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MONOTONE PROPER INTERVAL DIGRAPHS AND MIN-MAX ORDERINGS*

PAVOL HELL† AND ARASH RAFIEY‡

Abstract. We introduce a class of digraphs analogous to proper interval graphs and bigraphs. They are defined via a geometric representation by two inclusion-free families of intervals satisfying a certain monotonicity condition; hence we call them *monotone proper interval digraphs*. They admit a number of equivalent definitions, including an ordering characterization by so-called Min-Max orderings, and the existence of certain graph polymorphisms. Min-Max orderings arose in the study of minimum cost homomorphism problems: if H admits a a Min-Max ordering (or a certain extension of Min-Max orderings), then the minimum cost homomorphism problem to H is known to admit a polynomial time algorithm. We give a forbidden structure characterization of monotone proper interval digraphs, which implies a polynomial time recognition algorithm. This characterizes digraphs with a Min-Max ordering; we also similarly characterize digraphs with an extended Min-Max ordering. In a companion paper, we shall apply this latter characterization to derive a conjectured dichotomy classification for the minimum cost homomorphism problems—namely, we shall prove that the minimum cost homomorphism problem to a digraph that does not admit an extended Min-Max ordering is NP-complete.

Key words. interval digraphs, Min-Max orderings, forbidden structure characterizations, minimum cost homomorphisms

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1. Introduction.

A graph H is an *interval graph* if it admits an interval representation, where an *interval representation* of H is a family of closed intervals $I_v, v \in V(H)$, such that u and v are adjacent in H if and only if I_u intersects I_v. (We take this definition to apply in the case $u = v$ as well, and hence an interval graph H is automatically *reflexive*, i.e., each vertex is adjacent to itself, via a *loop*.) If I is a closed interval, we denote by $\ell(I)$ (respectively, $r(I)$) the left (respectively, right) endpoint of I. Note that intervals I and I' intersect if and only if $r(I) \geq \ell(I')$ and $r(I') \geq \ell(I)$.

The study of interval graphs is one of the most beautiful and popular parts of graph theory, with interesting applications [12], elegant characterization theorems [31, 10], and ingenious and efficient recognition algorithms [2, 19, 5]. When it comes to digraphs, there is a consensus that instead of one family of intervals, one needs two families, one for the out-neighborhoods, and one for the in-neighborhoods [6, 33, 34]. A digraph H is an *interval digraph* if it admits a bi-interval representation, where a *bi-interval representation* consists of two families of closed intervals $I_v, v \in V(H)$, and $J_v, v \in V(H)$, such that uv is an arc of H if and only if I_u intersects J_v. This concept has been investigated [34], but the results do not seem to have the appeal

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†School of Computing Science, Simon Fraser University, Burnaby, BC V5A1S6, Canada (pavol@sfu.ca). This author was supported by an NSERC Canada discovery grant.
‡Informatics Department, University of Bergen, N-5020 Bergen, Norway (arash.rafiey@ii.uib.no). This author was supported by the NSERC Canada discovery grant of the first author; this author was also partially supported by ERC advanced grant PREPROCESSING 267959. The facilities of IRMACS are gratefully acknowledged.

of interval graphs—for instance, there is no known forbidden structure characterization analogous to [31], and the most efficient known algorithm has complexity $O(nm^6(n+m)\log n)$ [32]. In the special case of reflexive digraphs, the authors of [9] have proposed another class of interest: an interval digraph is called an *adjusted interval digraph* if it admits a bi-interval representation $I_v, J_v, v \in V(H)$, in which each pair of corresponding intervals I_v and J_v has the same left endpoint, $\ell(I_v) = \ell(J_v)$ for all $v \in V(H)$. Such a bi-interval representation is called *adjusted*. For adjusted interval digraphs, one can prove a forbidden structure characterization, which implies a faster recognition algorithm [9]. The clue that this class may be a better analogue of interval graphs came from the fact that interval graphs and adjusted interval digraphs have the same ordering characterization, by the so-called Min ordering [8, 9].

A linear ordering $<$ of $V(H)$ is a *Min ordering* of the digraph H if it satisfies the following property: if $u < w$ and $z < v$ and both uv, wz are arcs of H, then uz is also an arc of H. It is shown in [9] that a reflexive digraph H has a Min ordering if and only if it is an adjusted interval digraph. If we interpret a graph as a digraph by replacing each edge uv by the two opposite arcs uv, vu, then a reflexive graph has a Min ordering if and only if it is an interval graph [7]. Thus the concept of a Min ordering suggests that, for reflexive digraphs, one natural generalization of the class of interval graphs is the class of adjusted interval digraphs. For *general digraphs* (where possibly some vertices have loops and others do not) with a Min ordering, there is no known geometric representation, forbidden structure characterization, or polynomial time recognition algorithm. It would be very interesting to find such results if they exist. We note that for structures with two binary relations (digraphs with two kinds of arcs), the recognition problem of having a Min ordering is NP-complete, via a reduction (from a preliminary version of [1]) similar to that in the proof of Theorem 4.9.

Thus for general digraphs it is not clear what should be the best generalization of interval graphs. However, we have had more success generalizing the notion of proper interval graphs. In this case, we propose here an extension to general digraphs, for which we obtain a natural geometric representation, a forbidden structure characterization, and a polynomial time recognition algorithm.

An interval graph H is a *proper interval graph* if it admits an interval representation $I_v, v \in V(H)$, which is inclusion-free (no I_v is contained in I_w with $v \neq w$). Note that proper interval graphs are also reflexive by definition. Taking a cue from the above example of interval graphs, we seek an ordering characterization of proper interval graphs that can be applied to more general classes of digraphs. A linear ordering $<$ of $V(H)$ is a *Min-Max ordering* of the digraph H if it satisfies the following *Min-Max property*: if $u < w$ and $z < v$ and both uv, wz are arcs of H, then uz and wv are also arcs of H. A reflexive graph H has a Min-Max ordering if and only if it is a proper interval graph; cf. [14]. (We again interpret the graph H as a digraph by replacing each edge uv by the two opposite arcs uv, vu.)

This suggests a digraph analogue of proper interval graphs, namely, those digraphs that have a Min-Max ordering. It turns out that this concept does correspond to a natural class of interval digraphs. In analogy with the case of Min orderings described above, we first focus on the case of reflexive digraphs H: we say H is an *adjusted proper interval digraph* if it has a bi-interval representation by two inclusion-free families that are adjusted. It will follow from our results that a reflexive digraph H has a Min-Max ordering if and only if it is an adjusted proper interval digraph.

Consider now general digraphs H (where possibly some vertices have a loop and others do not). We will still be able to characterize the existence of a Min-Max

ordering by the possibility of a certain interval bi-representation. Before describing it, we wish to point out that in general it is convenient to allow some of the intervals $I_v, v \in V(H)$, or $J_v, v \in V(H)$, to be empty (if v has zero out-degree or zero in-degree). To avoid technicalities, we formally define a family that may contain empty members to be *inclusion-free* if there is no inclusion relationship between any two nonempty members of the family. Note that a family $I_v, v \in V(H)$, is inclusion-free if and only if we have $\ell(I_u) < \ell(I_w)$ if and only if $r(I_u) < r(I_w)$, for all nonempty pairs of intervals $I_u, I_w, u \neq w$ (and similarly for a family $J_v, v \in V(H)$).

We define two families $I_v, J_v, v \in V(H)$ to be *monotone* if the following condition is satisfied for any two pairs of nonempty intervals I_v, J_v, I_w, J_w:

- $\ell(I_u) < \ell(I_w)$ if and only if $\ell(J_u) < \ell(J_w)$.

We say a digraph H is a *monotone proper interval digraph* if it has a bi-interval representation by two inclusion-free families that are monotone. Note that a bi-interval representation that is adjusted is also monotone, and thus every reflexive adjusted proper interval digraph is a monotone proper interval digraph. We will show that a digraph H has a Min-Max ordering if and only if it is a monotone proper interval digraph.

It turns out that monotone proper interval digraphs also generalize another useful class of graphs. A *bigraph* is a bipartite graph H with a fixed bipartition of vertices $V(H) = B \cup W$. A bigraph H is a *proper interval bigraph* if it admits a representation by two inclusion-free families of closed intervals $I_v, v \in B$, $J_w, w \in W$, such that $v \in B$ and $w \in W$ are adjacent in H if and only if I_u intersects J_v. Note that a proper interval bigraph is *irreflexive*, i.e., no vertex has a loop. We shall view a bigraph as a digraph by orienting every edge $vw, v \in B, w \in W$, as an arc vw, i.e., oriented from v to w. Under such interpretation, it turns out that a bigraph H is a proper interval bigraph if and only if it (viewed as a digraph) has a Min-Max ordering [36] (cf. [14]), i.e., if and only if it is a monotone proper interval digraph. Thus the class of monotone proper interval digraphs generalizes both proper interval graphs and proper interval bigraphs.

Proper interval graphs (and bigraphs) are characterized by simple forbidden structures and recognized in polynomial time [35]; cf. [14, 22]. In this paper, we give a forbidden structure characterization of monotone proper interval digraphs. Our characterization implies forbidden structure theorems for both proper interval graphs and proper interval bigraphs. The characterization also leads to a polynomial time recognition algorithm for these digraphs. Thus it appears that the class of monotone proper interval digraphs is a sensible generalization of proper interval graphs and bigraphs.

We give a similar characterization of digraphs which admit certain extended Min-Max orderings, of interest in minimum cost homomorphism problems. A *homomorphism* of a digraph G to a digraph H is a mapping $f : V(G) \to V(H)$ such that $xy \in A(G)$ implies $f(x)f(y) \in A(H)$. The *minimum cost homomorphism problem* for H, denoted MinHOM(H), asks whether an input digraph G, with integer costs $c_i(u)$, $u \in V(G)$, $i \in V(H)$, and an integer k, admits a homomorphism to H of total cost $\sum_{u \in V(G)} c_{f(u)}(u)$ not exceeding k. The problem MinHOM(H) was first formulated in [18]; it unifies and generalizes several other problems [21, 27, 29, 30, 37]. The complexity of the problem MinHOM(H) for undirected graphs was classified in [14]: it is polynomial if each component of H is a reflexive interval graph or an irreflexive interval bigraph. A simple dichotomy classification of MinHOM(H) for reflexive digraphs can be found in [13]. It follows from [14, 13] that both for symmetric digraphs (undirected graphs) and for reflexive digraphs, MinHOM(H) is polynomial time solv-

able if H admits a Min-Max ordering and is NP-complete otherwise. This is not the case for general digraphs, as certain extended Min-Max orderings (defined in a later section) also imply a polynomial time algorithm [16]. However, it was conjectured by Gutin, Rafiey, and Yeo [16] that MinHOM(H) is NP-complete unless H admits an extended Min-Max ordering. Several special cases of the conjecture have been verified [13, 14, 15, 16, 17]. In a companion paper, we apply our characterization of digraphs with extended Min-Max ordering to prove the conjecture of [16].

2. Monotone proper interval digraphs.

THEOREM 2.1. *A digraph H admits a Min-Max ordering if and only if it is a monotone proper interval digraph.*

Proof. Suppose first that $<$ is a Min-Max ordering of the vertices of H. We define a monotone proper bi-interval representation of H as follows. We first lay out points p_v corresponding to the vertices $v \in V(H)$ at the integers $1, 2, \ldots, n = |V(H)|$ on the real line in the order corresponding to $<$. (We shall use $<$ for both the Min-Max ordering of vertices and the order of the corresponding points on the real line, as this will cause no confusion; we also sometimes don't distinguish between a vertex v and the corresponding point p_v.) We shall now construct intervals $I_v, v \in V(H)$: for each $v \in V(H)$, we consider the set S_v of points p_w where w is an out-neighbor of v in H not preceding v in $<$. (Note that S_v would include p_v if v has a loop in H.) If $S_v \neq \emptyset$, we let s_v be the minimum element of S_v and s'_v the maximum element of S_v, and we let I_v be the closed interval $[s_v, s'_v]$. If v has out-neighbors but they all precede v in $<$, let q_v be the rightmost out-neighbor of v and consider the point $p = p_{q_v}$. We let I_v be the singleton closed interval $[p, p]$. In the final case when v has no out-neighbors, we let $I_v = \emptyset$.

It follows from these definitions and the Min-Max property of $<$ that $v < w$ implies $\ell(I_v) \leq \ell(I_w)$ and $r(I_v) \leq r(I_w)$ if $I_v, I_w \neq \emptyset$. Indeed, if $v < w$ and $\ell(I_v) = s_v > \ell(I_w) = s_w$, then the Min-Max property for vs_v, ws_w implies that $vs_w \in A(H)$, contradicting the definition of s_v. If $v < w$ and $\ell(I_v) = q_v > \ell(I_w) = q_w$, then the Min-Max property for vq_v, wq_w implies that $wq_v \in A(H)$, contradicting the definition of q_w. Finally, if $v < w$ and $\ell(I_v) = s_v > \ell(I_w) = q_w$, then we obtain $ws_v \in A(H)$, contradicting the definition of $\ell(I_w)$. A similar analysis shows that if $v < w$ and $r(I_v) > r(I_w)$, then $vr(I_v), wr(I_w)$ implies that $wr(I_v) \in A(H)$, contradicting the definition of $r(I_w)$.

Next, we define the intervals $J_v, v \in V(H)$, as follows. Let T_v consist of all points p_w where w is an in-neighbor of v in H, not preceding v in $<$. If $T_v \neq \emptyset$, let t'_v be the maximum element of T_v, and let $t''_v = \max(p_v, \ell(I_{t'_v}))$. We let J_v be the closed interval $[p_v, t''_v]$. (Note that $t''_v = p_v$ if t'_v has no out-neighbors after p_v.) If $T_v = \emptyset$, then we let J_v be the singleton closed interval $[p_v, p_v]$ if v has any in-neighbors at all, and $J_v = \emptyset$ otherwise. Note that each $\ell(J_v) = p_v$, and thus $v < w$ implies $\ell(J_v) < \ell(J_w)$. Now we claim $v < w$ implies $r(J_v) \leq r(J_w)$. Indeed, if $T_v = \emptyset$, then $J_v = [p_v, p_v]$ and $p_v < p_w \leq r(J_w)$. So suppose $T_v \neq \emptyset$. If we also have $T_w \neq \emptyset$, then the Min-Max property on $t'_v v, t'_w w$ implies that $t'_v \leq t'_w$; thus $r(J_v) = \ell(I_{t'_v}) \leq \ell(I_{t'_w}) = r(J_w)$. In the last case when $T_v \neq \emptyset, T_w = \emptyset$, we note that $t'_v < w$, since $t'_v \geq w$ would imply $t'_v w \in A(H)$ by the Min-Max property. We have $r(J_v) \leq r(J_w)$ unless $s_{t'_v} > w$, because $r(J_v) = \ell(I_{t'_v})$. If $s_{t'_v} > w$, then considering an arc zw with $z < w$, together with one of the arcs $t'_v v$ or $t'_v s_{t'_v}$, we conclude that $t'_v w \in A(H)$, contradicting the definition of $s_{t'_v}$.

We now have $\ell(I_v) \leq \ell(I_w)$ if and only if $r(I_v) \leq r(I_w)$; $\ell(J_v) < \ell(J_w)$ if and only if $r(J_v) \leq r(J_w)$; and $\ell(I_v) \leq \ell(I_w)$ if and only if $\ell(J_v) < \ell(J_w)$. It is well

known [12] that an interval representation can be perturbed so that all intervals have distinct endpoints. The same argument applies to bi-interval representations, and it is easy to see that the changes can be made so that each family $I_v, v \in V(H)$, $J_v, v \in V(H)$ is inclusion-free and the monotonicity holds. (It only requires extending suitable intervals a little to the left or right, without intersecting any new intervals, and ensuring that we have the stronger equivalences $\ell(I_v) < \ell(I_w)$ if and only if $r(I_v) < r(I_w)$; $\ell(J_v) < \ell(J_w)$ if and only if $r(J_v) < r(J_w)$; and $\ell(I_v) < \ell(I_w)$ if and only if $\ell(J_v) < \ell(J_w)$.)

It remains to show that $uv \in A(H)$ if and only if I_u meets J_v. Note first that $x \in I_x$ if and only if x has a loop in H and $x \notin J_x$ if and only if $J_x = \emptyset$. Therefore I_x meets J_x if x has a loop. On the other hand, if I_x meets J_x, then we must have both $S_x \neq \emptyset$ and $T_x \neq \emptyset$, hence $xs_x, t'_x x$ with $s_x \geq x, t'_x \geq x$; therefore x has a loop by the Min-Max property. This verifies the claim in the case $u = v$.

If $uv \in A(H)$ with $u < v$, then both I_u and J_v contain the point p_v. If $uv \in A(H)$ with $u > v$, then we have $t'_v \geq p_u$ and hence $r(J_v) = \ell(I_{t'_v}) \geq \ell(I_u)$, as well as $r(I_u) \geq p_v = \ell(J_v)$. Thus I_u and J_v intersect.

On the other hand, suppose that I_u and J_v intersect. If $S_u \neq \emptyset$, then I_u is the closed interval $[s_u, s'_u]$, or a small perturbation of it, which does not contain any $p_x < s_u$ (and $p_x > s'_u$). Suppose first that $v < u$. Then we must have $T_v \neq \emptyset$; let $z = t'_v$. If $z = u$ then $uv \in A(H)$ as required; if $z > u$, then the Min-Max property for us_u, zv implies that $uv \in A(H)$ again. Finally, if $z < u$, then $\ell(I_z) < \ell(I_u)$, which is impossible, since $r(J_v) = \ell(I_z)$ and J_v intersects I_u. Suppose next that $v > u$. Note that $p_v \leq s'_u$ since I_u and J_v intersect. If $T_v \neq \emptyset$, then the Min-Max property applied to $us'_u, t'_v v$ implies $uv \in A(H)$. If $T_v = \emptyset$, then since $J_v \neq \emptyset$, the vertex v has an in-neighbor w and $J_v = [p_v, p_v]$. So we have $s_u \leq p_v \leq s'_u$ and the Min-Max property on wv with us_u or us'_u implies that $uv \in A(H)$. If $S_u = \emptyset$, then I_u is a small extension of $[p_w, p_w]$, where w is rightmost out-neighbour of u. Then we must have $v \leq w < u$, in which case the above argument (for $v < u$) applies verbatim.

If H is a monotone proper interval digraph with a bi-interval representation $I_v, J_v, v \in V(H)$, we define the ordering $<$ as follows. For vertices v, w with nonempty I_v, I_w, we let $v < w$ if $\ell(I_v) < \ell(I_w)$. For vertices with nonempty J_v, J_w, we let $v < w$ if $\ell(J_v) < \ell(J_w)$. The monotonicity of the bi-interval representation implies that these definitions agree if they are both applicable. It remains to define $<$ for pairs of vertices v, w, where I_v and J_w are empty. Taking the transitive closure of the current $<$ defines $v < w$ if there exists a vertex u with nonempty I_u, J_u such that $v < u$ and $u < w$ (respectively, $w < v$ if there exists a vertex u with nonempty I_u, J_u such that $v < u$ and $u < w$). At this point we almost have a linear order; $<$ is a partial order in which all antichains have at most two elements: two vertices v, w are incomparable if and only if I_v and J_w are empty and for all other vertices u we have $u < v$ if and only if $u < w$. We choose one of the options $v < w$ or $w < v$ arbitrarily. It is easy to check that $<$ is a linear order. We now claim that $<$ is a Min-Max ordering of H. Indeed, suppose $u < w, z < v$ and $uv, wz \in A(H)$, i.e., $I_u \cap J_v \neq \emptyset$, $I_w \cap J_z \neq \emptyset$. (In particular, I_u, I_w, J_v, J_z are all nonempty.) Note that $r(I_u) \geq \ell(J_v)$ as $uv \in A(H)$ and $\ell(J_v) > \ell(J_z)$ since $v > z$; thus $r(I_u) \geq \ell(J_z)$. Similarly, $r(J_z) \geq \ell(I_w) > \ell(I_u)$, so we have that I_u, J_z intersect and $uz \in A(H)$. By a similar calculation, $wv \in A(H)$. $\square$

COROLLARY 2.2. *A reflexive digraph H admits a Min-Max ordering if and only if it is an adjusted proper interval digraph.*

Proof. For reflexive digraphs, the above proof transforms a Min-Max ordering into an adjusted family of intervals (as long as we take care always extending both I_v and J_v by the same distance to the left, during the perturbation stage). ∎

3. Min-Max orderings. We first remark that Min-Max orderings correspond to a particular type of lattice polymorphisms [3]. The *product* $G \times H$ of digraphs G and H has the vertex set $V(G) \times V(H)$ and there is an arc in $G \times H$ from (u, v) to (u', v') if G has an arc from u to u' and H has an arc from v to v'. The *power* H^k is recursively defined as $H^1 = H$ and $H^{k+1} = H \times H^k$. A *polymorphism* of H is a homomorphism $f : H^k \to H$, for some positive integer k. Polymorphisms are of interest in the solution of constraint satisfaction problems [4, 28]. We say that polymorphisms $f, g : H^2 \to H$ are *lattice polymorphisms* of H if each f and g is associative, commutative, and idempotent and if, moreover, f and g satisfy the absorption identities $f(u, g(u, v)) = g(u, f(u, v)) = u$. It is easy to see that the usual operations of minimum $f(u, v) = \min(u, v)$ and maximum $g(u, v) = \max(u, v)$, with respect to a fixed linear ordering $<$, are polymorphisms if and only if $<$ is a Min-Max ordering. It is also clear that they satisfy the lattice axioms. A polymorphism f is *conservative* if $f(a_1, a_2, \ldots, a_k) \in \{a_1, a_2, \ldots, a_k\}$. Clearly, min and max are both conservative. We now make the following observation.

THEOREM 3.1. *A digraph H admits a Min-Max ordering if and only if it admits conservative lattice polymorphisms f, g.*

Proof. To see that any conservative lattice polymorphisms f, g yield a Min-Max ordering, note first that for $u \neq v$ we must have $f(u, v) \neq g(u, v)$ because of the absorption identities. Thus we may define $u < v$ whenever $f(u, v) = u, g(u, v) = v$: associative and commutative laws imply transitivity of $<$, hence $<$ is a Min-Max ordering. ∎

If $uv \in A(H)$, we say that uv is an arc of H, or that uv is a *forward arc* of H; we also say that vu is a *backward arc* of H. In any event, we say that u, v are *adjacent* in H if uv is a forward or a backward arc of H. A *walk* in H is a sequence $P = x_0, x_1, \ldots, x_n$ of consecutively adjacent vertices of H; note that a walk has a designated first and last vertex. A *path* is a walk in which all x_i are distinct. A walk is *closed* if $x_0 = x_n$ and a *cycle* if all other x_i are distinct. A walk is *directed* if all arcs are forward. The *net length* of a walk is the number of forward arcs minus the number of backward arcs. A closed walk is *balanced* if it has net length zero; otherwise it is *unbalanced*. Note that in an unbalanced closed walk we may always choose a direction in which the net length is positive (or negative). A digraph is *balanced* if it does not contain an unbalanced closed walk (or equivalently an unbalanced cycle); otherwise it is *unbalanced*. It is easy to see that a digraph is balanced if and only if it admits a *labeling* of vertices by nonnegative integers so that each arc goes from some level i to the level $i + 1$. The *height* of H is the maximum net length of a walk in H. Note that an unbalanced digraph has infinite height, and the height of a balanced digraph is the greatest label in a nonnegative labeling in which some vertex has label zero.

For any walk $P = x_0, x_1, \ldots, x_n$ in H, we consider the *minimum height* of P to be the smallest net length of an initial subwalk $x_0, x_1, \ldots, x_i$, and the *maximum height* of P to be the greatest net length of an initial subwalk $x_0, x_1, \ldots, x_i$. Note that when $i = 0$, we obtain the trivial subwalk x_0 of net length zero, and when $i = n$, we obtain the entire walk P. We shall say that P is *constricted from below* if the minimum height of P is zero (no initial subwalk $x_0, x_1, \ldots, x_i$ has negative net length) and *constricted* if moreover the maximum height is the net length of P (no initial subwalk $x_0, x_1, \ldots, x_i$ has greater net length than $x_0, x_1, \ldots, x_n$). We also say

that P is *nearly constricted from below* if the net length of P is minus one, but all proper initial subwalks $x_0, x_1, \ldots, x_i$ with $i < n$ have nonnegative net length. It is easy to see that a walk which is nearly constricted from below can be partitioned into two constricted pieces, by dividing it at any vertex achieving the maximum height.

A vertex x of H is called *extremal* if every walk starting in x is constricted from below, i.e., there is no walk starting in x with negative net length. It is clear that a balanced digraph H contains extremal vertices (we can take any x from which starts a walk with net length equal to the height of H) and a weakly connected unbalanced digraph does not have extremal vertices (from any x we can find a walk of negative net length by going to an unbalanced cycle and then following it long enough in the negative direction). Moreover, in a weakly connected digraph H, any extremal vertex x is the beginning of a constricted walk of net length equal to the height of H.

For walks P from a to b, and Q from b to c, we denote by PQ the walk from a to c which is the concatenation of P and Q, and by P^{-1} the walk P traversed in the opposite direction, from b to a. We call P^{-1} the *reverse* of P. For a closed walk C, we denote by C^a the concatenation of C with itself a times.

A cycle of H is *induced* if H contains no other arcs on the vertices of the cycle. In particular, an induced cycle with more than one vertex does not contain a loop.

The following lemma is well known. (For a proof, see [20, 38] or Lemma 2.36 in [23].)

LEMMA 3.2. *Let P_1 and P_2 be two constricted walks of net length r. Then there is a constricted path P of net length r that admits a homomorphism f_1 to P_1 and a homomorphism f_2 to P_2 such that each f_i takes the starting vertex of P to the starting vertex of P_i and the ending vertex of P to the ending vertex of P_i.*

We shall call P a *common preimage* of P_1 and P_2.

Suppose $<$ is a Min-Max ordering of H. If $xx', yy' \in A(H)$ but $xy' \notin A(H)$, then $x' \neq y'$ and so $x < y$ implies $x' < y'$ (since otherwise $x < y, y' < x'$ would violate the Min-Max property). A similar situation arises if $xx', yy' \in A(H)$ but $yx' \notin A(H)$. In other words, if $xx' \in A(H)$ and $yy' \in A(H)$, but $xy' \notin A(H)$ or $yx' \notin A(H)$, then $x < y$ if and only if $x' < y'$. This observation suggests the next two definitions.

We define two walks $P = x_0, x_1, \ldots, x_n$ and $Q = y_0, y_1, \ldots, y_n$ in H to be *congruent* if they follow the same pattern of forward and backward arcs, i.e., $x_i x_{i+1}$ is a forward (backward) arc if and only if $y_i y_{i+1}$ is a forward (backward) arc (respectively). Suppose the walks P, Q as above are congruent. We say an arc $x_i y_{i+1}$ is *a faithful arc from P to Q* if it is a forward (backward) arc when $x_i x_{i+1}$ is a forward (backward) arc (respectively), and we say an arc $y_i x_{i+1}$ is *a faithful arc from Q to P* if it is a forward (backward) arc when $x_i x_{i+1}$ is a forward (backward) arc (respectively). We say that P, Q *avoid each other* if there is no pair of faithful arcs $x_i y_{i+1}$ from P to Q, and $y_i x_{i+1}$ from Q to P, for some $i = 0, 1, \ldots, n$.

We define the *pair digraph* H^* as follows. The vertices of H^* are all ordered pairs (x, y) of distinct vertices of H, and $(x, y)(x', y') \in A(H^*)$ just if both $xx' \in A(H)$ and $yy' \in A(H)$ but at least one of $xy' \notin A(H), yx' \notin A(H)$. (Either just one xy', yx' is in $A(H)$ or neither is in $A(H)$.) Note that in H^* we have an arc from (x, y) to (x', y') if and only if there is an arc from (y, x) to (y', x').

We have used similar auxiliary digraphs in the study of list homomorphisms [8, 9] and interval graphs, bigraphs, and digraphs; cf. [24, 25].

Note that two congruent walks P, Q in H that avoid each other (as defined above) yield a walk in the pair digraph H^* from (x_0, y_0) to (x_n, y_n), and conversely, any walk

in the pair digraph H^* corresponds to a pair of congruent walks in H that avoid each other. According to our observation, having a walk in H^* from (x_0, y_0) to (x_n, y_n) means that $x_0 < y_0$ if and only if $x_n < y_n$ in any Min-Max ordering $<$ of H. Thus all pairs (x, y) in one weak component of H^* have $x < y$, or all have $x > y$, in any Min-Max ordering of H.

For brevity, we shall from now call a weak component of a digraph just a *component*.

A *circuit* in H^* is a set of vertices $(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, x_0)$ of H^*. Note that if a component of H^* contains a circuit, then H cannot have a Min-Max ordering, since $x_0 < x_1$ implies $x_0 < x_1 < x_2 < \ldots < x_n < x_0$ (and similarly for $x_0 > x_1$), contradicting the transitivity of $<$. We have proved the following fact.

THEOREM 3.3. *If some component of the pair digraph H^* has a circuit, then H does not admit a Min-Max ordering.*

We now single out two particular situations in which a circuit occurs in one component of the pair digraph H^*. A *symmetrically invertible pair* in H is a pair of distinct vertices u, v such that there exist congruent walks, P from u to v and Q from v to u, that avoid each other. (We have previously used a similar notion of so-called invertible pair in [8, 9]; thus we distinguish this notion by adding the adjective "symmetrically.") Obviously, a symmetrically invertible pair in H corresponds precisely to a circuit with $n = 2$ in one component of H^*. Another situation is described in the next theorem.

THEOREM 3.4. *If H contains an induced cycle of net length greater than one, then some component of H^* contains a circuit.*

Proof. Indeed, suppose C is an induced cycle of net length $k > 1$, and let x_0 be a vertex of C in which we can start a walk P around C which is constricted from below. It is easy to see that such a vertex must exist; in fact, we may even assume that $P \setminus x_0$ is constricted from below. Then following P let x_i $(1 \leq i \leq k-1)$ be the last vertex on P such that the walk from x_0 to x_i has net length i. It is easy to see that $x_i, i = 0, 1, \ldots, k-1$, are all found in the first pass around C. We show that $(x_0, x_1), (x_1, x_2), \ldots, (x_{k-1}, x_0)$ belong to the same component of H^*. Let X be the portion of P from x_{i-1} to x_i and Y be the portion of P from x_i to x_{i+1}.

First suppose the height of X is not greater than the height of Y. Let h be the last vertex of Y with the maximum height. Let h' be the first vertex of Y after x_i such that the portion of Y from h' to h has net length one. Now the portion of P from x_{i-1} to h' and the portion of P from x_i to h are constricted and have the same height. Thus by Lemma 3.6 they have a common preimage A. Also the portion of P^{-1} from h' to x_i and the portion of P from h to x_{i+1} are constricted and have the same height; thus they also have a common preimage A'. The walk in C from x_{i-1} to x_i corresponding to AA' and the walk in C from x_i to x_{i+1} corresponding to AA' avoid each other, since C is an induced cycle. This implies that (x_{i-1}, x_i) to (x_i, x_{i+1}) are in the same component of H^*.

Now suppose the height of X is greater than the height of Y. Let h be the last vertex of P with the maximum height, and let h' be the first vertex of P after x_{i+1} such that the walk from h' to h has net length zero. Now the portion of P from x_{i-1} to h and the portion of P from x_i to h' are constricted and have the same height, yielding a common preimage A, by Lemma 3.2. The portion of X from h to x_i and the the portion of P^{-1} from h' to x_{i+1} are constricted and have the same height, yielding a common preimage, A'. The walk in C from x_{i-1} to x_i corresponding to AA' and the walk in C from x_i to x_{i+1} corresponding to AA' again avoid each other,

since C is an induced cycle. This also implies that (x_{i-1}, x_i) to (x_i, x_{i+1}) are in the same component of H^*. ∎

We are now ready to claim the converse of Theorem 3.3.

THEOREM 3.5. *A digraph H admits a Min-Max ordering if and only if no component of the pair digraph H^* has a circuit.*

Thus we shall assume that no component of H^* has a circuit; in fact, it will turn out to be sufficient to assume that the digraph H has no induced cycle of net length greater than one and no symmetrically invertible pair.

We shall frequently use the following key lemma.

LEMMA 3.6. *Let a, b, c be three vertices of H such that the component of H^* which contains (a, b) contains neither of $(a, c), (c, b)$.*

Let A, B, C be congruent walks starting at a, b, c, respectively.

If A and B avoid each other, then B and C also avoid each other, and A and C also avoid each other.

Proof. By symmetry, it suffices to prove the claim about B and C.

Suppose $A = a_1, a_2, \ldots, a_n$, $B = b_1, b_2, \ldots, b_n$, and $C = c_1, c_2, \ldots, c_n$ (here $a_1 = a$, $b_1 = b$, and $c_1 = c$). For a contradiction, suppose that B and C do not avoid each other, and let i be the least subscript such that both $b_i c_{i+1}$ and $c_i b_{i+1}$ are faithful arcs in H. (Note that i could be equal to $n-1$.)

Since (a, b) and (a, c) are not in the same component of H^*, the congruent walks

$$R = a_1, \ldots, a_i, a_{i+1}, a_i, \ldots, a_1 \quad \text{and} \quad S = b_1, \ldots, b_i, b_{i+1}, c_i, \ldots, c_1$$

do not avoid each other. Since A and B do avoid each other, any faithful arcs between R and S must be between $b_{i+1}, c_i, \ldots, c_1$ and $a_{i+1}, a_i, \ldots, a_1$. Suppose first there exists a subscript $j < i$ such that $a_j c_{j+1}$ and $c_j a_{j+1}$ are faithful arcs, and let j to be chosen as small as possible subject to this. Note that there is a second possibility, that $a_i b_{i+1}$ and $c_i a_{i+1}$ are the only faithful arcs. We think of this case as having $j = i$ with the understanding that c_{j+1} is replaced by b_{j+1}, and we will deal with it at the end of this proof.

Since (a, b) and (c, b) are not in the same component of H^*, the congruent walks

$$R' = a_1, \ldots, a_j, a_{j+1}, c_j, \ldots, c_1 \quad \text{and} \quad S' = b_1, \ldots, b_j, b_{j+1}, b_j, \ldots, b_1$$

do not avoid each other. Since A and B do avoid each other and since $j < i$ while i was chosen to be minimal, the faithful arcs must be $b_j a_{j+1}, c_j b_{j+1}$. Similarly, the congruent walks

$$R'' = a_1, \ldots, a_j, c_{j+1}, c_j, \ldots, c_1 \quad \text{and} \quad S'' = b_1, \ldots, b_j, b_{j+1}, b_j, \ldots, b_1$$

yield the faithful arcs $a_j b_{j+1}$ and $b_j c_{j+1}$— contradicting the fact that A, B avoid each other.

Returning now to the special case when $j = i$, we observe that we can use the same pair of walks R', S' as above and then modify the walks

$$R'' = a_1, \ldots, a_i, a_{i+1}, c_i, \ldots, c_1 \quad \text{and} \quad S'' = b_1, \ldots, b_i, c_{i+1}, b_i, \ldots, b_1$$

to conclude that $b_i a_{i+1}$ is again an arc, yielding the same contradiction. ∎

We note that two congruent paths which avoid each other cannot intersect; thus the lemma implies that B and T are disjoint.

We now formulate a corollary which will also be used frequently.

COROLLARY 3.7. *Let a, b, c be three vertices of H, such that the component of H^* which contains (a, b) contains neither of $(a, c), (c, b)$.*

Let A, B, C be three constricted walks of the same net length, starting at a, b, c, respectively. Suppose that A and B are congruent and avoid each other.

Then there exists congruent common preimages A', B', C' of A, B, C starting at a, b, c, respectively, such that B' and C' avoid each other, and A' and C' also avoid each other.

We note that Corollary 3.7 will sometimes be applied to walks that are not constricted but can be partitioned into constricted walks of corresponding net lengths, and then the corollary is applied to each piece separately.

Since H has no symmetrically invertible pairs, we conclude that if a pair (u, v) is in a component C of H^*, then the corresponding reversed pair (v, u) is in a different component $C' \neq C$ of H^*. Moreover, if any (x, y) also lies in C, then the corresponding reversed (y, x) must also lie in C', since reversing all pairs on a walk between (u, v) and (x, y) results in a walk between (v, u) and (y, x). Thus the components of H^* come in pairs C, C' so that the ordered pairs in C' are the reverses of the ordered pairs in C. We say the components C, C' are *dual* to each other.

4. The algorithm. We now introduce an algorithm to construct a Min-Max ordering $<$, proving Theorem 3.5. As mentioned above, it will be sufficient to assume that H has no induced cycle of net length greater than one and no invertible pair.

At each stage of the algorithm, some components of H^* have already been *chosen*. Whenever a component C of H^* is chosen, its dual component C' is *discarded*. The objective is to avoid a circuit

$$(a_0, a_1), (a_1, a_2), \ldots, (a_n, a_0)$$

of pairs belonging to the chosen components. Our algorithm always chooses a component X of maximum height from among the as yet unchosen and undiscarded components. If X creates a circuit, then the algorithm chooses the dual component X'. We shall show that at least one of X and X' will not create a circuit. (Note that this implies that the component X does not contain a circuit.) Thus at the end of the algorithm we have no circuit. The chosen components define a binary relation $<$ as follows: we set $a < b$ if the pair (a, b) belongs to one of the chosen components. Since there was no circuit amongst the chosen pairs, the relation $<$ is transitive and hence a total order. It is easy to see that $<$ is a Min-Max ordering. Indeed, if $i < j, s < r$ and $ir, js \in A(H)$ but $is \notin A(H)$ or $jr \notin A(H)$, then (i, j) and (r, s) are adjacent in H^* — hence we have either $i < j, r < s$ or $j < i, s < r$, contrary to what was supposed.

THEOREM 4.1. *The algorithm does not create a circuit consisting of pairs from the chosen components.*

Thus suppose that at a certain time T there was no circuit with pairs from the chosen components, that X had the maximum height from all unchosen (and undiscarded) components, and that the addition of X to the chosen components created the circuit $(a_0, a_1), (a_1, a_2), \ldots, (a_n, a_0)$, and the addition of the dual component X' created the circuit $(b_0, b_1), (b_1, b_2), \ldots, (b_m, b_0)$. We may suppose that T was minimum for which this occurs, then n was minimum value for this T, and then m was minimum value for this T and n. We may also assume that X contains the pairs $(a_n, a_0), (b_0, b_m)$ and possibly other pairs (a_i, a_{i+1}) or (b_j, b_{j+1}).

Let A_i be the component of H^* containing the pair (a_i, a_{i+1}) and let B_j be the component containing the pair (b_j, b_{j+1}); subscripts are modulo n and m, respectively.

(Thus $X = A_n = B'_m$.) Note that the minimality of n implies that no A_i contains a pair (a_k, a_ℓ) for subscripts (reduced modulo $n+1$) $\ell \neq k+1$ (and similarly for B_j). (This is helpful when checking the hypothesis of Lemma 3.6 and Corollary 3.7, as in Case 2 below.)

The following lemma is our basic tool.

LEMMA 4.2. *Suppose that none of the pairs (a_i, a_{i+1}) is extremal in its component A_i.*

Then there exists another circuit $(a'_0, a'_1), (a'_1, a'_2), \ldots, (a'_n, a'_0)$, where each (a'_i, a'_{i+1}) can be reached from the corresponding (a_i, a_{i+1}) by a walk in A_i nearly constricted from below.

Proof. Since (a_i, a_{i+1}) is not extremal, there exists a walk W_i in A_i from (a_i, a_{i+1}) to some (p_i, q_i), which is nearly constricted from below. Corresponding to this walk in A_i, there are two walks P_i and Q_i in H, from a_i to p_i and from a_{i+1} to q_i, respectively, which avoid each other. Let L_i be the maximum height of W_i (which is the same as that of P_i and Q_i).

We now explain how to choose n of the $2n$ vertices p_i, q_i which also form a circuit. For any i, instead of a_i, we choose $a'_i = q_{i-1}$ if $L_{i-1} < L_i$, and we choose $a'_i = p_i$ otherwise. We now show that $(a'_0, a'_1), (a'_1, a'_2), \ldots, (a'_n, a'_0)$ is a circuit; it suffices to show that each (a'_i, a'_{i+1}) is in A_i.

Case 1. Suppose $L_i \leq L_{i-1}$ and $L_i \leq L_{i+1}$.

In this case, we have $a'_i = p_i, a'_{i+1} = q_i$, and (p_i, q_i) is in A_i by definition.

Case 2. Suppose $L_i \geq L_{i-1}$ and $L_i \geq L_{i+1}$, and Case 1 does not happen.

In this case, we have $a'_i = q_{i-1}, a'_{i+1} = p_{i+1}$. We may assume that $L_{i+1} \leq L_{i-1}$ (otherwise the argument is symmetric). Consider the congruent walks $A = P_{i-1}$ from a_{i-1} to p_{i-1} and $B = Q_{i-1}$ from a_i to q_{i-1}. They are nearly constricted from below and have maximum height L_{i-1}. Consider the following walk C from a_{i+1} to p_{i+1}: the walk C starts with a portion of Q_i, up to the maximum height L_{i-1} and then back down to a_{i+1}, followed by P_{i+1}. Note that C is also nearly constricted from below and has the same maximum height L_{i-1}. It follows that A, B, C can each be partitioned into two constricted pieces of corresponding net lengths. Since $(a_{i-1}, a_{i+1}), (a_{i+1}, a_i) \notin A_{i-1}$ by the minimality of n, Corollary 3.7 (applied to each of the constricted pieces) implies that B and C avoid each other. Since $a'_i = q_{i-1}, a'_{i+1} = p_{i+1}$, we have a walk in H^* from (a_i, a_{i+1}) to (a'_i, a'_{i+1}), hence $(a'_i, a'_{i+1}) \in A_i$.

Case 3. Suppose $L_{i-1} < L_i < L_{i+1}$ (or $L_{i-1} > L_i > L_{i+1}$).

In this case, we have $a'_i = q_{i-1}, a'_{i+1} = q_i$. Since the subscripts are computed modulo $n+1$, there must exist a subscript s such that $L_s \geq L_i \geq L_{s+1}$. Now we again apply Corollary 3.7 to the walks $A = P_i, B = Q_i$, and C from a_{s+1} to p_{s+1} using P_{s+1} and a portion of Q_s, to conclude that C avoids B. Finally, we once more apply Lemma 3.6 to the three walks B, C, and D from a_i to $a'_i = q_{i-1}$ using Q_{i-1} and a portion of P_i, to conclude that D avoids B. Hence there is a walk in H^* from (a_i, a_{i+1}) to (a'_i, a'_{i+1}), implying that $(a'_i, a'_{i+1}) \in A_i$. ∎

We now continue with the proof of Theorem 4.1.

We distinguish two principal cases, depending on whether the component X is balanced.

We first assume that the component X is balanced.

Suppose the height of X is h.

LEMMA 4.3. *Suppose some (a_k, a_{k+1}) is extremal in A_k.*

Let $(a_i, a_{i+1}), (a_j, a_{j+1})$ be distinct nonextremal pairs in A_i, A_j, respectively, and let W_i, W_j be walks in A_i, A_j starting from $(a_i, a_{i+1}), (a_j, a_{j+1})$, respectively, that

are nearly constricted from below. Let L_i, L_j be the maximum heights of W_i, W_j, respectively.

Then $L_i > h$ or $L_j > h$.

Proof. Suppose $L_i \leq h, L_j \leq h$, and assume, without loss of generality, that $L_i \leq L_j$. Since some (a_k, a_{k+1}) is extremal, we may assume that neither (a_{i-1}, a_i), nor (a_{j+1}, a_{j+2}) initiates walks of negative net length with maximum height at most h. Thus each of (a_{i-1}, a_i), (a_{j+1}, a_{j+2}) either is extremal, and thus initiate a constricted walk of net length h, or initiates a walk of negative net length, with maximum height greater than h, and hence again initiates a constricted walk of net length h. Thus we have

- a constricted walk U_i of net length h from a_i;
- a walk V_i, nearly constricted from below, from a_i to some p;
- a constricted walk U_{j+1} of net length h from a_{j+1};
- a constricted walk U_{j+2} of net length h from a_{j+2}, which avoids U_{j+1} and is congruent to it;
- a walk V_j, nearly constricted from below, from a_j; and
- a walk V_{j+1}, nearly constricted from below, from a_{j+1} to some q, which avoids V_j and is congruent to it.

Consider the three walks A, B, C, where A is the reverse of V_{j+1} (starting in q), B is the reverse of V_j, and C is the reverse of V_i followed by a suitable piece of U_i (and its reverse) as needed to have the same maximum height L_j as V_j. Each of these walks consists of two constricted pieces and hence we can apply Corollary 3.7 twice to conclude that there exist congruent preimages A' and C' of A and C, respectively, which avoid each other. We can also apply Corollary 3.7 to the constricted walks U_{j+1}, U_{j+2}, U_i to conclude that there are congruent preimages A'', C'' of U_{j+1}, U_j, respectively, which avoid each other. Concatenating A' with A'' and C' with C'', we conclude that (p, q) belongs to a component of H^* which has height greater than h; this means that before X we should have chosen the component of H^* containing (a_i, a_j), which is a contradiction. ∎

We remark that the algorithm's choice of a component of maximum height is crucial in the above argument.

We will use the following analogue of Lemma 3.2 for infinite walks which are constricted in the infinite sense, i.e., are constricted from below and have infinite height.

COROLLARY 4.4. *Let P_1 and P_2 be two walks of infinite height, constricted from below. Assume that P_i starts in p_i, $i = 1, 2$, and let q_i be a vertex on P_i such that the infinite portion of P_i starting from q_i is also constricted from below and the portions of P_i from p_i to q_i have the same net length for $i = 1, 2$.*

Then there is a path P that admits homomorphisms f_i to P_i taking the starting vertex of P to p_i and the ending vertex of P to q_i for $i = 1, 2$.

Proof of the corollary. Let P_i' be the portion of P_i from p_i to q_i, and suppose, without loss of generality, that the height h of P_1' is greater than or equal to the height of P_2'. Let r_i be the first vertex after q_i (or equal to q_i) on P_i such that the net length from p_i to r_i is h. Let R_i be the subwalk of P_i from p_i to r_i. Now Lemma 3.2 implies that there is a path R with homomorphisms f_i to R_i taking the beginning of R to p_i and the end of R to r_i. Suppose x is the last vertex on P_1' with $f_1(x) = q_1$: if $f_2(x) = q_2$, we are done, so suppose $f_2(x) = y \neq q_2$. Now consider the subwalk Y of P_2' joining y and q_2: it has net length zero and is constricted from below, because the portion of R between x and the end of R has net length zero and is constricted from

below. Let h' be the height of Y, and let X be the walk on P_1' from q_1 to the first vertex making a net length h' and then back to q_1. Since X and Y have the same height and have net length zero, we can split them into two constricted pieces, and so Lemma 3.2 implies that there is a path R' which is a common preimage of X and Y. Concatenating R with R' yields a path P and we can extend the homomorphisms f_i to P so that also the ending vertex of P is taken to q_i for $i = 1, 2$.

LEMMA 4.5. *If any (a_i, a_{i+1}) is extremal in A_i, then (a_n, a_0) is extremal in $X = A_n$.*

Proof. Suppose (a_n, a_0) is not extremal. By Lemma 4.3, it remains to consider the case when both (a_0, a_1) and (a_{n-1}, a_n) are extremal. Since (a_0, a_1) is extremal, there exists a constricted walk in H^* starting from (a_0, a_1) of net length equal to the height of A_0, which is at least h, according to our algorithm. Similarly, there exists a constricted walk from (a_{n-1}, a_n) of net length equal to the height of A_{n-1}, which is also at least h. From the walk in A_{n-1}, we extract a constricted walk A starting in a_{n-1} and a congruent constricted walk B starting in a_n such that A, B have net length h and avoid each other. From the walk in A_0 we moreover extract a walk C starting in a_0 which is also constricted and has net length h. Now Corollary 3.7 ensures that B and C have congruent preimages B' and C' which avoid each other. Let B'', C'' be two congruent walks of negative net length from a_n, a_0, respectively, which avoid each other; such walks exist since (a_n, a_0) is not extremal. Now taking the concatenations of $(B'')^{-1}$ with B' and $(C'')^{-1}$ with C' yields a walk in X of net length greater than h, which is a contradiction. ∎

Thus Lemma 4.2 ensures that we may assume that (a_n, a_0) is extremal in X (and similarly for (b_0, b_m)). The proof now distinguishes whether X contains another pair (a_i, a_{i+1}) (or similarly for (b_j, b_{j+1})).

Suppose first that some $(a_i, a_{i+1}) \in X$, and let W be a walk from (a_n, a_0) to (a_i, a_{i+1}) in X. We observe that the net length of W must be zero. Indeed, since (a_n, a_0) is extremal in X, the net length of W must be nonnegative. If the net length were positive, then W^{-1} would be a walk from (a_i, a_{i+1}) of negative net length and with maximum height less than h. Thus Lemma 4.3 implies that both $(a_{i-1}, a_i), (a_{i+1}, a_{i+2})$ initiate walks of net length h, yielding walks $U_{i-1}, U_i, U_{i+1}, U_{i+2}$ of net length h, from $U_{i-1}, U_i, U_{i+1}, U_{i+2}$, respectively. Here U_{i-1}, U_i are congruent constricted walks that avoid each other, and hence Corollary 3.7 implies that there are preimages of U_i, U_{i+1} of net length h that are congruent and avoid each other. This yields a walk in X from (a_i, a_{i+1}) of net length h—and concatenated with W we obtain a walk in X from (a_n, a_0) of net length strictly greater than h, which is impossible.

Thus the net length of W is zero, and hence it can be partitioned into two constricted pieces, U from (a_n, a_0) to some vertex (z_1, z_2) of maximum height and V from (z_1, z_2) to (a_i, a_{i+1}). Let U_1 (respectively, U_2) denote the corresponding walk from a_n to z_1 (respectively, from a_0 to z_2), and similarly for V_1, V_2. Then Lemma 3.6 applied to U_1, U_2, V_2 implies that (z_1, z_2) and (a_n, a_{i+1}) are in the same component of H^*; however, $(z_1, z_2) \in X$, so $(a_n, a_{i+1}) \in X$, contrary to the minimality of n.

Thus we conclude that X does not contain another (a_i, a_{i+1}) or (b_j, b_{j+1}). In other words, before time T we have chosen all the pairs

$$(a_0, a_1), \ldots, (a_{n-1}, a_n), (b_0, b_1), \ldots, (b_{m-1}, b_m),$$

and then at time T we chose the component X containing (a_n, a_0) as well as (b_0, b_m). Consider a fixed walk W in X from (a_n, a_0) to (b_0, b_m). Since (a_n, a_0), and by symme-

try also (b_0, b_m), is extremal, W must have net length zero. Moreover, we may assume that W reaches some vertex (z_1, z_2) of maximum height h. Thus W consists of two constricted walks U, V. Let again U_1 (respectively, U_2) be the corresponding walk in H from a_n (respectively, from a_0) to a vertex of maximum height, and similarly let V_1 (respectively, V_2) be the corresponding walks from the vertices of maximum height to b_0 (respectively, b_m).

We shall prove first that there is a constricted walk of net length h from a_1. Indeed, the component A_0 containing the vertex (a_0, a_1) must have height at least h, according to the rules of our algorithm. If (a_0, a_1) does not initiate a walk of net length h, it must not be extremal, i.e., it must initiate a walk of negative net length. The same argument yields a walk of negative net length from (a_1, a_2). Since such walks contain walks that are nearly constricted from below, we obtain a contradiction with Lemma 4.3. A similar argument applies to b_1.

Thus there are constricted walks of net length h from both a_1 and b_1, say, R and S, respectively. We can now use Corollary 3.7 on the walks $A = U_1, B = U_2, C = R$, and again on the walks $A = V_1, B = V_2, C = R^{-1}$ to deduce that U_2 concatenated with V_2 and R concatenated with R^{-1} avoid each other, hence (a_0, a_1) and (b_m, a_1) are in the same component of H^*. By a similar argument we also deduce that (b_0, b_1) and (a_n, b_1) are also in the same component of H^*. This is impossible, as it would mean that at time $T - 1$ there already was a circuit, namely, $(b_m, a_1), (a_1, a_2), \ldots, (a_n, b_1), (b_1, b_2), \ldots, (b_{m-1}, b_m)$.

This completes the proof of Theorem 4.1 in the case where X is balanced.

We now assume the component X is unbalanced. In this case, the rules of the algorithm imply that each component A_i and B_j is also unbalanced.

We shall define a vertex u in an unbalanced digraph to be *weakly extremal* if there is a walk starting from u which is constricted from below and has infinite maximum height. Each cycle of positive net length, and hence each unbalanced digraph, contains a weakly extremal vertex.

Let D_i be a fixed unbalanced cycle in the component A_i for all i. We claim that we may assume without loss of generality that each pair (a_i, a_{i+1}) lies in D_i and it is weakly extremal. Indeed, if some (a_i, a_{i+1}) is not weakly extremal, then there is a walk from a weakly extremal vertex in D_i to (a_i, a_{i+1}) which is constricted from below. (Such a walk can be obtained by first following D_i in the positive direction as many time as necessary and then proceeding to (a_i, a_{i+1}).)

Observe now that if (a_j, a_{j+1}) already is weakly extremal, there still is such a walk, constricted from below, to (a_j, a_{j+1}). This walk may start from another weakly extremal vertex on D_j; note that in an induced cycle of net length r there are r weakly extremal vertices.Thus we can apply Lemma 4.2 as many times as needed until every (a_i, a_{i+1}) becomes weakly extremal on the corresponding D_i. Naturally, the same conclusion holds for the components B_j.

We now claim that we may assume that each D_i is an oriented cycle of net length one.

Since (a_i, a_{i+1}) is in D_i and it is weakly extremal, there is a closed walk Y in H, from a_i to itself, which is constricted from below. Note that some of the vertices in Y may be repeated. If the pth vertex of Y is the same as the qth vertex of Y, with $p < q$, and the portion of Y from the pth vertex to the qth vertex has net length zero, then we delete all vertices from the $(p+1)$st vertex to the $(q-1)$st vertex of Y, yielding a new walk which has the same net length as Y. We repeat this process until such a situation no longer occurs, obtaining a final walk V.

Suppose now the pth vertex and the qth vertex of V, with $p < q$, are the same, and no vertex between them is repeated. Suppose first that the portion of V from the pth vertex to the qth vertex has net length one. Then we obtain a walk W in H from a_i to a_i, constricted below and of net length one as follows. We walk from a_i to the pth vertex of V along some walk S, then follow V to the qth vertex, and then walk back to a_i along S^{-1}.

Otherwise the portion of V from the pth vertex to the qth vertex has net length greater than one; note that it must have a chord, as otherwise there would be an induced oriented cycle of net length greater than one in H. In fact there must be a chord which forms, together with a portion of V, an induced oriented cycle Z of net length one. Now we obtain a walk W in H from a_i to a_i, constricted below and of net length one, as follows. We again walk from a_i to Z along some walk S, then follow Z once around in positive direction, and then follow S^{-1} back to a_i.

Let W' be the infinite walk in H obtained by repeating W. Let U' be the corresponding walk in H obtained by the same process in the component A_{i+1}. (Thus U' is an infinite walk obtained by repeating a walk U, constricted below, of net length one, from a_{i+1} to a_{i+1}.)

Since (a_{i+1}, a_{i+2}) is weakly extremal in D_{i+1}, we obtain two other infinite walks X_{i+1}, X_{i+2} in H , starting in a_{i+1}, a_{i+2}, respectively. Note that X_{i+1} and X_{i+2} are constricted from below, avoid each other, and have infinite height.

Now we apply Lemma 3.6 to X_{i+1}, X_{i+2}, W' and conclude that W', X_{i+2} avoid each other. Applying Lemma 3.6 to X_{i+2}, W', U', we conclude that U', W' also avoid each other. We now apply Corollary 4.4 to W', U' with the subwalks W, U. This allows us to conclude that there is a closed walk on net length one from (a_i, a_{i+1}) to (a_i, a_{i+1}) in H^*.

Recall our assumptions that X contains $(a_n, a_0), (b_0, b_m)$ and maybe other pairs, creating the circuit $(a_0, a_1), (a_1, a_2), \ldots, (a_n, a_0)$ in X and the circuit (b_0, b_1), (b_1, b_2), $\ldots, (b_m, b_0)$ in X'.

As in the balanced case, we first assume that some $(a_i, a_{i+1}) \in X$. Then there is a walk W from (a_n, a_0) to (a_i, a_{i+1}) in X of net length zero. Indeed, the argument above shows that both (a_n, a_0) and (a_i, a_{i+1}) have a walk of net length ℓ to (e_i, e_{i+1}), since in this case $A_i = A_n = X$. As before, W can be partitioned into two constricted pieces, U and V, and Lemma 3.6 implies that $(a_n, a_{i+1}) \in X$, contrary to the minimality of n.

If X does not contain another (a_i, a_{i+1}) or (b_j, b_{j+1}), we again proceed as in the balanced case. There exists a walk W in X of net length zero from (a_n, a_0) to (b_0, b_m). (Both (a_n, a_0) and (b_0, b_m) can reach (e_n, e_0) with walks of the same net length.) Let L be the maximum height of W. Thus W consists of two constricted walks U, V. Let again U_1 (respectively, U_2) be the corresponding walk in H from a_n (respectively, from a_0) to a vertex of maximum height, and similarly let V_1 (respectively, V_2) be the corresponding walks from the vertices of maximum height to b_0 (respectively, b_m). Since (a_0, a_1) is weakly extremal, there is a constricted walk of net length L from a_1, and for a similar reason, there is also such a walk from b_1.

We can now use Corollary 3.7 as in the balanced case to deduce that (a_0, a_1) and (b_m, a_1) are in the same component of H^* and that (b_0, b_1) and (a_n, b_1) are in the same component of H^*, yielding the same contradiction.

This completes the proof of Theorem 4.1. We observe that in the proof we only used the fact that H has no invertible pairs and no induced cycles of net length greater than one.

COROLLARY 4.6. *The following statements are equivalent for a digraph H:*

1. *H is a monotone proper interval digraph.*
2. *H admits conservative lattice polymorphisms.*
3. *H admits a Min-Max ordering.*
4. *No component of H^* contains a circuit.*
5. *H has no symmetrically invertible pair and no induced cycle of net length greater than one.*

Proof. The equivalence of 1 and 3 is Theorem 2.1, the equivalence of 2 and 3 is Theorem 3.1. We have proved that 3 implies 4 in Theorem 3.3. Theorem 3.4 and the remark preceding it tell us that 4 implies 5. Finally, 5 implies 3 because the above algorithm does not create a circuit among the chosen pairs, hence $<$ is a Min-Max ordering as explained above the statement of Theorem 4.1. ∎

Corollary 4.6 implies that a digraph H is a monotone proper interval digraph if and only if it does not contain an induced cycle of net length greater than one, or a symmetrically invertible pair. For reflexive graphs and digraphs the only induced cycles are the loops, which have net length one, and for bigraphs (which are digraphs with all edges oriented from one part to the other) all cycles have net length zero. Therefore the cycle condition is trivially satisfied for these special graph classes, and we obtain the following corollary.

COROLLARY 4.7. *If H is a reflexive digraph, then H is an adjusted proper interval digraph if and only if it has no symmetrically invertible pairs.*

If H is a reflexive graph, then H is a proper interval graph if and only if it has no symmetrically invertible pairs.

If H is a bigraph, then H is a proper interval bigraph if and only if it has no symmetrically invertible pairs.

Thus we also obtain new obstruction characterizations for the well-studied classes of proper interval graphs and bigraphs [12, 32, 35, 36].

Moreover, Corollary 4.6 implies Theorem 3.5 or, equivalently, the following characterization of monotone proper interval digraphs.

COROLLARY 4.8. *A digraph H is a monotone proper interval digraph if and only if no component of the pair digraph H^* contains a circuit.*

This result implies a polynomial time recognition algorithms for monotone proper interval digraphs, i.e., for digraphs admitting a Min-Max ordering. It suffices to construct H^*, find its weak components, and test each for circuits. Testing a weak component for circuits amounts to looking at a set of ordered pairs, i.e., a digraph, and looking for a directed cycle. Acyclicity of a digraph is checked in linear time by topological sort. In fact, what our algorithm accomplishes is to find a common topological sort of the set of acyclic digraphs corresponding to the pairs in the chosen components.

We close this section by noting that the existence of a Min-Max ordering in more general structures is not likely to admit nice characterizations by forbidden substructures such as in statement 5 of Corollary 4.6.

A *binary structure* H consists of a vertex set $V(H)$ and two arc sets $A_1(H), A_2(H)$, each being a binary relation. A *Min-Max ordering* of a binary structure H is a linear ordering $<$ of $V(H)$ such that $u < w, z < v$ and $uv, wz \in A_i(H)$ imply that $uz \in A_i(H)$ and $wv \in A_i(H)$, for all u, v, w, z, and i.

THEOREM 4.9. *It is NP-complete to decide whether a given binary structure admits a Min-Max ordering.*

Proof. The following problem is known to be NP-complete [11]. Given a finite set V and a collection of k triples (a_i, b_i, c_i), $i = 1, 2, \ldots, k$, of distinct elements of V, can the elements of V be ordered by $<$ so the ordering is *consistent* with each triple (a_i, b_i, c_i), i.e., so that $a_i < b_i < c_i$ or $c_i < b_i < a_i$ for each i? We will reduce this problem to the existence of a Min-Max ordering of a suitable binary structure H. Since the existence of a Min-Max ordering is clearly in NP, this will prove the theorem.

For the vertices of H we take the disjoint union of k copies V_i of the set V with $i = 1, 2, \ldots, k$. Note that each copy V_i corresponds to a different triple (a_i, b_i, c_i). For the first arc set A_1 we take, for each triple (a_i, b_i, c_i), the arcs a_ib_i, b_ic_i in the copy V_i. For the second arc set A_2 we take all arcs uv, where u and v are copies of the same vertex in V, and u lies in the ith copy and v in the $(i+1)$st copy, for $i = 1, 2, \ldots, k-1$. We claim that V has an ordering consistent with all triples (a_i, b_i, c_i), $i = 1, 2, \ldots, k$, if and only if H has a Min-Max ordering.

If V has an ordering $<$ consistent with all the triples, then we can order the vertices of H by taking this ordering on all copies V_i, in an arbitrary order. It is easy to see that the resulting ordering is a Min-Max ordering. Conversely, if $<$ is a Min-Max ordering of H_C, then the relation A_2 ensures that all copies V_i are ordered in the same way, i.e., if x precedes y in V_i, then it also precedes it in V_{i+1} and hence in all V_j. This means that there is an ordering $<$ of V corresponding to all of them. The relation A_1 ensures that each triple is consistent with respect to $<$. ∎

The reduction in Theorem 4.9 is due to Bagan, Durand, Filiot, and Gauwin, who used it in the context of Min orderings; it has appeared only in a preliminary version of [1].

5. Extended Min-Max orderings. We now discuss *extended Min-Max orderings*. In some cases when Min-Max orderings do not exist, there may still exist extended Min-Max orderings, which is sufficient for the polynomial solvability of MinHOM(H) [16]. We denote by $\vec{C}_k$ the directed cycle on vertices $0, 1, \ldots, k-1$. We shall assume in this section that H is weakly connected. Indeed the minimum cost homomorphism problem to H can be easily separated into subproblems corresponding to the weak components of H; moreover, any version of the Min-Max property also applies to each individual weak component of H separately. This assumption allows us to conclude that any two homomorphisms ℓ, ℓ' of H to $\vec{C}_k$ define the same partition of $V(H)$ into the sets $V_i = \ell^{-1}(i)$, and we will refer to these sets without explicitly defining a homomorphism ℓ.

Thus suppose H is homomorphic to $\vec{C}_k$, and let V_i be the partition of $V(H)$ corresponding to all such homomorphisms. A *k-Min-Max ordering* of H is a linear ordering $<$ of each set V_i, so that the Min-Max property ($u < w, z < v$ and $uv, wz \in A(H)$ imply that $uz \in A(H), wv \in A(H)$) is satisfied for u, w and v, z in any two circularly consecutive sets V_i and V_{i+1}, respectively, (subscript addition modulo k).

Note that any H is homomorphic to the one-vertex digraph with a loop $\vec{C}_k$, and a 1-Min-Max ordering of H is just the usual Min-Max ordering. Also note that a Min-Max ordering of a digraph H becomes a k-Min-Max ordering of H for any $\vec{C}_k$ that H is homomorphic to. However, there are digraphs homomorphic to $\vec{C}_k$ which have a k-Min-Max ordering but do not have a Min-Max ordering—for instance $\vec{C}_k$ (with $k > 1$). An *extended Min-Max ordering* of H is a k-Min-Max ordering of H for some positive integer k such that H is homomorphic to $\vec{C}_k$.

We observe for future reference that an unbalanced digraph H has only a limited range of possible values of k for which it could be homomorphic to $\vec{C}_k$, and hence a

limited range of possible values of k for which it could have a k-Min-Max orderings. It is easy to see that a cycle C admits a homomorphism to $\vec{C}_k$ only if the net length of C is divisible by k [23]. Thus any cycle of net length $q > 0$ in H limits the possible values of k to the divisors of q. If H is balanced, it is easy to see that H has a k-Min-Max ordering for some k if and only if it has a Min-Max ordering.

For a digraph H homomorphic to $\vec{C}_k$ we shall consider the following version of the pair graph. The digraph $H^{(k)}$ is the subgraph of H^* induced by all ordered pairs (x, y) belonging to the same set V_i. We say that (u, v) is a *symmetrically k-invertible pair* in H if $H^{(k)}$ contains a walk joining (u, v) and (v, u). Thus a symmetrically k-invertible pair is a symmetrically invertible pair in H in which u and v belong to the same set V_i. Note that H may contain symmetrically invertible pairs but no symmetrically k-invertible pair. Consider, for instance, the directed hexagon $\vec{C}_6$. The pair $0, 3$ is symmetrically invertible and symmetrically 3-invertible, but not symmetrically 6-invertible.

The extended version of our main theorem follows. Since we are interested in the minimum cost homomorphism problem, we focus only on the main parts of the characterization.

THEOREM 5.1. *The following statements are equivalent for a weakly connected digraph H:*

1. *H admits an extended Min-Max ordering.*
2. *There exists a positive integer k such that H is homomorphic to $\vec{C}_k$ and no component of $H^{(k)}$ contains a circuit.*
3. *There exists a positive integer k such that H is homomorphic to $\vec{C}_k$ and H contains no symmetrically k-invertible pair and no induced cycle of positive net length other than k.*

Proof. We shall in fact prove that the following statements are equivalent for a positive integer k such that H is homomorphic to $\vec{C}_k$:

1. H admits a k-Min-Max ordering.
2. No component of $H^{(k)}$ contains a circuit.
3. H contains no symmetrically k-invertible pair and no induced cycle of positive net length other than k.

Suppose H admits linear orderings $<$ of sets V_i satisfying the Min-Max property between consecutive sets V_i, V_{i+1}. Any circuit $(x_0, x_1), (x_1, x_2), \ldots, (x_n, x_0)$ in $H^{(k)}$ must have all vertices $x_0, x_1, \ldots, x_n$ in the same set V_i, and hence if all the pairs (x_i, x_{i+1}) were in the same component of $H^{(k)}$ we would obtain the same contradiction with transitivity of $<$ as above the statement of Theorem 3.3. This proves that 1 implies 2.

We now prove that 2 implies 3. Thus assume that no component of $H^{(k)}$ contains a circuit. Then there can be no symmetrically k-invertible pair, as it would again correspond to a circuit of length two in a component of $H^{(k)}$. Furthermore, H cannot contain an induced cycle C of positive net length $q \neq k$. Otherwise q would be a multiple of k since we assumed H is homomorphic to $\vec{C}_k$. If $q = rk$, we proceed as in the proof of Theorem 3.4, choosing x_0 as a vertex of C from which there is a walk P around C constricted from below, and then letting x_i be the last vertex on P such that the walk from x_0 to x_i has net length ik. Then the proof of Theorem 3.4 shows that the circuit $(x_0, x_1), (x_1, x_2), \ldots, (x_{r-1}, x_0)$ lies in the same component of $H^{(k)}$. It remains to observe that by the definition the vertices $x_0, x_1, \ldots, x_{r-1}$ are in the same set V_i.

We finally prove that 3 implies 1. Thus assume that H is homomorphic to $\vec{C}_k$ and H contains no symmetrically k-invertible pair and no induced cycle of positive net length other than k. We shall construct a k-Min-Max ordering of H. We have again the components of $H^{(k)}$ in dual pairs C, C', where C' consists of the reverses of the pairs in C, and we can proceed with a similar algorithm as before. At each stage of the algorithm, some component of $H^{(k)}$ is chosen and its dual component discarded. We again choose a component X of maximum height, unless X creates a circuit (among the chosen pairs), in which case we choose its dual X'; we claim that in such a case the dual X' does not create a circuit. The proof is analogous to the proof of Theorem 4.1: we suppose for contradiction that both X and X' create circuits $(a_0, a_1), \ldots, (a_n, a_0)$ and $(b_0, b_1), \ldots, (b_m, b_0)$, respectively, and assume that the time T when this occurs was minimum, and then the value of n and then of m was minimum. Recall that we must have $a_0, a_1, \ldots, a_n$ in the same set V_s, and $b_0, b_1, \ldots, b_m$ in the same set V_t. Denote again by A_i (respectively, B_j) the component of $H^{(k)}$ containing (a_i, a_{i+1}) (respectively, (b_j, b_{j+1})). Now Lemmas 4.2, 3.4, and 4.5 apply verbatim to the new situation, and in particular, we can conclude as stated after the proof of Lemma 4.5 that both (a_n, a_0) and (b_m, b_0) are extremal.

When X is balanced, we must furthermore have $V_s = V_t$, since otherwise either (a_n, a_0) or (b_m, b_0) would not be extremal. Now the conclusion of the proof of Theorem 4.1 in the case X is balanced applies verbatim.

When X is unbalanced, we proceed again as in the proof of Theorem 4.1, letting D_i be a fixed unbalanced cycle in A_i, and concluding that we may assume that each (a_i, a_{i+1}) is weakly extremal on the corresponding A_i (and similarly for the components B_j). Instead of the claim that each D_i has net length one, however, we now claim that each D_i has net length k. We leave the analogous proof to the reader—the only changes required are to replace each occurrence of "net length one" by "net length k." (There are seven such occurrences, all in the first four paragraphs of the proof.) The rest of the proof of Theorem 4.1 again applies verbatim. ∎

We again note that the theorem implies a polynomial time algorithm to test whether an input digraph H has an extended Min-Max ordering. As noted above, it suffices to check for each component of H separately, so we may assume that H is weakly connected. If H is balanced, we have already observed this is only possible if H has a Min-Max ordering, which we can check in polynomial time. Otherwise we find any unbalanced cycle in H, say, of net length q, and then test for circuits in components $H^{(k)}$ for all k that divide q.

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