomorphism of D to H of cost $|V(G)| - k$. Indeed, if I is an independent set in G, we define a homomorphism $\phi : D \to H$ by setting

$\phi(j) = y$ if $j \in I$, setting $\phi(j) = x$ if $j \in V(G) \setminus I$, and extending this mapping to a homomorphism of D to H, using the mappings f, g, h. It is clear that the cost of ϕ is exactly $|V(G)| - |I|$. Conversely, let f be any homomorphism of D to H of total cost less than $|V(G)|$. Thus the old vertices of G must map to either x or y. If two adjacent vertices of G are mapped to y, we incur a cost of at least 2. By mapping one of the two vertices instead to x we decrease the cost of the mapping by at least 2 and increase it by 1, giving a net decrease of at least 1. Thus we may assume that those vertices that map to y are independent. Since the old vertices of G that map to x contribute a cost of 1 each, we conclude that if there is a homomorphism of cost $|V(G)| - k$, then there is an independent set of size k in G. $\square$

One example in which we can easily use this lemma deals with a special case of symmetrically invertible pairs.

COROLLARY 3.2. *Suppose u, v is a symmetrically invertible pair in H with corresponding walks P, Q, such that there exists at least one faithful arc from P to Q, but there exist no faithful arcs from Q to P.*

Then the problem MinHOM(H) is NP-complete.

Proof. We assume $P = u = a_1 \dots a_n = v$ and $Q = v = b_1 \dots b_n = u$, and let $S = s_1 \dots s_n$ be a *path* (all vertices are distinct) congruent to P (and Q). Define the cost of mapping vertices of S to H as follows. Set $c_u(s_1) = c_u(s_n) = 1, c_v(s_1) = c_v(s_n) = 0$, and $c_{a_i}(s_i) = c_{b_i}(s_i) = 0$ for $1 < i < n$. In any other case the cost is n.

Clearly there are obvious homomorphisms $\phi : S \to P$ and $\psi : S \to Q$. Let $a_t b_{t+1}$ be a faithful arc from P to Q. Define also $\zeta : S \to H$ to be the homomorphism defined by $\zeta(s_i) = a_i$ for $1 \le i \le t$ and by $\zeta(s_i) = b_i$ for $t + 1 \le i \le n$. Suppose there is homomorphism $g : V(S) \to V(P) \cup V(Q)$ such that $g(s_1) = g(s_n) = v$. Then the cost of g is at least n unless $g(r_i)$ is a_i or b_i. Since $g(s_1) = g(s_n) = v$, there has to be a faithful arc from Q to P in H, which is a contradiction. Now by Lemma 3.1 the problem MinHOM(H) is NP-complete. $\square$

We next consider the case where some symmetrically invertible pair has faithful arcs both from P to Q and from Q to P.

It was noted in [15] that the following problem Π_3 is NP-complete. Given a three-colored graph G and an integer k, decide if there exists an independent set of k vertices. It is easy to see that this fact can be generalized to the following problem Π_{2m+1}: Given a graph G with a homomorphism $f : G \to C_{2m+1}$, decide if there exists an independent set of k vertices.

LEMMA 3.3. *Each problem Π_{2m+1} is NP-complete.*

Proof. Modify every instance G of Π_{2m-1} to an instance G' of Π_{2m+1} by replacing each edge of G between classes $f^{-1}(1)$ and $f^{-1}(2)$ by a path of length three. $\square$

We apply this result as follows.

LEMMA 3.4. *Suppose u, v is a symmetrically invertible pair in H with corresponding walks P, Q, such that there exists at least one faithful arc from P to Q as well as at least one faithful arc from Q to P.*

Then MinHOM(H) is NP-complete.

Proof. The walks $P = x_0, x_1, \dots, x_n$ and $Q = y_0, y_1, \dots, y_n$ can be organized into segments $P_1, \dots P_k, Q_1, \dots, Q_k$, where for each i all faithful arcs between P and Q go from P to Q or from Q to P. Assume $P_i = x_{r_{i-1}}, x_{r_{i-1}+1}, \dots, x_{r_i}$ and $Q_i = y_{r_{i-1}}, y_{r_{i-1}+1}, \dots, y_{r_i}$ with $r_0 = 0, r_k = n$, and assume, without loss of generality, that there are faithful arcs from P_1 to Q_1 but no faithful arcs from Q_1 to P_1, there are faithful arcs from Q_2 to P_2 but no faithful arcs from P_2 to Q_2, etc. Note that if k is odd, the faithful arcs of the last segment go from Q to P, and if k is even, they go from P to Q. Let R_i be a path congruent to P_i (and Q_i), and for simplicity assume that $R_i = r_{i-1}, \dots, r_i$.

Case 1. Assume k is odd.

We reduce Π_k to MinHOM(H) as follows. Consider an instance of Π_k, namely, a graph G with a homomorphism f to C_k. Suppose the vertices of C_k are $1, 2, \ldots, k$ (consecutively, and viewed modulo k). Replace each edge uv of G having $u \in f^{-1}(i)$ and $v \in f^{-1}(i+1)$ (modulo k) by a copy $R_i(u, v)$ of R_i, identifying r_{i-1} with u and r_i with v, obtaining a digraph D. The costs of mapping an old vertex (from G) u in $f^{-1}(i)$ with i odd will be $c_{x_{r_i}}(u) = 1, c_{y_{r_i}}(u) = 0$, while the costs of mapping an old vertex u in $f^{-1}(i)$ with i even will be $c_{x_{r_i}}(u) = 0, c_{y_{r_i}}(u) = 1$. For vertices inside the substituted copies of R, we proceed as above, defining their costs to be zero only for the corresponding vertices in $R(u, v)$. All other costs are $|V(G)|$.

Suppose i is odd. Each homomorphism of R_i to D taking r_{i-1} to $x_{r_{i-1}}$ and r_i to y_{r_i} has a very high cost, but all other possibilities (r_{i-1} to $x_{r_{i-1}}$ and r_i to x_{r_i}; r_{i-1} to $y_{r_{i-1}}$ and r_i to y_{r_i}; and r_{i-1} to $y_{r_{i-1}}$ and r_i to x_{r_i}) have cost 1. A similar analysis applies to i even. A special consideration is needed for the last segment R_k, where we use the fact that $x_{r_k} = x_n = y_0$ and $y_{r_k} = y_n = x_0$.

As in the proof of Corollary 3.2, these facts imply that G has an independent set of size ℓ if and only if D has a homomorphism to H of cost $|V(G)| - \ell$.

Case 2. Assume k is even.

In this case instead of the symmetrically invertible pair u, v with walks P, Q we consider the symmetrically invertible pair y_{r_1}, x_{r_1} with walks P', Q' where $P' = y_{r_1}, \ldots, y_{r_2}, \ldots, y_{r_{k-1}}, \ldots, y_{r_k} = y_n = x_0, \ldots, x_{r_1}$, and $Q' = x_{r_1}, \ldots, x_{r_2}, \ldots, x_{r_{k-1}}, \ldots, x_{r_k} = x_n = y_0, \ldots, y_{r_1}$. Note that there are no faithful arcs from $x_{r_{k-1}}, \ldots, x_{r_k} = x_n = y_0, \ldots, y_{r_1}$ to $y_{r_{k-1}}, \ldots, y_{r_k} = y_n = x_0, \ldots, x_{r_1}$. Thus we obtain an odd number of segments and can proceed as above, unless $k = 2$, in which case we have only one segment and Corollary 3.2 applies. ∎

We can now handle the case when H is balanced. Recall that this means that the vertices of H have levels $0, 1, \ldots, h$ so that each arc goes from some level i to level $i + 1$. It is easy to see that in a balanced digraph a symmetrically invertible pair u, v must have u and v on the same level. Thus all symmetrically invertible pairs u, v must have $\ell(u) = \ell(v)$ in any homomorphism $\ell : H \to \vec{C}_k$. Therefore, the NP-completeness part of Theorem 2.5 in the balanced case reduces to the following statement.

THEOREM 3.5. *If a balanced digraph H contains a symmetrically invertible pair, then MinHOM(H) is NP-complete.*

Proof. By Corollary 3.2 and Lemma 3.4, we may assume that we have a symmetrically invertible pair u, v and corresponding walks P, Q with no faithful arcs between P and Q. Consider the walk W in H^* from (u, v) to (v, u) corresponding to P and Q. If some (a, b) lies on W, then there is a walk in H^* from (a, b) to (b, a) (because H^* has an arc from (x, y) to (x', y') if and only it has an arc from (y, x) to (y', x')). Thus we may assume that u, v are on the lowest level of P and Q. Let z be a vertex on the highest level of P, and let w be the corresponding vertex on Q. Let R be the walk obtained by following Q from v to w and then following Q^{-1} back from w to v. Let the path S be the common preimage of P, Q, and R, obtained by applying Lemma 2.1 twice, since P, Q, R consist of two constricted pieces. Let f be the corresponding homomorphism of S to P, let g be the corresponding homomorphism of S to Q, and let h be the corresponding homomorphism of S to R. We define the cost of mapping an internal vertex j of S to a vertex i of H as 0 if $i \in \{f(j), g(j), h(j)\}$; the cost of mapping the first and the last vertex of S to v is 1 and to u is 0. In all other cases the cost is $|V(S)|$. Note that there is no homomorphism from S to H which maps both the beginning and the end of S to u of total cost smaller than $|V(S)|$, as otherwise there would be a faithful arc from P to Q. Now by applying Lemma 3.1 to S and f, g, h we conclude that MinHOM(H) is NP-complete. ∎

COROLLARY 3.6. *Theorem* 2.5 *holds for balanced digraphs* H.

Specifically, for a balanced digraph H *the problem MinHOM*(H) *is polynomial time solvable if* H *has a Min-Max ordering, and is NP-complete otherwise.*

We observe that the same proof also applies for an unbalanced digraph H as long as P (and hence Q) has net length zero. Specifically, if *any* digraph H has a symmetrically invertible pair u, v with corresponding walks P, Q which have net length zero, then MinHOM(H) is NP-complete.

We now focus on unbalanced digraphs H.

THEOREM 3.7. *Suppose* H *is weakly connected and contains two induced cycles* C_1, C_2, *with net lengths* $k, n > 0, k \neq n$.

Then MinHOM(H) is NP-complete.

Proof. Suppose $k > n$, so k does not divide n. We may assume that H is minimal, in the sense that no weakly connected subgraph H' of H with fewer vertices contains two induced cycles with different nonzero net lengths. Indeed, if H' were such a subgraph, then MinHOM(H') would be polynomially reduced to MinHOM(H) by setting the cost of mapping to vertices of H not in H' to be very high.

Each cycle $C_i, i = 1, 2$, contains a vertex u_i such that the walk starting in u_i and following C_i (in the positive direction) is constricted from below. Let U be a walk in H from u_1 to u_2, and let u be a vertex on U of minimum height. By minimality, we may assume $V(H) = V(C_1) \cup V(C_2) \cup U$. Let $P_i, i = 1, 2$, be the walk from u to u_i following U (or U^{-1}), then once around C_i (in the positive direction), and then back from u following U^{-1} (or U). It follows that each P_i is constricted from below. The net length of P_1 is k and the net length of P_2 is n. Let Q_i, $i = 1, 2$, be the infinite walk starting at u obtained by repeatedly concatenating P_i, and let Q'_i be the two-way infinite walk obtained by expanding Q_i in the opposite direction by repeatedly concatenating P_i^{-1}.

Let d be greatest common divisor of n and k, and let $a = k/d - 2$. Thus $(a+2)n$ is the smallest positive common multiple of n and k. We now define the following three walks W_1, W_2, W_3 in H of net length $(a+1)n$:

1. The walk W_1 starts at u and follows Q_1 going around P_1 until the last vertex v such that the net length of the resulting walk is $(a+1)n$.
2. W_2 also starts at u and follows Q_2 going around P_2 fully $(a+1)$ times, ending at u.
3. W_3 starts at v and follows P_1 until the first occurrence of u, and then continues a times around P_2, ending again at u.

Now we define, in analogy with Q_1, Q_2, also the infinite walk Q_3, obtained from W_3 by continuing to go around P_2. Because we chose v to be the last vertex on Q_1 with the right net length, the walk W_3 is constricted from below; of course W_1, W_2 are also constricted from below. Hence Q_1, Q_2, Q_3 are also constricted from below; they have infinite heights because C_1, C_2 have positive net length. Thus we can apply Corollary 2.2 to Q_1, Q_2, Q_3, obtaining a common preimage which is a path S, say, $s = s_0, s_1, \ldots, s_q = t$, with homomorphisms f, g, h of S to Q_1, Q_2, Q_3, respectively, such that

1. $f(s) = u, f(t) = v$,
2. $g(s) = g(t) = u$,
3. $h(s) = v, h(t) = u$.

Note that the walk W'_1 equal to $u = f(s_0), f(s_1), \ldots, f(s_q) = v$, the walk W'_2 equal to $v = g(s_0), g(s_1), \ldots, g(s_q) = u$, and the walk W'_3 equal to $v = h(s_0), h(s_1), \ldots, h(s_q) = u$ are congruent.

Assume first that W_1', W_3' do not avoid each other; i.e., for some i we have both the faithful arcs (forward or backward) $f(s_i)h(s_{i+1})$, $h(s_i)f(s_{i+1})$. Note that $W_1' \cup W_2' \cup W_3'$ contains all the vertices of H, so the minimality of H easily implies that all four vertices $f(s_i), h(s_i), f(s_{i+1}), h(s_{i+1})$ must belong to $C_1 \cup C_2$. Since the cycles are induced, we must have two vertices in each cycle. Up to symmetry, we may assume we have forward arcs $ab \in C_1$ and $cd \in C_2$, as well as forward arcs ad, cb in H. Then, say, $a = f(s_i)$, $b = f(s_{i+1})$, $c = h(s_i)$, $d = h(s_{i+1})$.

We first claim that C_1, C_2 do not have common vertices, or arcs joining them other than ad, cb. Otherwise, let x on C_1 be the first vertex following b in the direction opposite to a, equal to or adjacent with some y on C_2, and assume that y is the first vertex of C_2 following d, in the direction opposite to c, adjacent to x. Consider the cycle D_1 with arcs ab, ad, xy; the portion of C_1 between b and x not containing a; and the portion of C_2 between d and y not containing c. Also consider the cycle D_2 with arcs cb, cd, xy, and the same portions of C_1, C_2. The cycles D_1, D_2 have the same net length m. If m is not zero and not k, we could delete c and obtain a smaller weakly connected H' with two different nonzero net lengths. If m is not zero and not n, we could likewise delete a. Thus $m = 0$. If x has no neighbors on C_2 other than y, then consider instead of D_2 the cycle D_2' obtained from D_2 by replacing the portion of C_2 between c and y containing d by the portion of C_2 between c and y not containing d. Since $m = 0$, the net length of D_2' is n, so we can delete d and obtain a smaller weakly connected H' with two different nonzero net lengths. Otherwise, let $y_1, y_2, \ldots, y_p$ be all the neighbors of x on C_2 after $y = y_0$, numbered consecutively in the direction from y to c, away from d. Consider the cycles Y_i containing x, y_i, y_{i+1} and the segment of C_2 between y_i and y_{i+1} not containing d. Each Y_i is an induced cycle in H, and the sum of their net lengths is n. Hence at least one Y_i has a nonzero net length and we similarly obtain a contradiction with the minimality of H.

Thus H consists of C_1, C_2, and the two extra arcs (forward or backward) ad, cb; in particular $u \in C_1 \cup C_2$, and the path U uses ad or bc. Without loss of generality, we may assume that it uses bc, since we can replace ad by ab, bc, cd. Suppose first that $u \in C_1$, whence we also have $v \in C_1$. Consider the initial portion of W_1' from v to $b = f(s_{i+1})$: it has net length equal to a multiple of k (corresponding to going full rounds around the cycle C_1) plus the net length of the portion X_1 of C_1 (in the positive direction) from u to b. Consider next the initial portion of W_3' from v to c followed by the arc joining c and b: it has net length equal to n (corresponding to going from v to u, which must precede $c \in C_2$) plus a multiple of n (corresponding to going full rounds around the closed walk P_2 from u to u) plus the net length of the portion X_2 of P_2 (in the positive direction) from u to c concatenated with the arc joining c and b. However, from u to c we must use the arc joining b and c. Thus X_2 uses the arc joining b and c first in one direction and then in the opposite direction, whence the net lengths of X_1, X_2 are the same. This means that a multiple of n, smaller than $(a+2)n$ is also a multiple of k, which is impossible, by our choice of a.

It remains to consider the case when W_1', W_3' avoid each other. We now assume that of all homomorphisms f, g, h of S to Q_1, Q_2, Q_3 satisfying properties (1, 2, 3) and such that the resulting walks W_1', W_3' avoid each other, we have chosen ones that maximize the number of vertices with $f(s_i) = g(s_i)$ or $g(s_i) = h(s_i)$.

If W_1', W_3' have at least some faithful arcs, then Corollary 3.2 and Lemma 3.4 imply MinHOM(H) is NP-complete. Thus we may assume that there are no faithful arcs between W_1' and W_3'.

We now define the costs of mapping vertices x of S to vertices j of H as follows: $c_j(x) = 2$ except for $c_u(s) = c_u(t) = 1$, $c_v(s) = c_v(t) = 0$, and $c_j(s_i) = 0$ when $j \in \{f(s_i), g(s_i), h(s_i)\}$, $j \neq u$.

By properties 1, 2, 3, we see that to apply Lemma 3.1 it remains to show that there is no homomorphism of S to H of cost less than 2, taking both s and t to v. Suppose, for a contradiction, that there is such a homomorphism ϕ. Then we must have $\phi(s_0) = h(s_0)$, $\phi(s_q) = f(s_q)$, and each $\phi(s_i) \in \{f(s_i), g(s_i), h(s_i)\}$. Since there are no faithful arcs between W_1' and W_3', we cannot have $h(s_i)$ and $f(s_{i+1})$ adjacent. Thus, because of the costs, some $h(s_i)$ and $g(s_{i+1})$ must be adjacent, and also some $g(s_j)$ and $f(s_{j+1})$ must be adjacent, with $i < j$. We now claim that this contradicts the maximality of f, g, h. Indeed, we could redefine f to equal g up to s_j (and then, continuing as before, taking advantage of the arc joining $g(s_j)$ and $f(s_{j+1})$), obtaining a new W_1' with at least one more vertex (namely s_{i+1}) having equality of f and g. (We need to observe that the new W_1' still avoids W_3', which also follows by maximality of f, g, h: there cannot be an arc between $g(s_p) \neq h(s_p)$ and $h(s_{p+1})$.) ∎

From the theorem we also derive the following corollary that will complete the proof of Theorem 2.5.

THEOREM 3.8. *Suppose H is a digraph containing an induced cycle of net length $k > 0$. If H is homomorphic to $\vec{C}_k$ and contains a symmetrically invertible pair u, v with u, v in the same set V_i, then MinHOM(H) is NP-complete.*

Proof. Recall that P is a walk from u to v and that Q is a congruent walk with P from v to u. Choose a homomorphism $\ell : H \to \vec{C}_k$, and note that $\ell(u) = \ell(v)$. It follows that the net length of P (and of Q) is divisible by k. If there are faithful arcs from P to Q or from Q to P, then by Corollary 3.2 or Lemma 3.4, MinHOM(H) is NP-complete. So we may assume that there are no such faithful arcs. We may also assume that the net length of P is greater than zero, as otherwise the remark following Corollary 3.6 implies that MinHOM(H) is NP-complete. We now proceed to find congruent walks from u to v and from v to u that avoid each other, and another congruent walk from u to u, so that we can apply Lemma 3.1 in a fashion similar to what was done in the proof of Theorem 3.7.

We may assume that P is constricted from below, as otherwise we replace u, v by vertices $u' \in P$, $v' \in Q$, where u' is a vertex of P with the minimum height, and v' is the corresponding vertex of v' in Q. We have observed that u', v' is also a symmetrically k-invertible pair; thus there are walks P' from u' to v' and Q' from v' to Q' that avoid each other. It is easy to see that the minimality of u' implies that this new P' is constricted from below. Let C be a walk in H from u to a cycle of net length k, followed by going around the cycle once in the positive direction and then returning back on the same walk to u. Note that the net length of this walk is k. We may again assume that C is constricted from below, as otherwise instead of P, Q we could use P_1, Q_1, where P_1 is obtained by concatenating P with $(QP)^a$ and Q_1 is obtained by concatenating Q with $(PQ)^a$ for some positive a, such that the walk from u (at the beginning of P_1) to the $(a-1)$th appearance of u in P_1 followed by C is a walk constricted from below.

Let the net length of P be ℓk, with $\ell > 0$. Let W be the infinite walk obtained by repeatedly concatenating C; note that W is constricted from below. Let P' be the infinite walk obtained by concatenating P with infinitely many repetitions of QP. Let Q' be the infinite walk congruent to P' obtained by similarly concatenating Q with repetitions of PQ. Let C' be the walk in W, from u to a vertex u' that is the ℓth occurrence of u in W. Now we apply Corollary 2.2 to obtain a path $S = s_0, s_1, \ldots, s_t$ which is the common preimage of P, C', Q. In this application, we use P', W, Q' as the infinite walks and use the ends of P, C', Q as the vertices q_i. (Note that P, C', Q all have net length ℓk.) Corollary 2.2 also yields homomorphisms f, g, h of S to P', W, Q' taking s_0 to the beginnings of P', W, Q' (also to the beginnings of P, C', Q), and taking s_t to the ends of P, C', Q. Let P'' be the walk $f(s_0), f(s_1), \ldots, f(s_t)$, let Q'' be the walk $h(s_0), h(s_1), \ldots, h(s_t)$, and let C'' be the walk $g(s_0), g(s_1), \ldots, g(s_t)$. Observe that P'', Q'' avoid each other, and between the walks P'', Q'' there are no faithful arcs, because that was the case for P, Q.

Note that $f(s_0) = u$ and $f(s_t) = v$, $g(s_0) = g(s_t) = u$, and $h(s_0) = v, h(s_t) = u$. We define the costs as follows: $c_u(s_0) = c_u(s_t) = 1$, and $c_v(s_0) = c_v(s_t) = 0$, and $c_i(x) = 0$ when $i \in \{f(x), g(x), h(y)\}$, $x \neq u$. For any other case the cost is $|V(S)|$.

We now conclude the proof as in Theorem 3.7, assuming that the homomorphisms f, g, h of S to $V(P'') \cup V(C'') \cup V(Q'')$ satisfy properties 1, 2, 3, and maximize the number of vertices with $f(s_i) = g(s_i)$ or $g(s_i) = h(s_i)$. ∎

We are finally ready to conclude the Proof of Theorem 2.5, i.e., to prove Gutin's conjecture [17].

Recall that the polynomial case of the theorem has been established in [17]. For the NP-completeness claim, the case when H is balanced is handled by Corollary 3.6. Thus we may assume that H has an induced cycle of some positive net length k. It is a well-known fact (e.g., Corollary 1.17 in [23]) that H has a homomorphism to $\vec{C}_k$ if and only if it does not contain a closed walk of net length not divisible by k. Suppose first that H does not admit a homomorphism to $\vec{C}_k$. Then the above fact implies that H contains an induced cycle of net length not divisible by k. Hence the problem MinHOM(H) is NP-complete by Theorem 3.7. If, on the other hand, H does admit a homomorphism to $\vec{C}_k$, with a symmetrically invertible pair u, v from the same set V_i, then MinHOM(H) is NP-complete by Theorem 3.8. This completes the proof. ∎

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