Bi-arc Digraphs: Recognition Algorithm and Applications

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Abstract. We study the class of *bi-arc digraphs*, important from two seemingly unrelated perspectives. On the one hand, they are a broad generalization of interval graphs and includes other generalizations of interval graphs, such as co-threshold tolerance graphs and adjusted interval digraphs. On the other hand, they are precisely the digraphs that admit certain polymorphisms of interest in the study of constraint satisfaction problems. These digraphs were first considered in H-coloring problems by Woeginger et al. in 1992 [13] under the name of X-underbar digraphs. Since then, they have appeared in many studies on graph homomorphism and constraint satisfaction problems.
Our main result is a forbidden obstruction characterization of, and a polynomial recognition for, the class of bi-arc digraphs.
We also show that bi-arc digraphs are precisely the digraphs that admit a conservative semilattice polymorphism, also known as min ordering.
Moreover, we show that bi-arc digraphs are also precisely the digraphs that admit certain other kinds of conservative polymorphisms, thus collapsing these polymorphism types on the class of digraphs.
The complexity of the recognition problem for digraphs with conservative semilattice polymorphisms was an open problem. We complete the dichotomy classification of all general relational structures for the existence of conservative semilattice polymorphisms.

Keywords: Min ordering · Polymorphisms · Graph Homomorphism · Interval digraphs

1 Background and Motivation

1.1 Graph Theoretic Motivation

Digraph Generalization of Interval Graphs: Part of our motivation stems from a wish to generalize interval graphs. A graph H is an *interval graph* if there is a family of intervals $I_v, v \in V(H)$, such that $uv \in E(H)$ if and only if $I_u \cap I_v \neq \emptyset$. Interval graphs constitute one of the most important graph classes; they admit efficient recognition algorithms, and elegant obstruction characterizations and frequently occur in applications [1, 5, 11, 12, 24]. The classical digraph version of interval graphs [6] lacks many of these desirable attributes. A more successful generalization is given in [8]: we say that H is an *adjusted interval digraph* if there are two families of real intervals $I_v, J_v, v \in V(H)$, where for each $v \in V(H)$ the intervals I_v, J_v have the same left endpoint, such that $uv \in A(H)$ if and only if $I_u \cap J_v \neq \emptyset$. Adjusted interval digraphs have many of the desirable algorithmic attributes of interval graphs, including efficient recognition algorithms and forbidden structure characterizations [8].

It is useful to view both interval graphs and adjusted interval digraphs as being *reflexive*, i.e., each vertex having a loop. (This is consistent with their definition as each I_v intersects itself, or the corresponding J_v.) The adjusted interval digraphs appear to be the right generalization of interval graphs for reflexive digraphs. For general (not necessarily reflexive) digraphs, the right analog was less clear. Another special class of digraphs is *bipartite* digraphs, which are just bipartite graphs with all edges oriented from one part of the bipartition to the other part. It turns out there is a natural generalization of interval graphs amongst bipartite digraphs, namely the two-directional orthogonal ray digraphs [28], which have many equivalent definitions [15, 17], and also share several of the desirable properties of interval graphs.

One particular property that has been noticed in studying these classes of graphs and digraphs is the notion of *min ordering.* An ordering $<$ of the vertices of digraph H is min ordering if whenever uv and $u'v'$ with $u < u'$ and $v' < v$ are arcs of H then uv' is also an arc of H. For graph H, ordering $<$, in min ordering, if whenever $u < v < w$ and uw is an edge, then uv is also an edge of H. A reflexive graph has a *min ordering* if and only if it is an interval graph; a reflexive digraph has a min ordering if and only if it is an adjusted interval digraph, and a bipartite digraph has a min ordering if and only if it is a two-directional orthogonal ray graph [8, 15, 17, 28]. Thus it was long believed that *min-orderable* digraphs are the right overall generalization of interval graphs. However, it was not known whether this class of digraphs could be recognized in polynomial time, whether it has an obstruction characterization, and whether it has any geometric meaning. Recently, two geometric representations of the class of digraphs with a min ordering have been given in [16]. Min-orderable digraphs are shown there to be exactly the same as signed-interval digraphs, which arise as a natural extension of another well-studied graphs class, the complements of so-called threshold tolerance graphs. They are also shown to be exactly the same digraphs as *bi-arc digraphs*, which are defined as a digraph analogue of the previously studied class of bi-arc graphs [7]. Both these classes are defined by the intersection or inclusion of intervals or circular arcs. Thus it remained to find a forbidden structure characterization for, and a polynomial time recognition algorithm of, min-orderable digraphs. This is what we accomplish in this paper, thus contributing to the argument that min-orderable digraphs are the right general digraph analog of interval graphs.

1.2 CSPs, Meta-question and Algebraic Motivation

Other part of our motivation stems from the study of *Constraint Satisfaction Problems (CSPs)* and the so-called algebraic approach to them. A CSP involves deciding, given a set of variables and a set of constraints on the variables, whether or not there is an assignment to the variables satisfying all of the constraints.

A *relational structure* is a tuple $\mathbb{H} = \langle V, R_1, \ldots, R_s \rangle$ where V is a non-empty finite set, called the universe, and each R_i is a relation of arity r_i on V. For instance, a digraph H with vertex set $V(H)$ and arc set $A(H)$ is a relational structure with universe $V(H)$ and a single binary relation $A(H)$ i.e., $H = \langle V(H), A(H) \rangle$. A *homomorphism* from a relational structure $\mathbb{G}$ to relational structure $\mathbb{H}$ is a mapping from the vertex set of $\mathbb{G}$ to the vertex set $\mathbb{H}$ that the image of every r-tuples in $\mathbb{G}$ is an r-tuple in $\mathbb{H}$.

The CSP can be formulated in terms of homomorphisms as follows. Given a pair $(\mathbb{G}, \mathbb{H})$ of (similar) relational structures, decide whether or not there is a homomorphism from the first structure to the second structure. A common way to restrict this problem is to fix the second structure $\mathbb{H}$ so that each structure $\mathbb{H}$ gives rise to a problem CSP($\mathbb{H}$). The most effective approach to the study of the CSP($\mathbb{H}$) is the so-called algebraic approach that associates every $\mathbb{H}$ with its *polymorphisms*.

A polymorphism of a structure $\mathbb{H}$ is defined as a finite operation $f : V^k \to V$ that is a homomorphism from $\mathbb{H}^k$ to $\mathbb{H}$. That is for every k tuples $\tau_1, \ldots, \tau_k$ from relation R_i (of arity r_i), we have $(x_1, x_2, \ldots, x_{r_i}) \in R_i$ such that x_j, $1 \le j \le r_i$ is of form $x_j = f(\tau_1[j], \tau_2[j], \ldots, \tau_k[j])$ where $\tau_t[j]$, $1 \le t \le k$ is the j-element of τ_t. A polymorphism f is *conservative* if each value $f(x_1, x_2, \ldots, x_k)$ is one of the arguments $x_1, x_2, \ldots, x_k$. A binary (arity two) polymorphism $f : V^2 \to V$ that is conservative and *commutative* ($f(x, y) = f(y, x)$ for all vertices x, y) is called a CC polymorphism. Notice that by definition any binary CC polymorphism is *idempotent* i.e, $f(x, x) = x$. If f is additionally *associative* then it is called a *conservative semilattice* or a CSL polymorphism. That is, it satisfies the following *identities*, $f(f(x, y), z) = f(x, f(y, z))$, and $f(x, y) = f(y, x) \in \{x, y\}$ for all $x, y, z \in V$.

Roughly speaking, the presence of nice enough polymorphisms leads directly to the polynomial time tractability of CSP($\mathbb{H}$), while their absence leads to hardness. Besides decision CSPs, polymorphisms have been used extensively for approximating CSPs, robust satisfiability of CSPs, testing solutions (in the sense of property testing), and the study of the Ideal Membership Problems [3, 22, 27].

An interesting question arising from these studies, is known as the *meta-question*. Given a relational structure $\mathbb{H}$, decide whether or not $\mathbb{H}$ admits a polymorphism from a class–for various classes of polymorphisms. In many cases, hardness results are known. One particular case, that is the study of this paper, is deciding whether or not $\mathbb{H}$ admits a CSL polymorphism. The presence of semilattice polymorphisms leads to many positive results. As an example, it is now a classic theorem in the area that for any structure $\mathbb{H}$ having a semilattice polymorphism, the problem CSP($\mathbb{H}$) is polynomial time decidable [21]. In terms of approximation algorithms, the Minimum Cost Homomorphism problem to $\mathbb{H}$ (when $\mathbb{H}$ is a digraph) is approximable within a constant factor if $\mathbb{H}$ admits a CSL polymorphism [17, 27]. In terms of robust satisfiability, given a $(1 - \varepsilon)$-satisfiable instance of CSP($\mathbb{H}$), it is easy to find a $(1 - O(1/\log(1/\varepsilon)))$-satisfying assignment if $\mathbb{H}$ admits a semilattice polymorphism (in fact, the result holds for width-1 CSPs). However, on the negative side, there are instances where $\mathbb{H}$ admits a semilattice polymorphism and it is hard to find a $(1 - o(1/\log(1/\varepsilon)))$-satisfying assignment [22].

For a single binary relation, i.e., a digraph, the meta-question often turns out to be better behaved. For instance, there are forbidden induced structure characterizations for the existence of conservative *majority* [19] and conservative *Maltsev* [4, 19] polymorphisms in digraphs. The question of whether the existence of conservative semilattice polymorphism is polynomial was explicitly raised in [20]. This problem is polynomial for reflexive digraphs [8] and bipartite digraphs [17]. In this paper, we give forbidden obstruction characterization for digraphs admitting a conservative semilattice polymorphism. Observe that if a digraph H admits a CSL polymorphism then the CSL polymorphism naturally defines an ordering on the vertices of H. It turns out that a digraph admits a CSL polymorphism if and only if it has a min ordering. Other questions about the existence of polymorphisms of various kinds have turned out to also be interesting [2, 9, 19, 25]. In particular, the existence of conservative polymorphisms is a hereditary property (if H has a particular kind of conservative polymorphism, then so does any induced subgraph of H). Thus, these questions present interesting problems in graph theory.

1.3 Our Contributions

In this paper, we study the problem of deciding if a relational structure $\mathbb{H}$ admits a conservative semilattice (CSL) polymorphism. That is, we study for which relational structures Problem 1 is polynomial time decidable and for which ones it is NP-complete.

Problem 1.
Input: A relational structure $\mathbb{H} = \langle V, R_1, \ldots, R_s \rangle$,
Goal: Decide if $\mathbb{H}$ admits a conservative semilattice (CSL) polymorphism.

Note that any unary relation R admits a CSL polymorphism. This is because if $a, b \in R$, then applying CSL polymorphism f on a, b, would give either a or b, and hence, R is closed under f. So the interesting cases are when the arity of R is at least two. On the positive side, we present a polynomial time algorithm that, given a relational structure with a single binary relation $\mathbb{H} = \langle V, A(V) \rangle$ i.e., digraph, decides if $\mathbb{H}$ admits a CSL polymorphism.

Theorem 1 (Main Theorem). *There exists a polynomial time algorithm that, given a digraph H, decides if H admits a CSL polymorphism or not.*

We also have a structural characterization of digraphs with a CSL polymorphism, in terms of a forbidden structure we call a *strong circuit*. Recall that the class of digraphs that admit a CSL polymorphism is exactly the class of digraphs admitting a min ordering (also called bi-arc digraphs).

The class of digraphs admitting a min ordering coincides with the class of signed-interval digraphs. We therefore have the following corollary.

Corollary 1. *The class of min-orderable digraphs, bi-arc digraphs and signed-interval digraphs can be recognized in polynomial time.*

Furthermore, we show that there is quite a bit of collapse for digraph classes in the conservative case. We will point out that the class of digraphs with a min ordering is included in the class of digraphs with a *conservative set* polymorphism, which is included in the class of digraphs with a *conservative and commutative* polymorphism (called CC polymorphism). Formally, we prove the following (see appendix Section 11).

Theorem 2. *Let $\mathbb{H}$ be a digraph, then $\mathbb{H}$ admits a CSL polymorphism if and only if $\mathbb{H}$ admits a conservative set polymorphism if and only if $\mathbb{H}$ admits conservative cyclic polymorphisms of all arities.*

On the negative side, we prove that it is NP-complete to decide if a relational structure $\mathbb{H} = \langle V, R \rangle$ where R is a ternary relation (arity of R is three) admits a CSL polymorphism.

Theorem 3. *Deciding if a relational structure with a single ternary relation admits a CSL polymorphism is NP-complete.*

Moreover, we prove Problem 1 remains NP-complete even for two binary relations i.e., two digraphs. This leads us to the following dichotomy classification of the complexity of Problem 1 (details are given in appendix Section 12).

Theorem 4 (Dichotomy Theorem). *Deciding if a relational structure $\mathbb{H} = \langle V, R_1, \dots, R_k \rangle$ admits a CSL polymorphism is polynomial-time solvable if all relations R_i are unary, except possibly one binary relation. In all other cases, the problem is NP-complete.*

2 Bi-arc digraphs and Min-orderable Digraphs

A *digraph* H consists of a finite vertex set $V(H)$ and an arc set $A(H)$, each arc being an ordered pair of vertices. We say that $uv \in A(H)$ is an arc from u to v. Sometimes we emphasize this by saying that uv is a *forward arc* of H, and also say vu is a *backward arc* of H. We say that u, v are *adjacent* in H if uv is a forward or a backward arc of H (either $uv \in A(H)$ or $vu \in A(H)$). A *symmetric* arc is an arc $uv \in A(H)$ such that $vu \in A(H)$; thus, a symmetric arc is both a forward arc and a backward arc.

A *graph* H is a symmetric digraph (the binary relation $A(H)$ is symmetric), where we identify each pair of opposite arcs ab, ba into one edge $ab = ba$.

Let C be a circle with two distinguished points N and S. A *bi-arc* is a pair of arcs I, J on C such that I contains N but not S and J contains S but not N. The following definition unites many disparate geometric representations, although we know little about the corresponding class of digraphs.

A *weak bi-arc representation* of a digraph H is a family of bi-arcs $I_v, J_v, v \in V(H)$, such that $ab \in A(H)$ if and only if I_a and J_b are disjoint. A digraph H is a *weak bi-arc digraph* if it admits a weak bi-arc representation.

As mentioned above, we do not know which digraphs admit a weak bi-arc representation and believe they may be interesting. However, several well-studied graph and digraph classes are characterized by the existence of special kinds of weak bi-arc representations. A weak bi-arc representation is *consistent* if the clockwise end of I_a precedes, in the clockwise order on C, the clockwise end of I_b if and only if the clockwise end of J_a precedes (in the clockwise order) the clockwise end of J_b. A consistent weak bi-arc representation will be called simply a *bi-arc representation*, and a digraph admitting a bi-arc representation will be called a *bi-arc digraph*.

It turns out bi-arc digraphs are precisely the digraphs that admit a min ordering [7, 16] (see Figure 1, for min ordering definition see page 2). We add further statements, namely, we prove the following theorem.

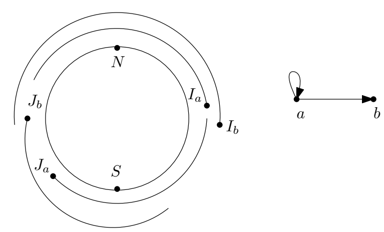

Fig. 1. A min ordering $a < b$ for digraph H and its bi-arc representation

Theorem 5. *Let H be a bi-arc digraph. Then H admits a conservative semi-lattice polymorphism, admits cyclic polymorphisms of all arities, and admits a conservative set polymorphism.*

A *set polymorphism* of H is a mapping f of the non-empty subsets of $V(H)$ to $V(H)$, such that $f(S)f(T) \in A(H)$ whenever S, T are non-empty subsets of $V(H)$ with the property that for each $s \in S$ there is a $t \in T$ with $st \in A(H)$ and also for every $t \in T$ there is an $s \in S$ with $st \in A(H)$. It is easy to see, cf. [10], that H has a conservative set polymorphism if and only if it has conservative totally symmetric (CTS) polymorphisms of all arities k. A polymorphism f of arity k on digraph H is called *cyclic* if $f(x_1, x_2, \ldots, x_k) = f(x_2, x_3, \ldots, x_k, x_1)$ for all $x_1, x_2, \ldots, x_k \in V(H)$.

3 Obstructions to Min Ordering

A *walk* in H is a sequence $P = x_0, x_1, \ldots, x_n$ of consecutively adjacent vertices of H; note that a walk has a designated first and last vertex. A *path* $P = x_0, x_1, \ldots,$

x_n is a walk in which all x_i are distinct. A walk $P = x_0, x_1, \ldots, x_n$ is *closed* if $x_0 = x_n$ and a *cycle* if all other x_i are distinct. A walk is *directed* if all its arcs are forward. We define two walks $P = x_0, x_1, \ldots, x_n$ and $Q = y_0, y_1, \ldots, y_n$ in H to be *congruent*, if they follow the same pattern of forward and backward arcs, i.e., $x_i x_{i+1}$ is a forward (backward) arc if and only if $y_i y_{i+1}$ is a forward (backward) arc (respectively). Suppose the walks P and Q as above are congruent. We say an arc $x_i y_{i+1}$ is *a faithful arc from P to Q*, if it is a forward (backward) arc when $x_i x_{i+1}$ is a forward (backward) arc (respectively), and we say an arc $y_i x_{i+1}$ is *a faithful arc from Q to P*, if it is a forward (backward) arc when $x_i x_{i+1}$ is a forward (backward) arc (respectively). We say that P *avoids* Q if there is no faithful arc from P to Q at all. We now introduce a basic tool for this paper.

Definition 1 (The pair digraph H^+). *The vertices of H^+ are all ordered pairs (x, y) of distinct vertices of H. There is an arc from pair (x, y) to pair (x', y') if and only if*

1. *$xx', yy' \in A(H)$ but $xy' \notin A(H)$, or*
2. *$x'x, y'y \in A(H)$ but $y'x \notin A(H)$.*

To avoid confusion with the vertices of H, we will refer to the vertices of H^+ as *pairs*. Arcs $(x, y)(x', y') \in A(H^+)$ arising from case (1) are called *positive arcs*, and those arising from case (2) are called *negative arcs*.

Note that in H^+ we have an arc from (x, y) to (x', y') if and only if there is an arc from (y', x') to (y, x). We call this the *skew property* of H^+, and call the pair (y, x) the *dual* of the pair (x, y). From the skew property, $(x, y)(x', y')$ is a positive arc in H^+ if and only if $(y', x')(y, x)$ is a negative arc. Note that when $(x, y)(x', y')$ is an arc of H^+ then in any min ordering $<$ of H, if $x < y$ then $x' < y'$. More generally, if there is a directed path from (x, y) to (x', y') in H^+, then in any min ordering $<$ (of H) having $x < y$ implies that $x' < y'$.

Definition 2 (Circuit, Strong Circuit). *Let D be a subset of $V(H^+)$. A* circuit *in D is a set of pairs $(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, x_0)$ in D. A* strong circuit *of H^+ is a circuit in C, where C is a strongly connected component (in short, strong component) of H^+. When $n = 1$, and $(x_0, x_1), (x_1, x_0)$ form a strong circuit, then (x_0, x_1) is called an* invertible *pair.*

Thus, in a strong circuit, there are directed paths (in H^+) from (x_{i-1}, x_i) to (x_i, x_{i+1}) for all $i = 1, 2, \ldots, n+1$, modulo $n+1$. If H^+ contains a strong circuit, then H cannot have a min ordering, since $x_0 < x_1$ implies $x_0 < x_1 < x_2 < \cdots < x_n < x_0$ (and similarly for $x_0 > x_1$) contradicting the transitivity of $<$. We have proved that if a digraph H admits a min ordering, then H^+ does not contain a strong circuit. It turns out that the converse also holds.

Theorem 6. *A digraph H admits a min ordering if and only if H^+ does not contain a strong circuit.*

This is our main result, giving a polynomially testable characterization of min-orderable digraphs. We provide an algorithm that outputs a min ordering

when H^+ does not have a strong circuit to prove the theorem. We can also use the algorithm to find a min ordering if one exists by pre-processing the input digraph to check for strong circuits. The time complexity of our algorithm is $O(|A(H)|^2)$. Detecting a strong circuit in H^+ amounts to testing, for each strong connected component C of H^+, the acyclicity of a digraph on $V(H)$ whose arcs are the pairs in C.
From now on, we write strong component for strongly connected component.

4 Algorithm

In this section, we introduce an algorithm to construct a min ordering $<$ of H, provided H^+ contains no strong circuit. We first give the necessary definitions and terminology in the following subsection and provide the algorithm's descriptions in the subsequent subsection.

4.1 Necessary Definitions

Paths and walks. A vertex u' is said to be *reachable* from a vertex u in H if there is a directed path from u to u' in H; a set U' is *reachable* from a set U if every vertex of U' is reachable from some vertex of U. Note that every vertex is reachable from itself by a directed path of length zero. A path can also be a graph on its own, consisting of all the vertices and arcs needed for the definition. We note that our terms *path* and *walk* correspond to what is sometimes called *oriented path* and *oriented walk*.

Net length and (un)balanced digraphs. The *net length* of a walk is the number of forward arcs minus the number of backward arcs. A closed walk is *balanced* if it has net length zero; otherwise, it is *unbalanced*. Note that in an unbalanced closed walk, we may always choose a direction in which the net length is positive (or negative). A digraph is *unbalanced* if it contains an unbalanced closed walk (or, equivalently, an unbalanced cycle); otherwise, it is *balanced*. It is easy to see that a digraph is balanced if and only if it admits a *labeling* of vertices by non-negative integers so that each arc goes from a vertex with a label i to a vertex with a label $i+1$.

We now focus on properties of H^+. Reachability in H^+ is defined in the usual way by the existence of directed paths in H^+. We use the following notation.

Definition 3 (Reachability Notation). *We write $(u,v) \rightsquigarrow (u',v')$ in H^+ if (u',v') is reachable from (u,v) in H^+, and, otherwise, we write $(u,v) \not\rightsquigarrow (u',v')$ in H^+.*

Definition 4 (Closure of S). *Suppose $S \subseteq V(H^+)$. The* closure *of S, denoted by $\widehat{S}$, is the set of all pairs in H^+ that are reachable from S in H^+.*

Note that $\widehat{S}$ contains S. We say S is *closed under reachability* if $\widehat{S} = S$.

Net value of a path in H^+. In H^+ when we mention a path, we mean a directed path. A (directed) path W in H^+ corresponds precisely to a pair of congruent walks P, Q in H such that P avoids Q. We occasionally write $W = (P, Q)$ and also denote the path W from (x, y) to (u, v) by $W : (x, y) \rightsquigarrow (u, v)$. The *net value* of the path W is defined to be the net length of the walk P (or equivalently the net length of Q). It is the difference between the number of positive and negative arcs of W. Walk W is called symmetric if P and Q avoid each other.

(Un)balanced components in H^+. A closed walk of H^+ is *balanced* if has net value zero, and *unbalanced* otherwise. A strong component of H^+ is *balanced* if it does not contain an unbalanced closed walk, and *unbalanced* otherwise. A strong component S of H^+, is balanced if every directed cycle of S has net value zero. Finally, a pair is called *balanced* if it is in a balanced strong component otherwise, it is called *unbalanced*.

4.2 Description of the Algorithm

We will be choosing pairs of H^+ to decide the ordering. Specifically, if a pair (x, y) of H^+ is chosen, we will set $x < y$. Note that choosing a pair requires choosing all pairs reachable from it. The process of choosing is different for pairs with balanced and unbalanced strong components. However, the chosen pairs will be closed under reachability in each case. Then all the duals of the chosen pairs will be discarded. At any stage of the algorithm, we will have a set V_c of *chosen pairs*, and a set V_d of *discarded pairs*; the pairs in the set $\mathcal{R} = V(H^+) \setminus (V_c \cup V_d)$ will be called the *remaining pairs*. Initially, we will have $V_c = V_d = \emptyset$, and throughout the algorithm, we will maintain the following properties:

1. $(a, b) \in V_c$ if and only if $(b, a) \in V_d$;
2. if $(a, b) \in V_c$ and $(a, b)(a', b') \in A(H^+)$ then $(a', b') \in V_c$;
3. V_c does not contain a circuit.

Note that we will always have $V_c \cap V_d = \emptyset$, and each strong component of H^+ lies entirely in one of the three sets $V_c, V_d, \mathcal{R}$. Moreover, at the end of the algorithm, the set $\mathcal{R}$ will be empty; this ensures that $<$ is a total ordering. Therefore, property (3) will then imply the following transitivity on the chosen pairs:

– if $(a, b) \in V_c$ and $(b, c) \in V_c$ then $(a, c) \in V_c$.

This fact, together with property (2), ensures that the chosen pairs do define a min ordering, by setting $x < y$ for all chosen pairs (x, y).

Algorithm 1 has two phases.

$\boxed{\text{PHASE ONE:}}$ In the first phase we reduce the problem to a balanced subdigraph $H^{\#}$ of H^+. We accomplish this by dealing with all the unbalanced strong components of H^+ first.

At each step, we consider an unbalanced strong component $C \not\subset (V_c \cup V_d)$ and its dual component C'. In Theorem 8, we prove that if $\widehat{C} \cup V_c$ contains a circuit, then $\widehat{C'} \cup V_c$ does not contain a circuit. Therefore, if $(\widehat{C} \cup V_c)$ does not contain a circuit, then we add $\widehat{C}$ into V_c and add the dual pair of $\widehat{C}$ into V_d, update $\mathcal{R}$, and proceed to the next unbalanced strong component. Otherwise, we remove C from further consideration and add $\widehat{C'}$ into V_c and update V_d and $\mathcal{R}$ accordingly.

$\boxed{\text{Phase Two:}}$ For the balanced strong components we need a different strategy because of the different structural properties of balanced and unbalanced strong components. In particular, unbalanced strong components have walks of *unbounded* net value.

Now consider the induced sub-digraph $H^{\#}$ of H^+ consisting of all pairs in the balanced strong components of H^+. Thus, $H^{\#}$ is itself balanced. (This is true, since each closed walk lies in a strong component of $H^{\#}$; recall that in H^+ balance refers to the equality of the number of positive and negative arcs in each closed walk.)

We partition the vertices of $H^{\#}$ into *layers* as follows. Consider an auxiliary digraph D with $V(D) = V(H^{\#})$ and $(a,b)(c,d) \in A(D)$ if and only if (c,d) is reachable from (a,b) by a path in $H^{\#}$ with negative net value. Since all directed cycles in $H^{\#}$ are balanced, D is acyclic. Layer 0 of $H^{\#}$, denoted by $\mathcal{L}_0$, consists of all vertices that have out-degree zero in D. Having defined layers $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_j$, layer $\mathcal{L}_{j+1}$ of $H^{\#}$ consists of all vertices of out-degree zero in the digraph obtained from D by removing all the vertices in layers $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_j$. We handle the pairs in $\mathcal{L}_0, \mathcal{L}_1, \ldots,$ consecutively, one at a time.

To proceed with the current layer $\mathcal{L}_k, k \geq 0$, we seek a vertex $p \in V(H)$ such that there exists **no** $q' \in V(H)$ so that $(q',p) \in V_c \cap \mathcal{L}_k$, and

– there exists a q such that $(p,q) \in \mathcal{R} \cap \mathcal{L}_k$ and $(p,q) \not\rightsquigarrow (q,p)$,

The existence of such p is justified in Lemma 1. For each choice of p, as long as there exists some pair $(p,r) \in \mathcal{R} \cap \mathcal{L}_k$ so that $(p,r) \not\rightsquigarrow (r,p)$ we add (p,r) into V_c. (This process can start with r being the vertex q from above and then continue as long as further r can be found.) We now define the transitivity-reachability (TR) closure of V_c as follows.

Definition 5 (Transitivity+Reachability (TR) Closure). *The* transitivity+reachability closure *of V_c, $Tr(V_c)$, is the smallest set of pairs containing V_c that is closed under reachability and transitivity. In other words, if $(x,y) \in Tr(V_c)$, and $(x,y) \rightsquigarrow (x',y')$ then $(x',y') \in Tr(V_c)$. Moreover, if $(x,y),(y,z) \in Tr(V_c)$ then $(x,z) \in Tr(V_c)$.*

Note that $V_c \subseteq Tr(V_c)$. Then we update the set V_c to be $Tr(V_c)$. Of course, we also update $\mathcal{R}$ by removing all the dual pairs of V_c from $\mathcal{R}$, and all the pairs of V_c from $\mathcal{R}$. Note that during the computation of $Tr(V_c)$ we may add $(q',p) \in \mathcal{L}_k$ into $Tr(V_c)$ and p no longer satisfies the condition that there exists no $(q',p) \in V_c \cap \mathcal{L}_k$. In the next section, we will prove that $Tr(V_c)$ does not contain a circuit (Lemma 2).

Once we are done with p, we look for another vertex p_1 on layer $\mathcal{L}_k$ satisfying the aforementioned conditions and repeat. Once we finish processing all the pairs in $\mathcal{L}_k \cap \mathcal{R}$, we go on to the next layer and consider the remaining pairs in $\mathcal{L}_{k+1}$. The details are provided in Algorithm 1.

Algorithm 1 Algorithm to find a min ordering of input digraph H

1: **function** MinOrdering(H) ▷ Phase One: Handling unbalanced components
2: Construct H^+ and compute its strong components
3: **if** H^+ contains a strong circuit **then return** False
4: Set $V_c = V_d = \emptyset$ and let $\mathcal{R} = V(H^+)$
5: **while** $\mathcal{R}$ contains an unbalanced strong component C **do**
6: **if** $\widehat{C} \cup V_c$ has no circuit **then**
7: Add $\widehat{C}$ into V_c, and add all the dual pairs of $\widehat{C}$ into V_d.
8: Remove from $\mathcal{R}$ all the pairs in $\widehat{C}$ and their dual pairs.
9: **else** ($\widehat{C'} \cup V_c$ has no circuit)
10: Add $\widehat{C'}$ into V_c, and add all the dual pairs of $\widehat{C'}$ into V_d.
11: Remove from $\mathcal{R}$ all the pairs in $\widehat{C'}$ and their dual pairs.
▷ Phase Two: Handling the remaining balanced components
12: Let $H^{\#}$ be the set of all balanced pairs, and let $\mathcal{R} = V(H^{\#}) \setminus V_c$
13: Compute the layers of $H^{\#}$; $\mathcal{L}_0, \mathcal{L}_1, \ldots$, and set $k = 0$
14: **while** $\mathcal{R} \neq \emptyset$ **do**
15: **while** $\mathcal{R} \cap \mathcal{L}_k \neq \emptyset$ **do**
16: Find $p \in V(H)$ s.t. no $(q', p) \in V_c \cap \mathcal{L}_k$ and $\exists (p, q) \in \mathcal{R} \cap \mathcal{L}_k$ with $(p, q) \not\rightsquigarrow (q, p)$
17: **while** $\exists (p, r) \in \mathcal{R} \cap \mathcal{L}_k$ s.t. $(p, r) \not\rightsquigarrow (r, p)$ **do**
▷ at least one (p, r) exists, i.e. $r = q$ in line 16, and empty while loop avoided
18: Add (p, r) into V_c and set $V_c = Tr(V_c)$
19: Remove all the dual pairs of V_c from $\mathcal{R}$, and add them into V_d.
20: Set $\mathcal{R} = \mathcal{R} \setminus V_c$.
21: Increase k by one
return V_c

5 Correctness

To justify the correctness of Phase One we first define the concept of a dual-free set and a minimal circuit. A subset of H^+ is called *dual-free* if it does not contain a pair and its dual.

Definition 6 (Minimal Circuit). *Suppose $S_0, S_1, \ldots, S_n$ (not necessarily distinct) are strong components in $T \subseteq V(H^+)$ where $\widehat{T}$ is dual-free. Let C : $(a_0, a_1), (a_1, a_2), \ldots, (a_n, a_0)$ be a circuit where $(a_i, a_{i+1}) \in \widehat{S}_i$, $0 \leq i \leq n$. We say C is minimal if there is no other circuit $(a'_0, a'_1), (a'_1, a'_2), \ldots, (a'_m, a'_0)$, $m < n$, where each (a'_j, a'_{j+1}) belongs to some $\widehat{S}_i$, $0 \leq i \leq n$.*

We need some technical definition to state and prove Theorem 8. For walks P from a to b, and Q from b to c, we denote by $P + Q$ the walk from a to c which is the concatenation of P and Q. We denote by P^{-1} the walk P traversed in the opposite direction, from b to a; we call P^{-1} the *reverse* of P. Notice that if walk P avoids walk Q then Q^{-1} avoids P^{-1}.

For a closed walk C, we denote by C^a the concatenation of C with itself a times. The *height* of H is the maximum net length of a walk in H. Note that an unbalanced digraph has infinite height, and the height of a balanced digraph is the greatest label in non-negative labeling in which some vertex has label zero.

For a walk $P = x_0, x_1, \dots, x_n$ and any $i \leq j$, we denote by $P[x_i, x_j]$ the walk $x_i, x_{i+1}, \dots, x_j$. We call $P[x_i, x_j]$ a *prefix* of P if $i = 0$. Suppose $P = x_0, x_1, \dots, x_n$ is a walk in H of net length $k \geq 0$. We say that P is *constricted from below* if the net length of any prefix $P[x_0, x_j]$ is non-negative and is *constricted from above* if the net length of any prefix is at most k. We also say that P is *constricted* if it is constricted both from below and from above. Moreover, we say that P is *strongly* constricted from below or above if the corresponding net lengths are strictly positive or smaller than k. For a walk P of net length $k < 0$, we say that P is (strongly or not) constricted from below, or above, or both if the above definitions apply to the reverse walk P^{-1}.

Definition 7 (Extremal Vertex). *Consider a cycle C in H of positive net length k. A vertex v is* extremal *in C if traversing C,in the positive direction, from v yields a walk constricted from below.*

A cycle of H is *induced* if H contains no other arcs on the vertices of the cycle. An induced cycle with more than one vertex does not contain a loop.

Let $W = (P, Q)$ be a path in H^+. We say W is *constricted* if the walk P (or Q) is constricted, i.e., if each prefix of W has a net value between zero and the net value of W. Paths (in H^+) constricted from below or above are defined similarly. Other notions for H^+ are also defined in the manner corresponding to the notions in H. Consider, for instance, the above notion of an extremal vertex. We define *extremal pair* of a cycle C in H^+ as a pair $\bar{v}$ such that traversing C from $\bar{v}$ in the positive direction yields a walk with values constricted from below.

The correctness of the first phase relies on the following technical theorem.

Theorem 7. *Let T be the vertices of a set of unbalanced strong components (in H^+) where $\widehat{T}$ is dual-free, and assume that $\widehat{T}$ contains a minimal circuit $C : (a_0, a_1), (a_1, a_2), \dots, (a_n, a_0)$. Then $n > 1$ and the following statements hold.*

1. *There exits some minimal circuit (Definition 6) with extremal pairs (b_0, b_1), $(b_1, b_2), \dots, (b_{n-1}, b_n), (b_n, b_0)$ in T such that $(a_i, a_{i+1}), (b_i, b_{i+1})$, $0 \leq i \leq n$, are in the same strong component, and (a_i, a_{i+1}) is reachable from (b_i, b_{i+1}) by a symmetric walk of non-negative net value, and constricted from below.*
2. *For each i, $0 \leq i \leq n$, there exists an infinite walk P_i that starts from b_i and has unbounded positive net length. P_i is obtained by winding around a cycle in H containing b_i. Furthermore, for every i, j, $0 \leq i < j \leq n$, P_i and P_j avoid each other [4].*
3. *In statement 1, for a given $0 \leq i \leq n$, we can choose (b_i, b_{i+1}) to be any given extremal pair from its corresponding strong component.*
4. *There is no path in H^+ from (b_i, b_{i+1}) to any of (b_j, b_{j+1}) $i \neq j$, and to any of (b_{j+1}, b_j).*
5. *There is no path in H^+ from any of (b_{i+1}, b_i), $0 \leq i \leq n$ to (b_i, b_{i+1}).*

[4] When we say two infinite walks P , Q avoid each other it means for every prefix of P there exists a prefix of Q that avoid each other.

Theorem 8. *Suppose $C \not\subset (V_c \cup V_d)$ is an unbalanced strong component and V_c does not contain a circuit. If $\widehat{C} \cup V_c$ contains a circuit, then $\widehat{C'} \cup V_c$ does not contain a circuit.*

Proof. Since $C \not\subset (V_c \cup V_d)$, skew property implies $C' \not\subset (V_c \cup V_d)$. Suppose for contradiction that $\widehat{C} \cup V_c$ contains a circuit $(b_0, b_1), (b_1, b_2), \ldots, (b_n, b_0)$ and $\widehat{C'} \cup V_c$ contains a circuit $(d_0, d_1), (d_1, d_2), \ldots, (d_m, d_0)$. We may assume that both are minimal circuits. Notice that Algorithm 1 selects unbalanced strong components one at a time and adds their closure into V_c. Thus, if $\widehat{C} \cup V_c$ contains a circuit, then that circuit would be at $\widehat{T}$ where T is a set of unbalanced strong components in $C \cup V_c$. A similar statement is true for $\widehat{C'} \cup V_c$. Observe that since V_c does not contain a circuit, at least one of the (b_i, b_{i+1}) pairs should be in $\widehat{C}$. The same holds for $\widehat{C'}$, and at least one of the (d_j, d_{j+1}) pairs is in $\widehat{C'}$. Hence, without loss of generality, we assume that $(b_n, b_0) \in \widehat{C}$, and $(d_m, d_0) \in \widehat{C'}$.

We first assume that both $m, n > 1$. Thus, there is no $(p, q) \in C \cup V_c$ so that $(p, q) \rightsquigarrow (q, p)$, as otherwise, we have $(p, q), (q, p) \in \widehat{C} \cup V_c$ which contradicts the minimality assumption and the assumption that $n > 1$. Similarly, there is no $(p', q') \in C' \cup V_c$ so that $(p', q') \rightsquigarrow (q', p') \in \widehat{C'} \cup V_c$. Therefore, $\widehat{C} \cup V_c$, and $\widehat{C'} \cup V_c$ are dual-free. Thus, according to the statement (1) of Theorem 7, we may also assume that all the pairs on these two circuits are extremal pairs in H^+. Moreover, by statement (3) of Theorem 7, we assume that $(b_n, b_0) \in C$ and $(d_m, d_0) \in C'$, i.e., $(d_0, d_m) \in C$, and that $(b_n, b_0) = (d_0, d_m)$.

Moreover, according to statement (4) of Theorem 7, we may assume that (b_n, b_0) is the only pair of the first circuit in C and (d_m, d_0) is the only pair of the second circuit in C'. Now, consider the following circuit (where $(b_{n-1}, b_n) = (b_{n-1}, d_0)$, $(d_{m-1}, d_m) = (d_{m-1}, b_0)$)

$$(b_0, b_1), (b_1, b_2), \ldots, (b_{n-1}, d_0), (d_0, d_1), (d_1, d_2), \ldots, (d_{m-1}, b_0)$$

all pairs of which are in V_c . This contradicts the assumption that V_c has no circuit. In what follows we consider separately the cases when n or m is 1.

Observation 9 *If $\widehat{C} \cup V_c$ contains a circuit $(b_0, b_1), (b_1, b_0)$ (i.e., $n = 1$) then by definition we have $C \cup V_c \rightsquigarrow (b_0, b_1)$, and $C \cup V_c \rightsquigarrow (b_1, b_0)$. Now by skew property, we have $(b_1, b_0) \rightsquigarrow C' \cup V_d$ and $(b_0, b_1) \rightsquigarrow C' \cup V_d$. Therefore, $C \rightsquigarrow C'$, and hence, there is also a circuit $(p, q), (q, p)$ where $(p, q) \in C$, and $(p, q) \rightsquigarrow (q, p)$ (it is not possible that, (p, q) or (q, p) in V_c because $C \cup C' \not\subset V_c \cup V_d$).*

If both circuits have $n = m = 1$ then by the above observation we have $C \rightsquigarrow C'$ and also $C' \rightsquigarrow C$, implying a strong circuit in H^+, a contradiction. Finally, if $n = 1$, but $m > 1$, then the first circuit is $(b_0, b_1), (b_1, b_0)$ and by Observation 9 and skew property we have $(b_0, b_1) \rightsquigarrow C'$ and $C' \rightsquigarrow (b_0, b_1)$. Now again since $m > 1$, by statement (3) of Theorem 7 we may assume that C' contains (b_1, b_0). This means $(b_0, b_1) \rightsquigarrow (b_1, b_0)$ which is in contradiction to statement (5) of Theorem 7 (i.e., reverse of a pair on the circuit does not reach that pair).

The following two lemmas justify the computation in Phase Two. (Lemma 1 justifies Line 16, and Lemma 2 justifies Line 17.)

Lemma 1. *Suppose V_c does not contain a circuit, and furthermore, $\mathcal{R} \cap \mathcal{L}_k \neq \emptyset$. Then there exists a vertex $p \in V(H)$ such that there exists **no** $(q', p) \in V_c \cap \mathcal{L}_k$, and the following condition is satisfied:*

- *there exists a q such that $(p, q) \in \mathcal{R} \cap \mathcal{L}_k$ and $(p, q) \not\rightsquigarrow (q, p)$.*

Lemma 2. *Suppose V_c does not contain a circuit, and furthermore, $\mathcal{R} \cap \mathcal{L}_k \neq \emptyset$. Then after executing the entire while loop at line 17, V_c does not contain a circuit. In other words, after adding all the (p, r) pairs on line 17 and computing $Tr(V_c)$ and setting $V_c = Tr(V_c)$, there will not be a circuit in V_c.*

Theorem 10. *Algorithm 1 correctly decides if a digraph H admits a min ordering or not and it correctly outputs a min ordering for H if one exists.*

Proof. The proof follows from Theorem 8, Lemma 1, and Lemma 2.

6 Conclusions

We have provided polynomial time algorithm, obstruction characterizations, for digraphs admitting a min ordering, i.e., a CSL polymorphism. We believe they are a useful generalization of interval graphs, encompassing adjusted interval digraphs, monotone proper interval digraphs, complements of circular arcs of clique covering number two, two-directional orthogonal ray graphs, and other well-known classes. We also study this problem beyond digraphs, and consider the general case of relational structures. We fully classify the polynomial-time cases (see Theorem 4). Due to space limit, this part is presented in the appendix.

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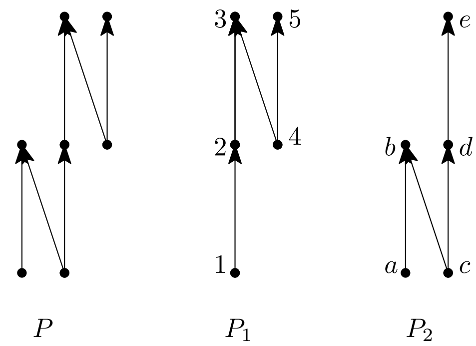

Fig. 2. Example for Lemma 3. The path P (an abstract path on its own) is a common pre-image of paths P_1 and P_2 in a digraph H. It admits a homomorphism f_1 to P_1 that maps the vertices of P, in order, to 1, 2, 1, 2, 3, 4, 5, and a homomorphism f_2 that maps the vertices of P, in order, to a, b, c, d, e, d, e. Thus the walk $f_1(P) = 1, 2, 1, 2, 3, 4, 5$ is the embedded pre-image of P_1, and the walk $f_2(P) = a, b, c, d, e, d, e$ is the embedded pre-image of P_2.

7 Structure of walks in H

The *pre-image* of a walk P' in a digraph H is any path P (a digraph that is a path) that admits a homomorphism f to P', taking the first vertex of P to the first vertex of P' and the last vertex of P to the last vertex of P'. If P is a pre-image of P' (under a homomorphism f), then we say that the sequence $f(P)$, which is a walk on the vertices of P', is the *embedded pre-image* of P' under f. Note that $f(P)$ starts with the starting vertex of P' and ends with the ending vertex of P'. The following lemma is well known. (For a proof, see [14, 29] or Lemma 2.36 in [18]).

Lemma 3. *Let P_1 and P_2 be two constricted walks of net length r in a digraph H. There exists a constricted path P of net length r that admits a homomorphism f_1 to P_1 and a homomorphism f_2 to P_2, such that each $f_i, i = 1, 2$ takes the starting vertex of P to the starting vertex of P_i and the ending vertex of P to the ending vertex of P_i.*

We call P a *common pre-image* of P_1 and P_2. If P is a common pre-image of P_1 and P_2, under homomorphisms f_1 and f_2, then their embedded pre-images $f_1(P)$ and $f_2(P)$ are congruent walks on the vertices of P_1 and P_2, respectively. We illustrate this in Figure 2.

Corollary 2. *Let P_1 and P_2 be two constricted walks of net length r. Then there exist congruent walks W_1 and W_2 that are embedded pre-images of P_1 and P_2 respectively.*

We note for future reference that if two congruent walks P, Q avoid *each other*, then the same is true for any congruent embedded pre-images P', Q' of

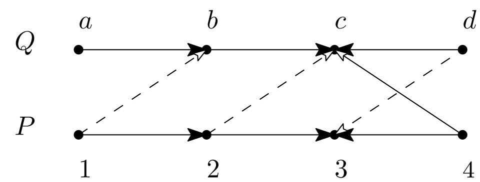

Fig. 3. Here P avoids Q, but $P' = 12, 23, 34, 43, 34$ (the embedded pre-image of P) does not avoid $Q' = ab, bc, cd, dc, cd$ (the embedded pre-image of Q).

P, Q respectively. However, note that if P avoids Q, it is not necessarily true that P' avoids Q' because of the back steps involved in the pre-images, see Figure 3.

We often consider a constricted walk or constricted from below walk and refer to its verities with maximum/minimum height. The following observation illustrates the existence of such vertices.

Observation 11 *We treat every oriented walk W as a balanced digraph. Corresponding to W, we consider an oriented path P with distinct vertices, which is congruent to W. We give level to the vertices of P, where the first vertex of P has level zero. Suppose uv is a forward arc of P, then $level(u) = level(v)+1$, and if uv is a backward arc, then $level(u) = level(v) - 1$. Clearly, if P is constricted from below, then the level of each vertex is at least zero. If P is constricted from above, then the last of P, say h, has the maximum level. Moreover, if P is constricted, then level of h is the same as the height of P and the same as the net length of P.*

7.1 Four Congruent Walks in H

Consider four walks A, B, C, D in H that start in four distinct vertices, p, q, r, s, and end respectively in a, b, b, d, i.e., the end vertices of walks B and C coincide. Assume that A avoids B, and C avoids D. Note that B does not avoid C, and C does not avoid B: at the last step, there is a faithful arc.

Lemma 4. *Let A, B, C, D be four congruent walks in H, from p, q, r, s to a, b, b, d respectively, such that A avoids B and C avoids D. Suppose in H^+ we have the following:*

1. $(p, q) \not\rightsquigarrow (a, d)$ *and* $(p, q) \not\rightsquigarrow (d, b)$,
2. $(r, s) \not\rightsquigarrow (a, d)$ *and* $(r, s) \not\rightsquigarrow (b, a)$.

Then all pairs from A, B, C, D avoid each other, except the pair B, C.

Proof. Let A be the walk $p = a_1, a_2, \dots, a_n = a$, B the walk $q = b_1, b_2, \dots, b_n = b$, C the walk $r = c_1, c_2, \dots, c_n = b$, and D the walk $s = d_1, d_2, \dots, d_n = d$. Let S_i denote the statement that all pairs from

$$A[a_{i+1}, a], B[b_{i+1}, b], C[c_{i+1}, b], D[d_{i+1}, d]$$

avoid each other, except possibly $B[b_{i+1}, b], C[c_{i+1}, b]$. The lemma claims that S_0 holds, while S_{n-1} holds vacuously. Therefore, let $i, 0 \leq i \leq n-1$ be the first index such that S_i holds.

Note that $a_i d_{i+1}$ is not a faithful arc. Otherwise, $(P, Q) : (p, q) \rightsquigarrow (d, b)$ in H^+ where $P = A[p, a_i] + a_i d_{i+1} + D[d_{i+1}, d]$ and $Q = B[q, b_i] + b_i b_{i+1} + B[b_{i+1}, b]$. In more details, note that $A[p, a_i]$ avoids $B[q, b_i]$ (since A avoids B), $a_i d_{i+1}$ avoids $b_i b_{i+1}$ since $a_i b_{i+1}$ is not a faithful arc, and finally $D[d_{i+1}, d]$ avoids $B[b_{i+1}, b]$ because S_i holds. Therefore, we have $(P, Q) : (p, q) \rightsquigarrow (a_i, b_i)(d_{i+1}, b_{i+1}) \rightsquigarrow (d, b)$ which contradicts the assumption of the lemma that $(p, q) \not\rightsquigarrow (d, b)$.

This implies that $b_i d_{i+1}$ is also not a faithful arc, since otherwise, (P, Q) : $(p, q) \rightsquigarrow (a, d)$ where $P = A$ and $Q = B[q, b_i] + b_i d_{i+1} + D[d_{i+1}, d]$ (using the fact that $a_i d_{i+1}$ is not a faithful arc). By a similar line of reasoning, we conclude that $c_i a_{i+1}$ is not a faithful arc (as otherwise $(r, s) \rightsquigarrow (a, d)$), and then $d_i a_{i+1}$ is not a faithful arc (otherwise $(r, s) \rightsquigarrow (b, a)$).

Now, $d_i b_{i+1}$ is not a faithful arc. Otherwise, $(P, Q) : (p, q) \rightsquigarrow (a, d)$ in H^+ where $P = A[p, a_{i+1}] + a_{i+1} a_i + A[a_i, a]$ and $Q = B[q, b_{i+1}] + b_{i+1} d_i + D[d_i, d]$. (using the fact that none of $d_i a_{i+1}, a_i d_{i+1}$ is a faithful arc). Similarly $a_i c_{i+1}$ is not a faithful arc, as otherwise $(r, s) \rightsquigarrow (a, d)$.

Now, $b_i a_{i+1}$ is not a faithful arc. Otherwise, $(R, S) : (r, s) \rightsquigarrow (a, d)$ in H^+ where $R = C + (B[b, b_i])^{-1} + b_i a_{i+1} + A[a_{i+1}, a]$ and $S = D + (D[d, d_i])^{-1} + d_i d_{i+1} + D[d_{i+1}, d]$ (using the fact that none of $d_i b_{i+1}$, $d_i a_{i+1}$ is a faithful arc). Similar argument implies that $d_i c_{i+1}$ is not a faithful arc (otherwise (P, Q) : $(p, q) \rightsquigarrow (a, d)$ where $P = A + A([a, a_{i+1}])^{-1} + a_{i+1} a_i + A[a_i, a]$ and $Q = B + (C[b, c_{i+1}])^{-1} + c_{i+1} d_i + D[d_i, d]$).

Together with the fact that $a_i b_{i+1}$ and $c_i d_{i+1}$ are not faithful arcs (because of the assumption that A avoids B and C avoids D), we obtain a contradiction with the minimality of i; therefore $i = 0$, and the lemma is proved.

7.2 Implication of Four Constricted Walks

A similar result applies to walks that are not all congruent as long as they are constricted and have the same net length. Of course, the pairs of walks where one avoids the other must be congruent by definition.

In the proof of the main result of this subsection, we will use the following generalization of Lemma 3, or more specifically, of Corollary 2. Consider two walks in a digraph H, a constricted walk $P_1 = a_1, a_2, \ldots, a_n$ of net length $r > 0$, and another walk $P_2 = b_1, b_2, \ldots, b_m$ which is constricted from below (but not necessarily from above), and has height r. Suppose P_2 has net length ℓ. Note that P_1 also has height r, and contains (possibly several) vertices a_x such that the walk $Q = P_1 + (P_1[a_n, a_x])^{-1}$ also has net length ℓ. The walk Q is a walk of height r and net length ℓ, on the vertices of P_1. We claim that for some a_x the paths P_2 and Q have a common pre-image and hence congruent embedded pre-images.

Lemma 5. *Let P_1 be a constricted walk of net length $r > 0$, and P_2 a walk constricted from below of net length $\ell \geq 0$ and height r, both in a digraph H. Then there exist congruent walks Q_1, Q_2 with the following properties :*

- *Q_1 is an embedded pre-image of a walk of net length ℓ and height r, on the vertices of P_1,*
- *the first vertex of Q_1 is the same as P_1,*
- *Q_2 is an embedded pre-image of P_2.*

Proof. Let the walks be $P_1 = a_1, a_2, \ldots, a_n$ and $P_2 = b_1, b_2, \ldots, b_m$. We proceed by induction on the sum $m + n$. Let h be any subscript such that the prefix $P_2^1 = P_2[b_1, b_h]$ has net length r (see Observation 6.3). Then P_2^1 is constricted and of net length r, as is P_1.

Let s be a subscript such that $P_2[b_h, b_s]$ has the minimum possible net length k. (Note that $k \leq 0$). The walk $P_2^2 = (P_2[b_h, b_s])^{-1}$ has net length $-k \geq 0$. Let t be the greatest subscript such that $P_1[a_t, a_n]$ has net length $-k$. Then $P_1^2 = P_1[a_t, a_n]$ and P_2^2 are constricted and have the same net length. The remaining walk $P_2^3 = P_2[b_s, b_m]$ has net length $\ell - (r + k)$, and is constricted from below. Suppose it has height z. Let $q \geq t$ be the smallest subscript, such that $P_1^3 = P_1[a_t, a_q]$ has net length z. Note that P_1^3 is constricted.

P_1, P_2^1 are constricted and have the same net length. Therefore, by Corollary 2, they have embedded pre-images X_1, Y_1 that are congruent . By Corollary 2, we obtain congruent walks X_2, Y_2 that are embedded-pre images of P_1^2, P_2^2.

Note that P_1^3 is constricted, whence the induction hypothesis applies to P_1^3 and P_2^3, and hence, we obtain embedded pre-images X_3, Y_3 (of P_1^3, P_2^3) that are congruent. Now $Q_1 = X_1 + X_2 + X_3$ and $Q_2 = Y_1 + Y_2 + Y_3$ are the desired walks.

Lemma 6. *Let A, B, C, D be four constricted walks of the same net length, from p, q, r, s to a, b, b, d respectively, such that A, B are congruent and A avoids B, and C, D are congruent and C avoids D. Suppose in H^+ we have:*

1. *$(p, q) \not\rightsquigarrow (a, d)$ and $(p, q) \not\rightsquigarrow (d, b)$,*
2. *$(r, s) \not\rightsquigarrow (a, d)$ and $(r, s) \not\rightsquigarrow (b, a)$.*

Then there exist congruent walks A', B', C', D' that are embedded pre-images of A, B, C, D respectively, such that all pairs from A', B', C', D' avoid each other, except the pair B', C'. Hence, A, B avoid each other, and C, D avoid each other.

Proof. Let A be the walk $p = a_1, a_2, \ldots, a_n = a$, B the walk $q = b_1, b_2, \ldots, b_n = b$, C the walk $r = c_1, c_2, \ldots, c_m = b$, and D the walk $s = d_1, d_2, \ldots, d_m = d$. Notice that $b_i \in B$, $1 \leq i \leq n$, is the corresponding vertex to $a_i \in A$, and $d_\ell \in D$, $1 \leq \ell \leq m$ is the corresponding vertex to $c_\ell \in C$. Furthermore, we point out the following easy observation.

Observation 12 *For every $1 \leq \ell \leq n$, neither of $(a, d), (d, b)$ is reachable (in H^+) from (a_ℓ, b_ℓ), otherwise, they would also be reachable from (p, q) because A avoids B. Similarly, neither (a, d) nor (b, a) is reachable from (c_t, d_t), $1 \leq t \leq m$.*

We prove the lemma by induction on the sum of the lengths $m+n$. If $m+n = 0$, i.e., $m = n = 0$, this holds trivially. First, we turn our attention to the case where all A, B, C, D are strongly constricted from below. After proving the lemma for this case, we proceed to the general case.

Suppose first that A, B, C, D are strongly constricted from below (no prefix of A has net length zero). This means that the first two arcs in each walk are forward, and the walks $A-p, B-q, C-r, D-s$ (here $A-p$ is the walk obtained from A by removing the first vertex p) are also constricted walks of the same net length, with the first two congruent and the last two congruent. Moreover, by Observation 12, in H^+, neither (a, d) nor (d, b) is reachable from (a_2, b_2). Similarly, neither (a, d) nor (b, a) is reachable from (c_2, d_2). By the induction hypothesis, $A - p, B - q, C - r, D - s$ have congruent embedded pre-images A'', B'', C'', D'' in which all pairs except B'', C'' avoid each other. Noting that A'' starts in a_2, we let A' consist of p concatenated with A'' (i.e., $A' = a_1a_2 + A''$), let B' be q concatenated with B'', and similarly for C' and D'. Since A', B', C', D' are all congruent, we can apply Lemma 4, and conclude that all pairs avoid each other, except the pair B', C'. This implies that A, B also avoid each other, and C, D also avoid each other.

Since we have already considered the case when all four walks A, B, C, D are strongly constricted from below, we may assume, up to symmetry, that A, B are not strongly constricted from below, i.e., that there exists a subscript $j > 1$ such that $A[p, a_j]$ and $B[q, b_j]$ have net length zero. We take the subscript j as large as possible, therefore $A[a_j, a], B[b_j, b]$ are strongly constricted from below and have the same net length as C, D. Now, by Observation 12, the conditions 1 and 2 of the lemma are satisfied, and by induction hypothesis for $A[a_j, a], B[b_j, b], C, D$, we conclude that $A[a_j, a], B[b_j, b], C, D$ have congruent embedded pre-images that pairwise avoid each other (except for the embedded pre-images of $B[b_j, b], C$). This implies that $A[a_j, a], B[b_j, b]$ also avoid each other, and C, D also avoid each other. If C, D were also not strongly constricted from below, we could draw a similar conclusion that A, B avoid each other, as claimed. However, in general, C, D may happen to be strongly constricted from below, and we proceed more carefully as follows: we first show the following claim.

Claim. For every $i < j$, $A[a_i, a_n], B[b_i, b_n]$ avoid each other (i.e., A, B avoid each other).

Once the above claim is established, the lemma follows. Notice that since the walks A, B, C, and D are constricted, by applying Lemma 3 on A, B we conclude there exists congruent walks A_1, B_1 from p, q to a, b (respectively), that are embedded pre-images of A, B respectively. By applying Lemma 3 on C, D we conclude there exists congruent walks C_1, D_1 from r, s to b, d (respectively), that are embedded pre-images of C, D respectively. Finally, by applying Lemma 3 on A_1, C_1 we conclude there exists congruent walks A', C' from p, r to a, c (respectively), that are embedded pre-images of A_1, C_1 respectively. Now it is easy to obtain congruent walks A', B', C', D' from p, q, r, s to a, b, b, d that are embedded pre-images of A, B, C, D respectively.

Since A and B are congruent and avoid each other, the walks A', B' follow the same sequence of back and forth steps inside A, B, and also avoid each other.

Similarly, C', D' also avoid each other. Therefore we can now apply Lemma 4 to A', B', C', D' and conclude that all pairs from A', B', C', D' avoid each other, except the pair B', C'.

Proof of Claim 7.2. Notice that we have already shown that $A[a_j, a], B[b_j, b]$ avoid each other, we proceed by backward induction on i, starting with $i = j$. Let $i < j$ be the minimum subscript such that $A[a_i, a], B[b_i, b]$ avoid each other, and assume that the height of the walk $(A[a_i, a_j])^{-1}$ is t. Let h be the first subscript such that the prefix $C[c_1, c_h]$ of walk C has net length t. Let $P_1 = C[c_1, c_h]$ and $P_2 = (A[a_i, a_j])^{-1}$. Lemma 5 implies that there exist congruent walks W_A and W_C where W_A is an embedded pre-image of P_2, and W_C is an embedded pre-image of a walk with height t on the vertices of P_1 starting at c_1 and ending at c_ℓ. Let W_B be the embedded pre-image of $(B[b_i, b_j])^{-1}$ corresponding to W_A, and similarly let W_D be walk on D corresponding to W_C (W_D is congruent with W_C and it starts from d_1 and end at d_ℓ). Combining these, we deduce that $W_A^{-1} + A[a_j, a]$, $W_B^{-1} + B[b_j, b]$, $W_C^{-1} + C$, $W_D^{-1} + D$ all have congruent embedded pre-images, say, X, Y, Z, U. Notice that X, Y avoid each other and Z, U avoid each other.

In order to continue our backward induction, we consider $A[a_{i-1}, a]$ and $B[b_{i-1}, b]$ and prove they avoid each other. The idea of the proof is to continue this way backwards on A, B until proving that they avoid each other in their entirety. In what follows, we consider different cases for the direction of $a_{i-1}a_i$ and a_ia_{i+1}, and in each case we prove $A[a_{i-1}, a]$ and $B[b_{i-1}, b]$ avoid each other.

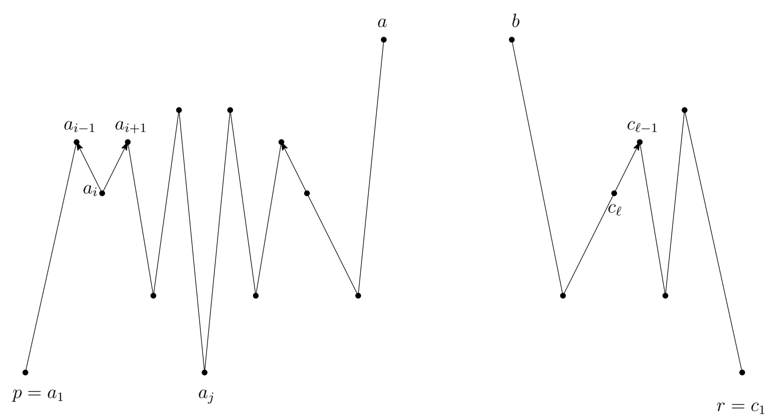

Fig. 4. The notation for Claim 7.2; each straight segment represents a constricted walk. The figure shows $A[a_1, a_i]$ plus the embedded pre-image of $A[a_i, a]$. It also shows $(C[c_\ell, c_1])^{-1} + C$. Here, a_ia_{i+1} is a forward arc, i.e., $c_{\ell-1}c_\ell \in C$ is a backward arc. Note that in general $A[a_1, a_{i-1}]$ may consist of several constricted segments.

Case 1. $a_{i-1}a_i, a_ia_{i+1}$ have different directions (see Figure 4). Notice that $c_{\ell-1}c_\ell$ have the same direction as $a_{i-1}a_i$ (since X, Z are congruent and the first arc of X is a_ia_{i+1} and the first arc of Z is $c_\ell c_{\ell-1}$, a_ia_{i+1} and $c_\ell c_{\ell-1}$ have the same direction). Thus, $a_{i-1}a_i + X$, $b_{i-1}b_i + Y$, $c_{\ell-1}c_\ell + Z$, $d_{\ell-1}d_\ell + U$ are congruent, and satisfy the conditions of Lemma 4 ($a_{i-1}a_i + X$ avoids $b_{i-1}b_i + Y$ and $c_{\ell-1}c_\ell + Z$ avoids $d_{\ell-1}d_\ell + U$), and hence, $a_{i-1}a_i + X$, $b_{i-1}b_i + Y$ avoid each other. This implies that $A[a_{i-1}, a], B[a_{i-1}, b]$ avoid each other.

Case 2. $a_{i-1}a_i, a_ia_{i+1}$ have the same direction and $c_\ell c_{\ell+1}$ has the opposite direction to $a_{i-1}a_i$. Now $a_{i-1}a_i + X$, $b_{i-1}b_i + Y$, $c_{\ell+1}c_\ell + Z$, $d_{\ell+1}d_\ell + U$ are all congruent (since $c_{\ell+1}c_\ell$ has the same direction as $a_{i-1}a_i$), and by Lemma 4 $a_{i-1}a_i + X$ and $b_{i-1}b_i + Y$ avoid each other. Therefore, $A[a_{i-1}, a_j], B[b_{i-1}, b_j]$ avoid each other.

Case 3. $a_{i-1}a_i, a_ia_{i+1}$ are forward arcs and $c_\ell c_{\ell+1}$ is a forward arc (see Figure 5). In this case we show that there exists some $c_{\ell'}$ in $C[c_1, c_\ell]$ such that $c_{\ell'-1}c_{\ell'}$ is a forward arc and $c_{\ell'}$ play the same role as c_ℓ in *Case 2*. The argument is as follows. Let $a_{\ell'}$ be the last vertex on $X[a_i, a_j]$ such that $X[a_i, a_{\ell'}]$ is constricted from below and has net length zero. Such a vertex $a_{\ell'}$ exists because a_ia_{i+1} is a forward arc. By the choice of $a_{\ell'}$, we observe that $a_{\ell'-1}a_{\ell'}$ is a forward arc. Let $c_{\ell'} \in Z$ be the corresponding vertex to $a_{\ell'}$ (notice that X, Z are congruent), and hence, $c_{\ell'-1}c_{\ell'}$ is a forward arc. Let a_h be a vertex on $X[a_i, a_{\ell'}]$ with a maximum height, and let $c_h \in Z$ be the corresponding vertex to a_h. Notice that $X[a_i, a_h]$ is constricted. Suppose the net length of $X[a_i, a_h]$ is L. By the choice of a_h, $X[a_h, a_{\ell'}]$ is constricted and has net length $-L$. $(Z[c_h, c_{\ell'}])^{-1}$ is also constricted and has net length L because $Z[c_h, c_{\ell'}]$ and $X[a_h, c_{\ell'}]$ are congruent and both have net length $-L$. By Corollary 2, $X[a_i, a_h]$ and $(Z[c_h, c_{\ell'}])^{-1}$ have embedded pre-images X_1, Z_1 that are congruent. We let $b_{\ell'} \in Y$ and $d_{\ell'} \in U$ be the corresponding vertices to $a_{\ell'}$, and let $b_h \in Y$, $d_h \in U$ be the corresponding vertices to a_h. We also let Y_1 to be an embedded pre-image of $Y[b_i, b_h]$ which is congruent to X_1, Z_1 and U_1 be an embedded pre-image of $U[d_i, b_h]$ which is congruent to X_1, Z_1. Now, let $X' = X_1 + X[a_h, a_n]$, $Y' = Y_1 + Y[b_h, b_n]$, $Z' = Z_1 + Z[c_h, c_n]$, and $U' = U_1 + U[d_h, d_n]$. Observe that since X, Y, Z, U are congruent, X', Y', Z', U' are congruent. Since X, Y avoid each other, X', Y' avoid each other and since Z, U avoid each other Z', U' avoid each other.

Finally, we recall that $a_{i-1}a_i$ and $c_{\ell'-1}c_{\ell'}$ are both forward arcs. Now, $a_{i-1}a_i + X'$ avoids $b_{i-1}b_i + Y'$ ($a_{i-1}b_i$ is not a faithful arc since A avoids B) and $c_{\ell'-1}c_{\ell'} + Z'$ avoids $d_{\ell'-1}d_{\ell'} + U'$. Notice that $a_{i-1}a_i + X'$, $b_{i-1}b_i + Y'$, $c_{\ell'-1}c_{\ell'} + Z'$, $d_{\ell'-1}d_{\ell'} + U'$ are congruent. Therefore, by Lemma 4 for $a_{i-1}a_i + X'$, $b_{i-1}b_i + Y'$, $c_{\ell'-1}c_{\ell'} + Z'$, $d_{\ell'-1}d_{\ell'} + U'$ we conclude that $a_{i-1}a_i + X', b_{i-1}b_i + Y'$ avoid each other. Therefore, $A[a_{i-1}, a_j], B[b_{i-1}, b_j]$ avoid each other.

Case 4. $a_{i-1}a_i, a_ia_{i+1}$ are backward arcs and $c_\ell c_{\ell+1}$ is a backward arc. This case is analogues to *Case 3*.

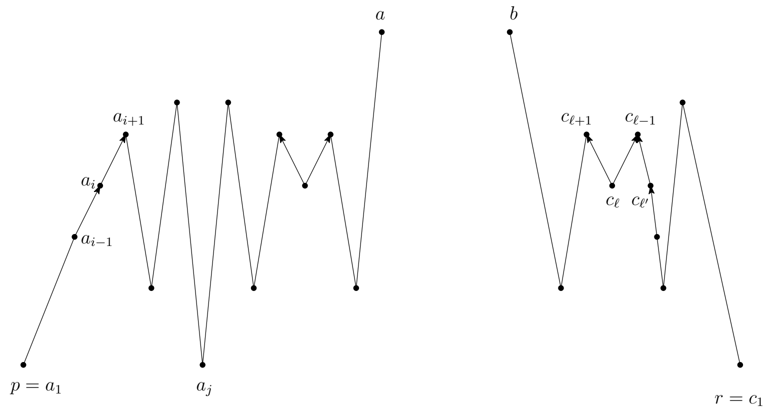

Fig. 5. The notation for Claim 7.2; each straight segment represents a constricted walk. The figure shows $A[a_1, a_i]$ and the embedded pre-image of $A[a_i, a]$. It also shows $(C[c_\ell, c_1])^{-1} + C$. Here, $a_i a_{i+1}$ is a forward arc, i.e., $c_{\ell-1} c_\ell \in C$ is a backward arc. Note that in general $A[a_1, a_{i-1}]$ may consist of several constricted segments.

We repeat this argument until $i = 1$, and hence, Claim 7.2 is proved. This finishes the proof of Lemma 6. Note that in the above arguments we assumed A, B, C, D all have the same positive net length. Of course, the argument for the case when A, B, C, D all have the same negative net length is analogous.

Corollary 3. *Let A, B, C, D be four congruent walks in H from p, q, r, s to a, b, c, d (all distinct) respectively. Suppose A avoids B and C avoids D and the following conditions hold.*

1. $(p, q) \not\rightsquigarrow (a, c), (a, d), (c, b), (d, b), (d, c)$
2. $(r, s) \not\rightsquigarrow (a, d), (a, c), (b, d), (c, a), (c, b)$

Then A, B, C, D pairwise avoid each other except A, B and C, D. Furthermore, if $(b, c) \not\rightsquigarrow (a, c), (b, d)$ then A, B, C, D pairwise avoid each other.

Proof. Let $A : p = a_0, a_1, \dots, a_{n-1}, a_n = a$, $B : q = b_0, b_1, \dots, b_{n-1}, b_n = b$, $C : r = c_0, c_1, \dots, c_{n-1}, c_n = c$, and $D : s = d_0, d_1, \dots, d_{n-1}, d_n = d$. Let j be the minimum index such that $A_j[a_j, a], B[b_j, b], C[c_j, c], D[a_j, d]$ are pairwise avoid each except except A_j, B_j and C_j, D_j. If $j = 0$ then we are done. Otherwise, let $j > 0$. Now $a_{j-1} c_j$ is not faithful arc otherwise, we have $(p, q) \rightsquigarrow (a_j, b_j)(c_{j+1}, b_{j+1}) \rightsquigarrow (c, b)$, a contradiction. Consequently $b_{j-1} c_j$ is not a faithful arc as otherwise, $(p, q) \rightsquigarrow (a_{j-1}, b_{j-1})(a_j, c_j) \rightsquigarrow (a, c)$. A similar argument implies that $c_{j-1} b_j, c_{j-1} a_{j+1}$ are not faithful arcs, as otherwise, $(r, s) \rightsquigarrow (b, d)$ or $(r, s) \rightsquigarrow (a, d)$. Now since, $c_{j-1} a_j$ is not a faithful arc, we have $d_{j-1} a_j$ is not a faithful arc, as otherwise, $(r, s) \rightsquigarrow (c_{j-1}, d_{j-1}(c_j, a_j) \rightsquigarrow (c, a)$. By the same line of reasoning, the absence of $c_{j-1} b_j$ implies that $d_{j-1} b_j$ is not a faithful arc, as otherwise, $(r, s) \rightsquigarrow (c, b)$.

Now suppose $(b,c) \not\rightsquigarrow (a,c),(b,d)$. If $b_{j-1}a_j$ is a faithful arc then $(b_{j-1},c_{j-1})(a_j,c_j) \in A(H^+)$, and hence, $(b,c) \rightsquigarrow (b_{j-1},c_{j-1})(a_j,c_j) \rightsquigarrow (a,c)$, a contradiction. Similarly, $d_{j-1}c_j$ is not a faithful arc. These show that $j=0$, and the corollary is proved.

Corollary 4. *Let A,B,C,D be four constricted walks in H from p,q,r,s to a,b,c,d(all distinct) respectively. Suppose A avoids B and C avoids D with the following conditions.*

1. $(p,q) \not\rightsquigarrow (a,c),(a,d),(c,b),(d,b),(d,c)$
2. $(r,s) \not\rightsquigarrow (a,d),(a,c),(b,d),(c,a),(c,b)$

Then A,B,C,D have embedded pre-images A',B',C',D' that pairwise avoid each other expect A',B' and C',D'. Furthermore, if $(b,c) \not\rightsquigarrow (a,c),(b,d)$ then A,B,C,D have embedded pre-images A',B',C',D' that pairwise avoid each other.

Proof. Add distinct vertices a',d' to the end of A,D by adding forward arcs aa',dd' and obtain walks A_1,D_1 respectively. Moreover, add vertex b' (distinct from a',d') to the end of both B,C by adding arcs bb',cb' and obtain walks B_1,C_1 respectively. Observe that A_1,B_1,C_1,D_1 are all congruent, and A_1 avoids B_1 and C_1 avoids D_1. Moreover, since $(p,q) \not\rightsquigarrow (a,d),(d,b)$, we have $(p,q) \not\rightsquigarrow (a',d'),(d',b')$. Similarly, $(r,s) \not\rightsquigarrow (a',d'),(b',d')$.

Now by applying Lemma on 6 on A_1,B_1,C_1,D_1 we conclude that A_1,B_1,C_1,D_1 have embedded pre-images A_1',B_1',C_1',D_1' (they are all congruent) where A_1' and B_1' avoid each other, and C_1' and D_1' avoid each other. Now by applying Corollary 3, on $A_1'-a'$, $B_1'-b'$, $C_1'-b'$, and $D_1'-d'$ we conclude that they all avoid each other. This implies that A,B,C,D all avoid each other.

8 Structure of circuits in H^+

8.1 M-Lemma

We now consider two particular situations where a circuit occurs in one strong component of H^+. We need to deal with extremal vertices. We observe that *a cycle C of positive net length k has at least k extremal vertices.* Namely, we can obtain such vertices $v_0,v_1,\ldots,v_{k-1}$ as follows: starting at any vertex x and following C, the net length of the prefix $C[x,v]$ varies with v from 0 to a possibly negative minimum m, but ending with $k>0$. We let v_0 be the last vertex with the net length of $C[x,v_0]$ equal to the minimum m (possibly $v_0=x$ if $m=0$). We can let v_i be the last vertex with the net length of $C[v_0,v_i]$ equal to $i, i=1,2,\ldots,k-1$. Note that each walk $C[v_i,v_{i+1}]$ is strongly constricted from below and has net length one. We also note for future reference that any other extremal vertex of C has a walk of net length zero to one of $v_0,v_1,\ldots,v_{k-1}$. We say vertex x is an *extremal vertex in a digraph H* if there exists a cycle C in H that x is an extremal vertex in C.

Theorem 13. *Suppose C is a closed walk in H of net length greater than one, and x, y are two extremal vertices of C such that the net length of $C[x, y]$ is positive. Let P_x be the walk starting at x, obtained by continuously following the cycle C in the positive direction $|C| + 1$ times and let P_y obtained the same way starting at y. Suppose H contains two congruent walks X, Y such that X avoids Y and such that X is an embedded pre-image of P_x and Y is an embedded pre-image of P_y. Then H^+ contains a strong circuit.*

Proof. Let $f : X \to P_x$ be a homomorphism taking the first vertex of X to the first vertex of P_x, i.e., x, and similarly for $g : Y \to P_y$. It is easy to see that X contains vertices $x_0, x_1, ...$ and Y contains vertices $y_1, y_2, ...$ such that (for all i)

1. x_0 is the first vertex of X, and y_1 is the first vertex of Y,
2. x_i is the vertex on X corresponding to y_{i+1} on Y,
3. $f(x_i) = g(y_i)$, and the vertex $v_i = f(x_i) = g(y_i)$ is extremal on C,
4. each segment $X[x_i, x_{i+1}]$ and $Y[y_i, y_{i+1}]$ has the same net length as $C[x, y]$.

Since C has at most $|C|$ extremal vertices, we must eventually have $f(x_i) = f(x_{i+j})$ for some positive i and j. It is now clear that $(v_i, v_{i+1}), (v_{i+1}, v_{i+2}), \ldots, (v_{i+j}, v_i)$ is a circuit in a strong component of H^+, since $X[x_i, x_{i+1}] = C[v_i, v_{i+1}]$ avoids $Y[y_{i+1}, y_{i+2}] = C[v_{i+1}, v_{i+2}]$ (subscripts reduced modulo j).

We say a positive arc $(x, y)(x', y') \in H^+$, is *symmetric* if and only if $(y, x)(y', x')$ is a positive arc in H^+. Similarly, a negative arc $(x, y)(x', y') \in H^+$, is *symmetric* if and only if $(y, x)(y', x')$ is a negative arc in H^+. In other words, when $(x, y)(x', y')$ is a symmetric arc we have $xx', yy' \in A(H)$ but $xy', yx' \notin A(H)$, or $x'x, y'y \in A(H)$ but $y'x, x'y \notin A(H)$. Symmetric arcs of H^+ will play an important role. A walk, strong component, or subgraph of H^+ is called *symmetric* if all its arcs are symmetric.

In the rest of the paper we stick to the following notation whenever we consider a circuit. Depending on our need we may choose a specific pair (p_i, q_{i+1}) or assume Z_i is more specific.

Notation 14 *Let $C : (a_0, a_1), (a_1, a_2), \ldots, (a_n, a_0)$ be a circuit in H^+. To each pair (a_i, a_{i+1}) of the circuit we associate the following (see Figure 6)*

1. *A pair $(p_i, q_{i+1}) \in H^+$, and a pair $(p'_i, q'_{i+1}) \in H^+$ such that $(p'_i, q'_{i+1})(p_i, q_{i+1}) \in A(H^+)$ ((p'_i, q'_{i+1}) may not exist),*
2. *A walk Z_i in H^+ from (p_i, q_{i+1}) to (a_i, a_{i+1}) which is constricted from below, has net value zero, and has height $L_i \geq 0$,*
3. *A pair $(g_i, h_{i+1}) \in Z_i$ such that $Z_i[(p_i, q_{i+1}), (g_i, h_{i+1})]$ is constricted and has net value L_i,*
4. *A constricted walk A_i from p_i to g_i with net length L_i, and a constricted walk B_{i+1} from q_{i+1} to h_{i+1} such that A_i avoids B_{i+1},*
5. *A constricted walk A'_i from g_i to a_i with net length $-L_i$, , a constricted walk B'_{i+1} from h_{i+1} to a_{i+1} such that A'_i avoids B'_{i+1},*
6. *$Z_i = (A_i + A'_i, B_{i+1} + B'_{i+1})$.*

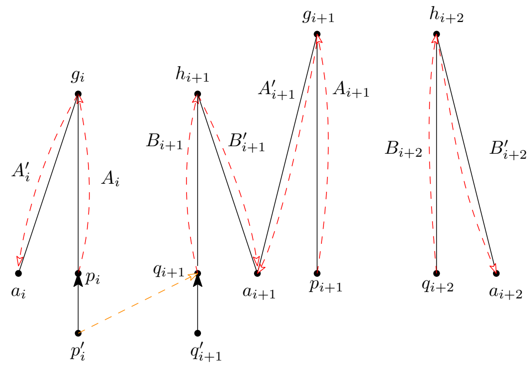

Fig. 6. This figure refers to circuit C in Notation 14. Each straight segment represents a constricted walk. The dotted line shows the direction of the walks. Here, A_i (from p_i to g_i) avoids B_{i+1}(from q_{i+1} to h_{i+1}), and A'_i (from g_i to a_i) avoids B'_{i+1} (from h_{i+1} to a_{i+1}).

One of our basic tools is the following lemma illustrated in Figure 7; because each quadruple of walks resembles the shape of the letter M, we call it the M-Lemma.

Lemma 7 (M-Lemma). *Let $(a_0, a_1), (a_1, a_2), \dots , (a_n, a_0)$ be a circuit in H^+. Consider Notation 14 for this circuit. Suppose that $1 < t \leq n, 0 < \ell$ are integers such that the following hold*

1. *for each $i = 0, 1, \dots, t$, $L_i = \ell$ (i.e. A_i, B_{i+1} have net length ℓ), and A'_i, B'_{i+1} have net length $-\ell$.*
2. *if $k \neq j+1$, then $(p_i, q_{i+1}) \not\rightsquigarrow (a_j, a_k)$, $0 \leq j, k \leq t$ (here $q_{n+1} = q_0$).*

Let C be any one of the walks A'_i or B'_i or A_i^{-1} or of B_i^{-1}, and let D be any one of the walks A'_j or B'_j or A_j^{-1} or of B_j^{-1}, with $i \neq j$. Then C, D have embedded pre-images that avoid each other.

Proof. We prove the lemma with $t = n$, and it is easy to check that the proof allows any smaller $t, t \geq 2$. We first prove that any B'_i, B'_j with $i \neq j$ have embedded pre-images that avoid each other.

Observation 15 *For every $0 \leq i \leq n$, $(g_i, h_{i+1}) \not\rightsquigarrow (a_r, a_s)$, $r \neq s-1, s$, $0 \leq r, s \leq n$. Otherwise, $(p_i, q_{i+1}) \rightsquigarrow (g_i, h_{i+1}) \rightsquigarrow (a_r, a_s)$; violating the condition (2) of the lemma.*

Recall that $A'_{i-1}, B'_i, A'_i, B'_{i+1}$ are constricted and have the same net length, namely $-\ell$. Furthermore, A'_{i-1} avoids B'_i and A'_i avoids B'_{i+1} (by notation 14 (5)). Now by Observation 15, the conditions of Lemma 6 are satisfied for walks $A'_{i-1}, B'_i, A'_i, B'_{i+1}$, and hence, all pairs from $A'_{i-1}, B'_i, A'_i, B'_{i+1}$ have embedded pre-images that avoid each other, except the pair B'_i, A'_i. Now by applying Corollary 4 on $A'_i, B'_{i+1}, A'_j, B'_{j+1}$ ($j \neq i+1$ and $i \neq j+1$) (and the fact

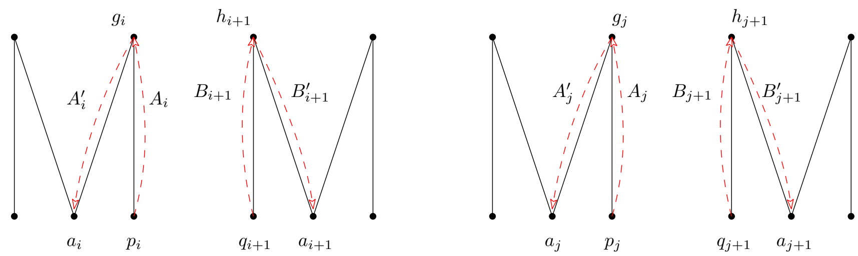

Fig. 7. The notation for Lemma 7; each straight segment represents a constricted walk. The dashed line shows the direction of the walks. Here, A_i (from p_i to g_i) avoids B_{i+1}(from q_{i+1} to h_{i+1}), and A'_i (from g_i to a_i) avoids B'_{i+1} (from h_{i+1} to a_{i+1}).

that A'_i, B'_{i+1} avoid each other and A'_j, B'_{j+1} avoid each other) we conclude that $A'_i, B'_{i+1}, A'_j, B'_{j+1}$ have embedded pre-images that pairwise avoid each other.

Notice that since A'_r, A'_s (for $s \neq r$) have embedded pre-images that avoid each other, we conclude that $(g_r, g_s) \rightsquigarrow (a_r, a_s)$. Therefore, $(p_i, q_{i+1}) \not\rightsquigarrow (a_r, a_s)$, $s \neq r, r+1$; as otherwise, $(p_i, q_{i+1}) \rightsquigarrow (a_r, a_s)$; contradicting the condition 2 of the lemma. We also notice that $(a_i, a_{i+1}) \rightsquigarrow (g_i, h_{i+1})$ since A'_i, B'_{i+1} avoid each other, and hence, $(A'_i)^{-1}, (B'_{i+1})^{-1}$ avoid each other. Now we have the following.

Observation 16 *For every $0 \leq i \leq n$, $(p_i, q_{i+1}) \not\rightsquigarrow (g_r, g_s), (h_r, h_s), (g_r, h_r)$, $r \neq s, s+1$, $0 \leq r, s \leq n$. Moreover, $(a_i, a_{i+1}) \not\rightsquigarrow (g_r, g_s), (h_r, h_s), (g_r, h_r)$, $r \neq s, s+1$.*

By Observation 15, the conditions of Corollary 4 on walks $A_i, B_{i+1}, A_j, B_{j+1}$ (with $a = a_i$, $b = a_{i+1}$, $c = a_j$, $d = a_{j+1}$, $p = g_i$, $q = h_{i+1}$, $r = g_j$, and $s = h_{j+1}$, $0 \leq i < j \leq n$) are satisfied, and hence, we conclude that A_i, B_j have embedded pre-images that avoid each other. Moreover, by Observation 16 the conditions of Corollary 4 on $(A'_i)^{-1}, (B'_{i+1})^{-1}, A_j, B_{j+1}$ are satisfied, and hence, we conclude that: $(A'_i)^{-1}, A_j$ have embedded pre-images that avoid each other, $(B'_{i+1})^{-1}, A_j$ have embedded pre-images that avoid each other, and finally $(B'_{i+1})^{-1}, B_{j+1}$ have embedded pre-images that avoid each other.

Remark 1. Another way to state the conclusion of the Lemma 7 is the following. There are embedded pre-images of all $A'_i, B'_i, A_i^{-1}, B_i^{-1}, i = 0, 1, \ldots, t$, such that any two embedded pre-images of walks with different subscripts avoid each other. The lemma will often be used for walks where $A'_i = A_i^{-1}$ and/or $B'_i = B_i^{-1}$ (or even $A'_i = A_i^{-1} = B'_i = B_i^{-1}$).

8.2 Basic Tools for Minimal Circuits

We now analyze a minimal circuit in H^+ under certain conditions and we derive properties of H^+.

Definition 8 (UL-pair). *Let S be a subset of $V(H^+)$. We say (x, y) is an UL-pair (upper layer pair) with respect to S if there exists a pair $(x', y') \in S$ that* *reaches (x,y) via a path $U_{x,y}$ in H^+ which is strongly constricted from below and has net value one.*

Lemma 8. *Let S be a set of pairs in H^+ and let $\widehat{S}$ contain a minimal circuit $(a_0,a_1),(a_1,a_2),\dots,\ (a_{n-1},a_n),(a_n,a_0),\ n>1$. Consider $Z_i, L_i,\ (p_i,q_{i+1})$ according to notations 14 for this circuit. Then the following hold.*

1. *If $L_{i-1}\le L_i$ then Z_{i-1} is symmetric, $(a_{i-1},a_i),(a_{i-1},q_i),(p_{i-1},q_i),(p_{i-1},a_i)$ are in the same strong component, and they are all reachable from each other via symmetric and constricted from below paths of net value zero. Moreover, $(a_i,a_{i+1}),(q_i,a_{i+1})$ are in the same strong component, and $(a_{i-1},a_{i+1}),(p_{i-1},a_{i+1})$ are in a same strong component. Circuit $(a_0,a_1),\dots,(a_{i-1},q_i),(q_i,a_{i+1}),\dots,(a_n,a_0)$ is a minimal circuit belong to $\widehat{S}$.*
2. *If $L_i\le L_{i-1}$ then Z_i is symmetric, $(a_i,a_{i+1}),(a_i,q_{i+1}),(p_i,q_{i+1}),(p_i,a_{i+1})$ are in the same strong component, and they are all reachable from each other via symmetric and constricted from below paths of net value zero. Moreover, $(a_{i-1},a_i),(a_{i-1},p_i)$ are in the same strong component, and $(a_{i-1},a_{i+1}),(a_{i-1},q_{i+1})$ are in a same strong component. Circuit $(a_0,a_1),\dots,(a_{i-1},p_i),(p_i,a_{i+1}),\dots,(a_n,a_0)$ is a minimal circuit belong to $\widehat{S}$.*
3. *If $(a_{i-1},a_i),(a_{i+1},a_{i+2})$ are UL-pairs and $L_i\ge L_{i-1},L_{i+1}$ then (a_i,a_{i+1}) is also an UL-pair and (a_i,a_{i+1}) is reachable from (q_i,p_{i+1}) by a symmetric and constricted from below path of net value zero. Moreover, $(q'_i,p'_{i+1})(q_i,p_{i+1})$ is a symmetric arc in H^+.*
4. *If (a_i,a_{i+1}) and (a_{i-1},a_i) ((a_{i+1},a_{i+2})) are UL-pairs and $L_{i-1}\le L_i\le L_{i+1}$ ($L_{i+1}\le L_i\le L_{i-1}$) then (a_i,a_{i+1}) and $(q_i,q_{i+1}),(a_i,q_{i+1})$ ($(p_i,p_{i+1}),(a_i,p_{i+1})$) are in the same strong component, and (q_i,q_{i+1}) ((p_i,p_{i+1})) is reachable from (a_i,a_{i+1}) by a constricted symmetric path of net value zero. Moreover, $p'_{i-1}q_{i+1},q'_iq_{i+1}\notin A(H)$ ($p'_{i+1}p_i,q'_{i+2}p_i\notin A(H)$).*
5. *If $(a_i,a_{i+1}),(a_{i+1},a_{i+2})$ are UL-pairs and $L_i\le L_{i-1},L_{i+1}$ then $(p_i,q_{i+2}),(a_i,a_{i+2})$ are reachable from each other via a symmetric and constricted from below path of net value zero, and $(q_{i+1},q_{i+2}),(a_{i+1},a_{i+2})$ are reachable from each other via a symmetric and constricted from below path of net value zero. Moreover, $(p'_i,q'_{i+2})(p_i,q_{i+2})$, $(q'_{i+1},q'_{i+2})(q_{i+1},q_{i+2})$, and $(p'_i,q'_{i+1})(p_i,q_{i+1})$ are symmetric arcs of H^+.*

Proof. Proof of 1. Let u be a vertex on A'_i and v be the corresponding vertex to u on B'_{i+1} such that $C=A'_i[u,a_i]$, $D=B'_{i+1}[v,a_{i+1}]$ are constricted and have net length $-L_{i-1}$ (C,D shown by green in Figure 8). Since the circuit is minimal, we have $(p_{i-1},q_i)\not\rightsquigarrow(a_{i-1},a_{i+1})$, and $(p_i,q_{i+1})\not\rightsquigarrow(a_{i-1},a_{i+1})$, as otherwise, we get a shorter circuit, violating the minimality of the circuit. Similarly, $(p_{i-1},q_i)\not\rightsquigarrow(a_{i+1},a_i)$, and $(p_i,q_{i+1})\not\rightsquigarrow(a_i,a_{i-1})$.

As a consequence, since $(p_{i-1},q_i)\rightsquigarrow(g_{i-1},h_i)$, we have $(g_{i-1},h_i)\not\rightsquigarrow(a_{i-1},a_{i+1})$, and $(g_{i-1},h_i)\not\rightsquigarrow(a_{i+1},a_i)$. Since $(p_i,q_{i+1})\rightsquigarrow(u,v)$, we have $(u,v)\not\rightsquigarrow(a_{i-1},a_{i+1}),(u,v)\not\rightsquigarrow(a_i,a_{i-1})$. Therefore, for $p=g_{i-1}$, $q=h_i$, $r=u$, and $s=v$ and $a=a_{i-1}$, $b=a_i$, $c=a_i$, $d=a_{i+1}$, and walks A'_{i-1},B'_i,C,D (all constricted and have same net length) Lemma 6 is applied to conclude that A'_{i-1},B'_i avoid each other and C,D

avoid each other. Moreover, A'_{i-1}, C have embedded pre-images avoiding each other and B'_i, D have embedded pre-images avoiding each other.

Now we observe that $C^{-1} + C$ avoids $D^{-1} + D$. $C^{-1} + C$ is a walk from a_i to a_i, and in the condition of the Lemma 7, p_i is assumed to be a_i, and q_{i+1} is assumed to be a_{i+1}. $D^{-1} + D$ is a walk from a_{i+1} to a_{i+1}, and in the condition of the Lemma 7. $A_{i-1}, B_i, C^{-1}, D^{-1}$ has net $Li - 1$, and A'_{i-1}, B'_i, C, D have net length $-L_{i-1}$. Since the circuit is minimal, condition 2 of Lemma 7 is satisfied for $A_{i-1}, A'_{i-1}, B_i, B'_i, C^{-1}, C, D^{-1}, D$ and we conclude that A'_{i-1}, B_i^{-1} have congruent embedded pre-images A, B (respectively) that avoid each other. Moreover, B'_i, D have embedded pre-images B_1, D_1 that avoid each other and B_i, D^{-1} (shown by green in Figure 8) have embedded pre-images B_2, D_2 that avoid each other.

From the above conclusions, $A_{i-1} + A$ and $B_i + B$ are constricted from below path with net value zero that avoid each other, and hence, (p_{i-1}, q_i), (a_{i-1}, q_i), (a_{i-1}, a_i) are in the same strong component. Moreover, $B_2 + B_1$ and $D_1 + D_2$ avoid each other, and hence, (q_i, a_{i+1}), (a_i, a_{i+1}) are in the same strong components. These allow us to replace a_i by q_i in the circuit $(a_0, a_1), \ldots, (a_{n-1}, a_n), (a_n, a_0)$ and obtain $\mathfrak{C}_1 : (a_0, a_1), \ldots, (a_{i-1}, q_i), (q_i, a_{i+1}), \ldots, (a_n, a_0)$ in $\widehat{S}$.

Proof of 2. Analogous to proof of 1.

Proof of 3. Since $L_{i-1} \leq L_i$, by (1) we replace a_i by q_i and obtain a minimal circuit $\mathfrak{C}_1 : (a_0, a_1), \ldots, (a_{i-1}, q_i), (q_i, a_{i+1})$,
$\ldots, (a_n, a_0)$. Notice that by (1) (q_i, a_{i+1}) and (a_i, a_{i+1}) are reachable from each other via constricted from below paths of net value zero. According to (2) (argument on blue walks in Figure 8 for circuit $\mathfrak{C}_1$) we conclude that $\mathfrak{C}_2$: $(a_0, a_1), \ldots, (a_{i-1}, q_i), (q_i, p_{i+1}), (p_{i+1}, a_{i+2}) \ldots, (a_n, a_0)$ is a minimal circuit in $\widehat{S}$.

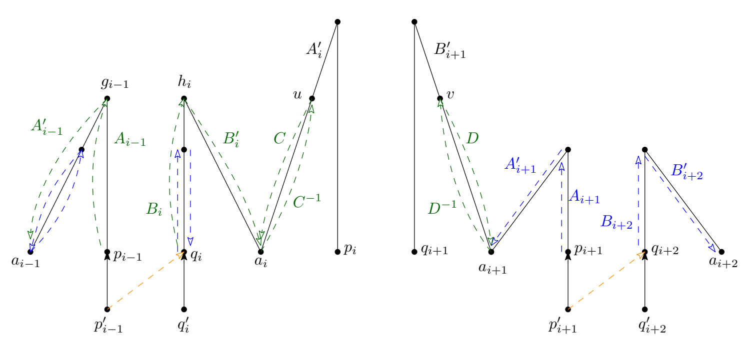

Fig. 8. Illustration of the proof of Lemma 8. The orange dashed arcs are the missing arcs. The green, blue dashed walks show the direction of the walks.

Moreover, $(q_i, a_{i+1}), (q_i, p_{i+1})$ are reachable from each other via constricted from below paths of net value zero. Therefore, $(a_i, a_{i+1}), (q_i, p_{i+1})$ are reachable from each other via constricted from below paths of net value zero. It also follows that $(p_{i-1}, p_{i+1}) \rightsquigarrow (a_{i-1}, a_{i+1})$ and $(q_i, q_{i+2}) \rightsquigarrow (q_i, a_{i+2})$.

Now $p'_{i-1}p_{i+1} \not\in A(H)$ (see Figure 8), as otherwise, $(p'_{i-1}, q'_i)(p_{i+1}, q_i) \in A(H^+)$, and hence $(p_{i+1}, q_i) \in \widehat{S}$; contradicting the minimality of the circuit $\mathfrak{C}_2$. Since $p'_{i-1}p_{i+1} \not\in A(H)$, we have $q'_ip_{i+1} \not\in A(H)$, otherwise, $(p'_{i-1}, q'_i)(p_{i-1}, p_{i+1}) \in A(H^+)$, and since $(p_{i-1}, p_{i+1}), (a_{i-1}, a_{i+1})$ are in the same strong component, we have $(p'_{i-1}, q'_i) \rightsquigarrow (a_{i-1}, a_{i+1})$, contradicting the minimality of the original circuit. We also note that $p'_{i+1}q_i \not\in A(H)$, as otherwise, $(p'_{i+1}, q'_{i+1})(q_i, q_{i+2}) \in A(H^+)$; implying that $(q_i, q_{i+2}) \in \widehat{S}$, and since by (1) $(q_i, q_{i+2}), (q_i, a_{i+2})$ are in the same strong component, we have $(p'_{i+1}, q'_{i+2}) \rightsquigarrow (q_i, a_{i+2})$, contradicting the minimality of the circuit $\mathfrak{C}_2$. Therefore, $(q'_i, p'_{i+1}) \rightsquigarrow (a_i, a_{i+1})$ via a symmetric path constricted from below and with net value one.

Proof of 4. We assume $(a_i, a_{i+1}), (a_{i-1}, a_i)$ are UL-pair and the case where $(a_i, a_{i+1}), (a_{i+1}, a_{i+2}))$ are UL-pair is similar. By (1) one can replace a_i, a_{i+1} by q_i, q_{i+1} (respectively) and obtain minimal circuit $\mathfrak{C}_2 : (a_0, a_1), \dots, (a_{i-1}, q_i), (q_i, q_{i+1}), (q_{i+1}, a_{i+2}), \dots (a_n, a_0)$ in $\widehat{S}$. It also follows that $(q_i, q_{i+1}), (a_i, a_{i+1})$ are in the same strong component (using the red walks in Figure 9). Now $p'_{i-1}q_{i+1} \not\in A(H)$, otherwise, $(p'_{i-1}, q'_i)(q_{i+1}, q_i) \in A(H^+)$, and hence, $(q_{i+1}, q_i) \in \widehat{S}$ while $(q_i, q_{i+1}) \in \widehat{S}$; contradicting the minimality of circuit $\mathfrak{C}_2$. Now, the absence of the arc $p'_{i-1}q_{i+1}$ implies that q'_iq_{i+1} is not an arc of H, as otherwise, $(p'_{i-1}, q'_i)(p_{i-1}, q_{i+1})$ is an arc of H^+. Since, $(p_{i-1}, q_{i+1}) \rightsquigarrow (a_{i-1}, q_{i+1})$, we have $(p'_{i-1}, q'_i) \rightsquigarrow (a_{i-1}, q_{i+1}) \in \widehat{S}$; contradiction to minimality of $\mathfrak{C}_2$.

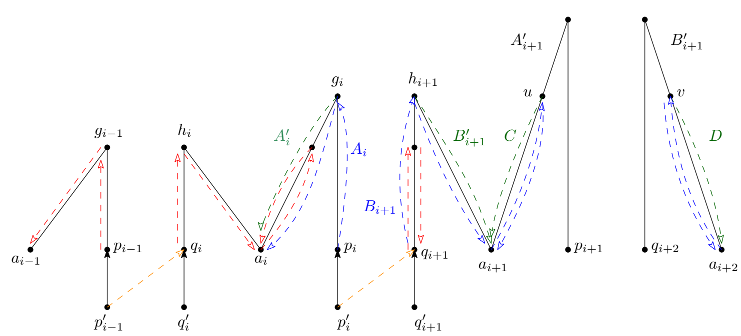

Fig. 9. Each straight segment represents a constricted walk. The dashed line shows the direction of the walks. Walks A, B avoid each other and walks C, D avoid each other. The orange dashed arrows are missing arc in H.

Proof of 5. Suppose $L_{i+1} \leq L_{i-1}$. The argument for the other case is similar. By (1) $A_i + A'_i, B_{i+1} + B'_{i+1}$ avoid each other (see Figure 10).

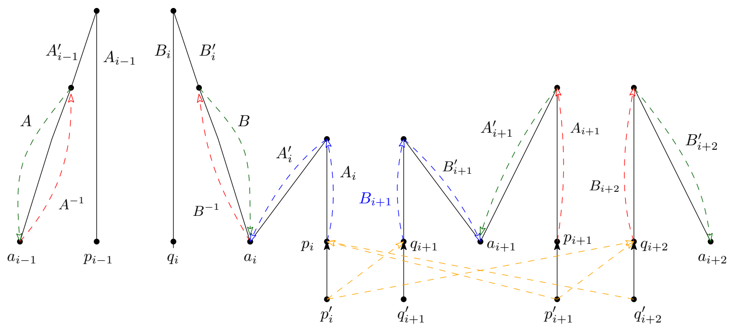

Fig. 10. Lemma 8 (5) where for contradiction we first suppose $L_{i-1} > L_i, L_{i+1}$ (the case $L_{i+1} > L_{i-1} > L_i$ is proved similarly). A'_j avoids B'_{j+1}, and A_j avoids B_{j+1}, $j = i-1, i, i+1$. The dashed arcs are missing arcs in H. In the proof we first we first show $p'_{i+1}p_i$ is a missing arc, then $q'_{i+1}p_i$ is a missing arc, and finally p'_iq_{i+2} is a missing arc.

We show that $A_{i+1} + A'_{i+1}$ and $B_{i+2} + B'_{i+2}$ avoid each other (see Figure 10). Since the circuit is minimal, the condition in Corollary 4 for A'_{i+1}, B'_{i+2}, A, B (where A, B) are constricted portion of A'_{i-1}, B'_i with net length $-L_{i+1}$ are satisfied, and hence, A, B, A'_{i+1}, B'_{i+2} pairwise avoid each other. Again by applying Corollary 4 on A_{i+1}, B_{i+2} and A^{-1}, B^{-1} we conclude that the embedded pre-images of A, B, B_{i+2}, A_{i+1} pairwise avoid each other.

We may assume $Z_i = (A_i + A'_i + (B')^{-1} + B', B_{i+1} + B'_{i+1} + A''_{i+1} + (A''_{i+1})^{-1})$ (here B', A''_{i+1} are the embedded pre-images of A, A'_{i+1}) ,and hence, $L_i = L_{i+1}$. By Lemma 7 we observe that $B_{i+1} + B'_{i+1} + (A'_{i+1})^{-1} + A'_{i+1}$ and $B_{i+2} + B'_{i+2}$ have embedded pre-images that avoid each other, and hence, $(q_{i+1}, q_{i+2}) \rightsquigarrow (a_{i+1}, a_{i+2})$, similarly $(p_i, q_{i+2}), (a_i, a_{i+2})$ are in the same strong component.

By 4, we have $p'_{i+1}p_i, q'_{i+2}p_i \notin A(H)$. We observe that $p'_iq_{i+2} \notin A(H)$, as otherwise, $(p'_i, q'_{i+1})(q_{i+2}, q_{i+1}) \in A(H^+)$, and hence, $(p'_i, q'_{i+1}) \rightsquigarrow (a_{i+2}, a_{i+1})$, contradicting the minimality of the circuit. Therefore, $(p'_i, q'_{i+2})(p_i, q_{i+2})$ is a symmetric arc, and hence, $(p'_i, q'_{i+2}) \rightsquigarrow (a_i, a_{i+2})$ via a symmetric path, constricted from below and with net value one. Now $q'_{i+2}q_{i+1} \notin A(H)$, as otherwise, $(p_i, q_{i+1})(p'_i, q'_{i+2}) \in A(H^+)$, and since $(p'_i, q'_{i+1})(p_i, q_{i+1}) \in A(H^+)$, $(p_i, q_{i+1}) \rightsquigarrow (p'_i, q'_{i+2}) \rightsquigarrow (a_i, a_{i+2})$ we get a contradiction to the minimality of the circuit. Moreover, $q'_{i+1}q_{i+2} \notin A(H)$, as otherwise, $(p'_i, q'_{i+1})(p_i, q_{i+2}) \in A(H^+)$ and since, $(p_i, q_{i+2}) \rightsquigarrow (a_i, a_{i+2})$, we have $(p'_i, q'_{i+1}) \rightsquigarrow (a_i, a_{i+2}) \in \widehat{S}$, contradicting the minimality of the circuit. Notice that $q'_{i+2}q_{i+1} \notin A(H)$, otherwise, $(p'_i, q'_{i+1}) \rightsquigarrow (p_i, q_{i+1}) \rightsquigarrow (p'_i, q'_{i+2}) \rightsquigarrow (a_i, a_{i+2})$, contradicting the minimality of the circuit. Now $(q'_{i+1}, q'_{i+2}), (q_{i+1}, q_{i+2}), (a_{i+1}, a_{i+2})$ are in the same strong component.

Finally, $q'_{i+1}p_i \notin A(H)$. Otherwise, $(q'_{i+1}, q'_{i+2})(p_i, q_{i+2}) \in A(H^+)$, and consequently, $(a_{i+1}, a_{i+2}) \rightsquigarrow (q'_{i+1}, q'_{i+2}) \rightsquigarrow (p_i, q_{i+2}) \rightsquigarrow (a_i, a_{i+2})$, a contradiction to minimality of the circuit.

Theorem 17. *Let S be a set of pairs in H^+ and let $\widehat{S}$ contain a minimal circuit $(a_0, a_1), (a_1, a_2), \dots, (a_{n-1}, a_n), (a_n, a_0)$, $n > 1$, such that each (a_i, a_{i+1}) is an UL-pair with respect to $\widehat{S}$. Then there exists another circuit $(a'_0, a'_1), (a'_1, a'_2), \dots, (a'_n, a'_0)$ in $\widehat{S}$, and walks $P_i, Q_i, i = 0, \dots, n$, in H, such that*

1. *P_i, Q_i are walks of net length one, and avoid each other,*
2. *P_i, Q_i are constricted from below, and P_i is from a'_i to a_i, Q_i is from a'_{i+1} to a_{i+1}.*

Proof. Since each (a_i, a_{i+1}) is an UL-pair with respect to $\widehat{S}$, there exists a path $U_i = U_{a_i, a_{i+1}}$ in H^+. We consider Notations 14, and let (p'_i, q'_{i+1}) be first vertex and (p_i, q_{i+1}) be the second vertex of U_i, and $Z_i = U_i[(p_i, q_{i+1}), (a_i, a_{i+1})]$ which is constricted from below and has net value zero.

We will find n vertices a'_i from amongst the $2n$ vertices p'_i, q'_i which satisfy the conclusion. As an intermediate step, we will find n vertices a^*_i of the $2n$ vertices p_i, q_i which also yield a circuit in $\widehat{S}$. For any i, if $L_{i-1} < L_i$ we let $a^*_i = q_i$ and $a'_i = q'_i$ and if $L_{i-1} \geq L_i$ we let $a^*_i = p_i$ and $a'_i = p'_i$. We first prove that each pair (a^*_i, a^*_{i+1}) in the circuit $(a^*_0, a^*_1), (a^*_1, a^*_2), \dots, (a^*_n, a^*_0)$ can be reached from the corresponding pair (a_i, a_{i+1}).

Case 1. $L_{i-1} \leq L_i$ and $L_{i+1} \leq L_i$. In this case we have $(a^*_i, a^*_{i+1}) = (q_i, p_{i+1})$ and $(a'_i, a'_{i+1}) = (q'_i, p'_{i+1})$. The proof follows from Lemma 8 (3). See Figure 8 for the walks: from q_i to a_i and p_{i+1} to a_{i+1} that avoid each other).

Case 2. $L_{i-1} < L_i < L_{i+1}$. In this case we have $(a^*_i, a^*_{i+1}) = (q_i, q_{i+1})$ and $(a'_i, a'_{i+1}) = (q'_i, q'_{i+1})$. We show that (a_i, a_{i+1}), (q_i, q_{i+1}) are reachable from each other via a constricted symmetric path of net value zero. The proof is symmetric for the case where $L_{i+1} < L_i < L_{i-1}$. In fact, we assume $L_{i-1} < L_i < L_{i+1} < \cdots < L_{i+j} \geq L_{i+j+1}$ for some $j \geq 1$. Note that this means that

$$a^*_i = q_i, a^*_{i+1} = q_{i+1}, \dots, a^*_{i+j-1} = q_{i+j-1}, a^*_{i+j} = q_{i+j}, a^*_{i+j+1} = p_{i+j+1}.$$

By Lemma 8 (1,2) we may replace $a_i, a_{i+1}, \dots, a_{i+j}$ by $q_i, q_{i+1}, \dots, q_{i+j-1}, p_{i+j}$(respectively) and obtain the following minimal circuit in $\widehat{S}$.

$$\mathfrak{C}_j : (a_0, a_1), \dots, (a_{i-1}, q_i), (q_i, q_{i+1}), \dots, (q_{i+j-2}, q_{i+j-1}), (q_{i+j-1}, p_{i+j}), (p_{i+j}, a_{i+j+1}), \dots, (a_n, a_0)$$

Observation 18 *Suppose $q'_{r+1}q_r \notin A(H)$ for some $r \leq i \leq i + j - 1$. Then $(q'_r, q'_{r+1})(q_r, q_{r+1})$ is a symmetric arc (because $q'_r q'_{r+1} \notin A(H)$ by Lemma 8 (4)), and hence, $(q'_r, q'_{r+1}) \in \widehat{S}$. As a consequence, $q'_r q_{r-1} \notin A(H)$, as otherwise, $(q'_r, q'_{r+1})(q_{r-1}, q_{r+1}) \in A(H^+)$ and consequently, $(q'_r, q'_{r+1})(q_{r-1}, q_{r+1})$, implying that $(q_{r-1}, q_{r+1}) \in \widehat{S}$, a contradiction to minimality of $\mathfrak{C}_j$.*

By Lemma 8 (3), $(q'_{i+j}, p'_{i+j+1})(q_{i+j}, p_{i+j+1})$ is a symmetric arc of H^+, and $(q_{i+j}, p_{i+j+1}), (a_{i+j}, a_{i+j+1})$ are in the same strong component. Now $q'_{i+j}q_{i+j-1} \notin A(H)$, as otherwise, $(q'_{i+j}, p'_{i+j+1})(q_{i+j-1}, p'_{i+j+1}) \in \widehat{S}$, a contradiction to minimality of $\mathfrak{C}_j$. Thus, by above Observation, we conclude that $q'_{i+1}q_i \notin A(H)$. Therefore, $(q'_i, q'_{i+1})(q_i, q_{i+1})$ is a symmetric arc of H^+. It remains to observe that by Lemma 8 (4) (q_i, q_{i+1}) reaches (a_i, a_{i+1}) by a symmetric path W of net value zero. Now $(P_i, Q_i) = (q'_i, q'_{i+1}) + W$.

Case 3. $L_i \leq L_{i-1}$ **and** $L_i \leq L_{i+1}$**.** We have $(a^*_i, a^*_{i+1}) = (p_i, q_{i+1})$ and $(a'_i, a'_{i+1}) = (p'_i, q'_{i+1})$.

It follows from Lemma 8 (1) that Z_i is symmetric and has net value zero. Moreover, by Lemma 8 (5), $(p'_i, q'_{i+1})(p_i, q_{i+1})$ is a symmetric arc in H^+. Therefore, $(P_i, Q_i) = (p'_i, q'_{i+1}) + Z_i$.

9 Tools for proving Theorem 8 (proof of Theorem 7)

In this section, we provide the tools needed to prove Theorem 8, which justified the correctness of Phase One. We use the structural properties of walks in H^+ (Section 7) and the structural properties of circuits in H^+ (Section 8). The proof relies on the following results, which we proceed to prove first.

Lemma 9. *Let T be a set of unbalanced strong components where $\widehat{T}$ is dual-free. Suppose $\widehat{T}$ contains a minimal circuit $C : (a_0, a_1), (a_1, a_2), \ldots, (a_n, a_0)$ with $n > 1$. Then each pair (a_i, a_{i+1}) is an UL-pair with respect to $\widehat{T}$.*

Proof. Recall that each unbalanced pair belongs to a strong component containing an unbalanced directed cycle. Plus, each unbalanced directed cycle contains an extremal pair. Now, for each i, let C_i be the strong component of H^+ containing an extremal pair in $\widehat{T}$ where (a_i, a_{i+1}) is reachable from C_i, and let D_i be the directed cycle in C_i containing that extremal pair.

When some D_i has positive net value. Then there is an infinite path W (starting at some extremal vertex in component C_i) continuously going around D_i in the positive direction, which is constricted from below with unbounded net value. By following W as far as necessary and then following a path that reaches (a_i, a_{i+1}) from C_i (such a path exists both when (a_i, a_{i+1}) is in C_i or reachable from C_i), we obtain a path W_i in H^+ that is constricted from below[5]. According to Notation 14, we let (p'_i, q'_{i+1}) be the last vertex on W_i such that the net value of $W_i[(p'_i, q'_{i+1}), (a_i, a_{i+1})]$ is one, and we set $Z_i = W_i[(p_i, q_{i+1}), (a_i, a_{i+1})]$, where $(p'_i, q'_{i+1})(p_i, q_{i+1})$ is a forward arc of W_i. Note that L_i (the height of Z_i) could be zero, in case we have $(p_i, q_{i+1}) = (a_i, a_{i+1})$. Thus, (a_i, a_{i+1}) is an UL-pair.

[5] We say a path X in H^+ is constricted from below when the walks A_1, A_2 in H corresponding to X are constricted from below

When some D_i has negative net value. Now let W be a walk in H^+ that starts from an extremal vertex on D_i and then following around D_i in negative direction (sufficiently many times) and then to (a_i, a_{i+1}). This way we can assume that W is constricted from above. If W is not constricted from below, then, we obtain a suffix of W_i of W which is constricted from below and has net value one, and hence, (a_i, a_{i+1}) is an UL-pair and we can use Notation 14 for the pair (a_i, a_{i+1}) and define $(p'_i, q'_{i+1}), (p_i, q_{i+1})$, Z_i. So we may assume that W is constricted, and hence, in this case, $W_i = W$. In what follows, we show that each D_i has positive net values, and hence, each (a_i, a_{i+1}) is an UL-pair.

Suppose next that there are two subscripts $i, i+1$ (addition modulo n) such that both D_i and D_{i+1} have negative net values, and both W_i and W_{i+1} are constricted (as we discussed above), from some pairs $(u, v), (w, x) \in \widehat{T}$ to $(a_i, a_{i+1}), (a_{i+1}, a_{i+2})$ respectively . We may assume that the pairs $(u, v), (w, x)$ are on the cycles D_i, D_{i+1}, and that the net values of W_i, W_{i+1} are the same and arbitrary (by choosing for their starting vertices a suitable extremal vertex on D_i, D_{i+1}). Now according to Lemma 8 (1) W_i, W_{i+1} are both symmetric. This implies that the reverse traversal of the cycles D_i, D_{i+1} is also a cycle in H^+, of positive net value, and we can proceed as in the case when D_i, D_{i+1} had positive net value.

Thus, it remains to consider the case when D_i has negative net value, W_i is constricted (both from below and from above), and D_{i-1}, D_{i+1} have positive net values. There exists, $(g, h) \in W_i$ so that C, D are constricted and have net length $-L_{i-1}$ (the same net length as B'_i) where $(C, D) = W_i[(g, h), (a_i, a_{i+1})]$. Notice that C avoids D. Now according to Lemma 8 (3) we conclude that (a_i, a_{i+1}) is an UL-pair.

Statement of Theorem 7 Let T be the vertices of a set of unbalanced strong components (in H^+) where $\widehat{T}$ is dual-free, and assume that $\widehat{T}$ contains a minimal circuit $C : (a_0, a_1), (a_1, a_2), \ldots, (a_n, a_0)$. Then $n > 1$ and the following statements hold.

1. There exists some minimal circuit (see Definition 6) with extremal pairs $(b_0, b_1), (b_1, b_2), \ldots, (b_{n-1}, b_n), (b_n, b_0)$ in T such that $(a_i, a_{i+1}), (b_i, b_{i+1}), 0 \leq i \leq n$, are in the same strong component, and (a_i, a_{i+1}) is reachable from (b_i, b_{i+1}) by a symmetric walk of non-negative net value, and constricted from below.
2. For each i, $0 \leq i \leq n$, there exists an infinite walk P_i that starts from b_i and has unbounded positive net length. P_i is obtained by going (in the clockwise direction) around a cycle in H containing b_i. Furthermore, for every i, j, $0 \leq i < j \leq n$, P_i and P_j avoid each other [6].
3. In statement 1, for a given $0 \leq i \leq n$, we can choose (b_i, b_{i+1}) to be any given extremal pair from its corresponding strong component.
4. There is no path in H^+ from (b_i, b_{i+1}) to any of (b_j, b_{j+1}) $i \neq j$, and to any of (b_{j+1}, b_j).

[6] When we say two infinite walks P , Q avoid each other it means for every prefix of P there exists a prefix of Q that avoid each other.

5. There is no path in H^+ from any of (b_{i+1}, b_i), $0 \leq i \leq n$ to (b_i, b_{i+1}).

Proof. We first show that $n > 1$. Otherwise, by definition $(x, y) \rightsquigarrow (a_0, a_1)$, and $(x', y') \rightsquigarrow (a_1, a_0)$ where $(x, y), (x', y') \in T$. Now by skew property we have $(a_0, a_1) \rightsquigarrow (y', x')$, and hence, $(x, y) \rightsquigarrow (y', x')$, a contradiction that $\widehat{T}$ is dual-free. Recall that each unbalanced pair belongs to a strong component containing an unbalanced directed cycle. Plus, each unbalanced directed cycle contains an extremal pair. Now, for each i, let C_i be the strong component of H^+ containing an extremal pair in $\widehat{T}$ where (a_i, a_{i+1}) is reachable from C_i, and let D_i be the directed cycle in C_i containing that extremal pair.

Proof of 1. According to Lemma 9, each (a_i, a_{i+1}) is an UL-pair. Therefore, it is possible to apply Theorem 17, and conclude that there exists another minimal circuit $(a'_0, a'_1), (a'_1, a'_2), \ldots, (a'_n, a'_0)$ in $\widehat{T}$ with the corresponding walks A_i, B_i where each A_i and B_i is constricted from below with net length one. Furthermore, each A_i is from a'_i to a_i, and each B_i is from a'_{i+1} to a_{i+1}. We can repeat the argument obtaining, at the k-th step, an ordered sequence of vertices $a^k_0, a^k_1, \ldots, a^k_n$ such that (a^k_0, a^k_1), $(a^k_1, a^k_2), \ldots, (a^k_n, a^k_0)$ is a circuit in $\widehat{T}$ with each (a^k_i, a^k_{i+1}) is reachable from $(a^{k+1}_i, a^{k+1}_{i+1})$ by a symmetric walk of net value 1 (symmetric follows from Theorem 17 since P_i and Q_i avoid each other). Thus, there exist $r \neq s$ such that $a^r_0 = a^s_0, a^r_1 = a^s_1, \ldots, a^r_n = a^s_n$. Thus, we obtain the circuit $(b_0, b_1), (b_1, b_2), \ldots, (b_n, b_0)$ where $(b_i, b_{i+1}) = (a^r_i, a^r_{i+1})$, $0 \leq i \leq n$, and note that $(b_i, b_{i+1}) = (a^r_i, a^r_{i+1})$ is extremal because for each k, $(a^{k+1}_i, a^{k+1}_{i+1}) \rightsquigarrow (a^k_i, a^k_{i+1})$ via a path constricted from below and net value 1.

Proof of 2. Continuing the last sentence from the proof of (1), let (W_i, W'_i) : $(a^r_i, a^r_{i+1}) = (b_i, b_{i+1}) \rightsquigarrow (a^s_i, a^s_{i+1}) = (b_i, b_{i+1})$, $0 \leq i \leq n$. Here, W_i is the closed walk from b_i to b_i of net length $|s - r| > 0$, and constricted from below, and W'_i is the closed walk from b_{i+1} to b_{i+1} such that W_i, W'_i avoid each other. For $0 \leq i \leq n$, let Q_i, be the walk going around W_i (repeating W_i), α times for arbitrary large positive integer α, and let Q'_i be the walk obtained by going around the closed walk W'_i, α times. Let $h_i \in Q_i$ such that $R_i = Q_i[b_i, h_i]$ is constricted and have net length $\alpha|r - s|$ (notice that h_i could be b_i). Let $h'_i \in Q'_i$ be the corresponding vertex to h_i and let $R'_i = Q'_i[b_{i+1}, h'_i]$. Notice that since R_i, R'_i avoid each other and are constricted, $(R_i)^{-1}, (R'_i)^{-1}$ also avoid each other.

For $0 \leq i \leq n$, let $A_i = R_i$, $A'_i = (R'_i)^{-1}$, $B_{i+1} = R_{i+1}$, $B'_{i+1} = (R'_{i+1})^{-1}$, $(p_i, q_{i+1}) = (b_i, b_{i+1})$, and $(a_i, a_{i+1}) = (b_i, b_{i+1})$ according to Notations 14. Now one can apply the Lemma 7 on $R_0, (R'_0)^{-1}, R_1, (R'_1)^{-1}, \ldots, R_n, (R'_n)^{-1}$ and conclude that $R_1, R_2, \ldots, R_n$ have congruent embedded pre-images $P_1, P_2, \ldots, P_n$ that all avoid each other. Notice that P_i starts at b_i, $0 \leq i \leq n$. This proves (2).

Proof of 3. By statement (2) of Theorem 7, there exist infinite walks P_i, $0 \leq i \leq n$, starting at b_i with unbounded positive net length. Moreover, all pairs P_i, P_j, $0 \leq i < j \leq n$, avoid each other. We prove the statment for $i = 0$ (the other cases are similar).

Let (x, y) be an arbitrary extremal pair in the strong component C_0 containing (b_0, b_1). We may assume there exists a path W (in H^+) from (x, y) to (b_0, b_1) which is constricted and has positive net value. This can be done by going around a directed cycle, in C_0 containing (x, y), in the positive direction as many times as needed. Note that by (2), the cycle has positive net value because of the walks P_0, P_1. This means that $W = (X, Y)$ where X is constricted and has positive net length, and avoids Y. Note that the net length of X could be arbitrarily large.

Let b'_i, $0 \leq i \leq n$, be a vertex on P_i such $P_i[b'_i, b_i]$ is constricted and has the same net length as X. Now, by applying Lemma 7 on $X, Y, P_1[b'_1, b_1], P_2[b'_2, b_2]$, and on $P_n[b'_n, b_n], P_0[b'_0, b_0], X, Y$ we conclude that X, Y have congruent embedded pre-images X', Y' that avoid each other. Moreover, we can choose X_0 from x to b_0, X_1 from y to b_1, X_2 from b'_2 to b_2, and X_n from b'_n to b_n in such a way that X_0, X_1, X_2, X_n are all congruent and all avoid each other. Note that X_0, X_1, X_2, X_n are congruent embedded pre-images of $X', Y', P_2[b'_2, b_2], P_n[b'_n, b_n]$, respectively. This means $(x, y), (y, b'_2), (b'_2, b'_3), ..., (b'_n, x)$ is also a circuit, and each (b'_i, b'_{i+1}) is an extremal pair. Now it is easy to see that there exist $P'_0, P'_1, \ldots, P'_n$ starting at $x, y, b'_2, b'_3, \ldots, b'_n$ (respectively) so that P'_i, P'_j avoid each other.

Proof of 4. Suppose there exists a path W in H^+ from (b_i, b_{i+1}) to (b_j, b_{j+1}). We may assume W has non-positive net value (the argument for the other case is similar). Now, define W' to be a walk in H^+ starting at (b_i, b_{i+1}) and then following the cycle D_i (containing (b_i, b_{i+1})) in negative direction sufficiently many times and then following W to (b_j, b_{j+1}) and then following cycle D_j (containing (b_j, b_{j+1}) in negative direction sufficiently many times such that W' is constricted. Notice that W' has negative net value. Let $W' = (X_1, X_2)$ and observe that X_1 is a walk in H from b_i to b_j and X_2 is a walk from b_{i+1} to b_{j+1} and X_1 avoids X_2. Now we can take vertices b'_{i+1}, b'_{i+2} on P_{i+1}, P_{i+2} such that $P_{i+1}[b'_{i+1}, b_{i+1}], P_{i+2}[b'_{i+2}, b_{i+2}]$ are constricted and have the same net length as X_1, X_2. Since the circuit is minimal, there is no path from $(b_i, i+1)$ to any of $(b_i, b_{i+2}), (b_{i+2}, b_{i+1})$ and there is no path from (b'_{i+1}, b'_{i+2}) (in the same component as (b_{i+1}, b_{i+2})) to any of $(b_i, b_{i+2}), (b_{i+1}, b_i)$. Therefore, by Lemma 6 we conclude that X_1, X_2 avoid each other. This allows us to consider a directed cycle (C_1, C_2) in H^+ going through both $(b_i, b_{i+1}), (b_j, b_{j+1})$ with net value positive and assume that X_1 is part of the closed walk C_1 in H containing both b_i, b_j.

Let $b'_0, b'_1, \ldots, b'_n$ be the vertices on $P_0, P_1, \ldots, P_n$ such that for every $0 \leq r \leq n$, $P'_r = P_r[b'_r, b_r]$ has the same net length as $(C_1)^a$ (here a is an arbitrary large positive integer). Let C_{b_i} be the walk going around C_1 (a time) starting at b_i, $C_{b_{i+1}}$ be the walk going around C_2 (a time) starting at b_{i+1}, C_{b_j} be the walk going around C_1 (a time) starting at b_j, $C_{b_{j+1}}$ be the walk going around C_2 (a time) starting at b_{j+1}. Now by applying Lemma 7 on walks $P'_0, P'_1, \ldots, P'_{i-1}, C_{b_i}, C_{b_{i+1}}, P'_{i+2}, \ldots, P'_{j-1}, C_{b_j}, C_{b_{j+1}}, P'_{j+1}, \ldots, P'_n$ we conclude that C_{b_i} and C_{b_j} avoid each other. Now it is easy to see that C_{b_i}, C_{b_j} satisfy the condition of Theorem 13, and hence, we obtain a strong circuit in H^+; contradiction to our assumption that H^+ does not have a strong circuit.

Proof of 5. Assume, without loss of generality, that $i = n$, the other cases are symmetric. By (2), we conclude that (b_0, b_n) is also an extremal pair.

Using (3), there is a circuit $(d_0, d_1), \ldots, (d_{n-1}, d_n), (d_n, d_0)$ with $(d_n, d_0) = (b_0, b_n)$ where $(d_i, d_{i+1}) \rightsquigarrow (b_i, b_{i+1})$ (via a symmetric path according to (1)) and (d_n, d_0) is in the same strong component as (b_n, b_0). Therefore, (b_n, b_0) and (b_0, b_n) are in the same strong component, a contradiction.

9.1 Auxiliary Results Obtained from Theorem 7

The results of this subsection follow from Theorem 7; some are used in Subsection 10 and some in Section 11.

Proposition 1. *Let T be a set of unbalanced pairs where $\widehat{T}$ is dual-free. Let $(b_0, b_1), (b_1, b_2), \ldots, (b_n, b_0)$ be any minimum circuit in $\widehat{T}$. By Theorem 7, let P_i be an infinite walk starting at b_i, $0 \leq i \leq n$. Suppose for some i, (b_i, b_{i+1}) lies on arbitrary cycle $D_i \in H^+$ with positive net value. Let $D_i = (C_1, C_2)$, (i.e. C_1, C_2 are closed walks in H and C_1 avoids C_2). Then P_i is a walk obtained by going around cycle C_1 as many times as necessary.*

Proof. By Theorem 7(2) walk P_i, $0 \leq i \leq n$, is constricted and has unbounded positive net length. Moreover, P_i, P_j, $0 \leq i < j \leq n$ avoid each other. Since b_i, b_{i+1} are extremal, there exists a vertex (b, c) on D_i, such that $C_1[b, b_i], C_2[c, b_{i+1}]$ are constricted. Now we can take vertices b'_{i+1}, b'_{i+2} on P_{i+1}, P_{i+2} such that $P_{i+1}[b'_{i+1}, b_{i+1}], P_{i+2}[b'_{i+2}, b_{i+2}]$ are constricted and have the same net length as $C_1[b, b_i]$. Since the circuit is minimal, there is no path from (b, c) (in the same component of (b_i, b_{i+1})) to any of $(b_i, b_{i+2}), (b_{i+2}, b_{i+1})$ and there is no path from (b'_{i+1}, b'_{i+2}) (in the same component as (b_{i+1}, b_{i+2})) to any of $(b_i, b_{i+2}), (b_{i+1}, b_i)$. Therefore, by Lemma 6 we conclude that C_1, C_2 avoid each other. Again one can apply the Lemma 7 on any prefix of walks $P_0, P_1, \ldots, P_{i-1}, C_1, C_2, P_{i+2}, \ldots, P_n$, where all have the same net length, and conclude that there exists, $P'_0, P'_1, \ldots, P'_n$ where P'_i, P'_j, $i \neq j$, avoid each other. Observe that P'_i goes around C_1.

The purpose of the following is to consider a circuit in H^+ and consider the case when it is not minimal.

Corollary 5. *Let A, B, C, D be congruent walks in H from p, q, r, s to a, b, b, d respectively where A avoids B and C avoids D. If A, B or A, C or A, D or B, D or C, D do not avoid each other then either $X_1 : (p, q) \rightsquigarrow (a, d)$ or $X_2 : (p, q) \rightsquigarrow (d, b)$ or $X_3 : (r, s) \rightsquigarrow (a, d)$ or $X_4 : (r, s) \rightsquigarrow (b, a)$ exists. If X_i $1 \leq i \leq 4$, exists then there are walks P_1, P_2 such that $X_i = (P_1, P_2)$, and P_1, P_2, A, B, C, D have congruent embedded pre-images, and P_1, A have the same net length.*

Proof. Let $A : p = a_1, a_2, \ldots, a_n = a$, $B : q = b_1, b_2, \ldots, b_n = b$, $C : r = c_1, c_2, \ldots, c_n = b$, and $D : s = d_1, d_2, \ldots, d_n = d$. Let S_i denote the statement that : all pairs from $A[a_{i+1}, a], B[b_{i+1}, b], C[c_{i+1}, b], D[d_{i+1}, d]$ avoid each other, except possibly $B[b_{i+1},b], C[c_{i+1},b]$. If there is a faithful arc a_id_{i+1} then $(P_1,P_2) : (p,q) \rightsquigarrow (a_i,b_i) \rightsquigarrow (d_{i+1},b_{i+1}) \rightsquigarrow (d,b)$. More precisely, $P_1 = A[a_1,a_i]+a_id_{i+1}+D[d_{i+1},d]$ and $P_2 = B[b_1,b_i] + b_ib_{i+1} + B[b_{i+1},b]$ where P_1 avoids P_2. Notice that P_1 is congruent to A,B,C,D, and hence, they have the same net length. Following the argument in the proof of Lemma 4, if d_ib_{i+1} is a faithful arc then $(p,q) \rightsquigarrow (a,d)$ which is composed of two walks $P_1 = A$, and $P_2 = B[q,b_i] + b_id_{i+1} + D[d_{i+1},d]$ that are congruent and congruent with A,B,C,D and we are done. By a similar line of reasoning, if any of c_ia_{i+1}, d_ia_{i+1} is a faithful arc, we reach the same conclusion. So we may assume c_ia_{i+1}, d_ia_{i+1} are not a faithful arcs. Now suppose d_ib_{i+1} is a faithful arc. Then $P_1 = A[p,a_{i+1}] + (a_{i+1}a_i)^{-1} + A[a_i,a]$ and $P_2 = B[q,b_{i+1}] + (d_ib_{i+1})^{-1} + D[d_i,d]$ where P_1 avoids P_2 yield a walk from (p,q) to (a,d). Notice that P_1 and P_2 have the same net length as A,B,C,D, and it is easy to see that by adding $(a_ia_{i+1})^{-1} + a_ia_{i+1}$ right after a_{i+1} we obtain an embedded pre-image of A that is congruent with P_1,P_2. By similar argument we conclude that if b_ia_{i+1} is a faithful arc then $P_1 = C + B^{-1}[b,b_i] + b_ia_{i+1} + A[a_{i+1},a]$ and $P_2 = D + D^{-1}[d,d_i] + D[d_{i+1},d]$ where P_1 avoids P_2 yields a walk from (r,s) to (a,d) in H^+. Notice that we can simply find an embedded pre-image of A that is congruent with P_1. A similar argument would imply when d_ic_{i+1} is a faithful arc.

From Corollary 5, and the proofs of Lemma 4 and Lemma 6 we have the following.

Corollary 6. *Suppose A,B,C,D are four constricted walks in H of the same net length from p,q,r,s to a,b,b,d (respectively) where A avoids B and C avoids D. Then one of the following occurs:*

- *There are embedded pre-images A',B',C',D' of A,B,C,D respectively such that A',B', and A',C', and A',D', and B',D', and C',D' avoid each other.*
- *$X_1 : (p,q) \rightsquigarrow (a,d)$ or $X_2 : (p,q) \rightsquigarrow (d,b)$ or $X_3 : (r,s) \rightsquigarrow (a,d)$ or $X_4 : (r,s) \rightsquigarrow (b,a)$ exists. If X_i exists, $1 \le i \le 4$, then X_i consists of two walks P_1,P_2 (P_1 avoids P_2) in H such that P_1,P_2,A,B,C,D have embedded pre-images that are congruent, and P_1,A has the same net length.*

When the four walks A,B,C,D all have distinct end points, a,b,c,d in Corollary 6, and following Corollary 4 we obtain the following corollary.

Corollary 7. *Suppose A,B,C,D are four constricted walks of the same net length from p,q,r,s to a,b,c,d (respectively) where A avoids B and C avoids D. Then one of the following occurs:*

- *There are embedded pre-images A',B',C',D' of A,B,C,D respectively such that A',C', and A',D', and B',D', and avoid each other.*
- *$X_1 : (p,q) \rightsquigarrow (a',b')$ for some $(a',b') \in \{(a,c),(a,d),(c,b),(d,b),(d,c)\}$ or $(r,s) \rightsquigarrow (c',d')$ for some $(c',d') \in \{(a,d),(a,c),(b,d),(c,a),(c,b)\}$. If X_i exists, $1 \le i \le 2$, then X_i consists of two walks P_1,P_2 (P_1 avoids P_2) in H such that P_1,P_2,A,B,C,D have embedded pre-images that are congruent, and P_1,A has the same net length.*

Proposition 2. *Let S be a set of pairs in H^+ such that $\widehat{S}$ is dual-free. Suppose $\widehat{S}$ contain a circuit $(a_0, a_1), (a_1, a_2), \ldots, (a_{n-1}, a_n), (a_n, a_0)$, $n > 1$. For every, $i = 0, 1, \ldots, n$, consider the path Z_i according to Notation 14 which is the suffix of a walk starting from a pair in S and ending at (a_i, a_{i+1}). Then one of the following occurs:*

1. *either there is no path from a vertex on Z_i to some (a_r, a_s), $r \neq s-1, s$,*
2. *or there is a path Y_i from beginning of Z_i to some (a_r, a_s), $r \neq s-1, s, s+1$ ($s+1$ is not possible as otherwise it would mean $(x, y) \rightsquigarrow (y, x)$) where Y_i is constricted from below and has the same net value as Z_i.*

Proof. Referring to Notation 14 for the circuit, consider the case when $L_i < L_j$ ($i \leq j$ then case where $j < i$ is similar). Now let (u, v) be a vertex on Z_j such that $Z_j[(u, v), (a_j, a_{j+1})]$ is constricted and has net value $-L_i$. Let $(A, B) = Z_j[(u, v), (a_j, a_{j+1})]$. When $j = i+1$, we apply Corollary 6 to A, B, A'_i, B'_{i+1}. If the first item of Corollary 6 does not occur, then either $(g_i, h_{i+1}) \rightsquigarrow (a_i, a_{i+2})$ via a path which is constricted and has net value which is the same as length A'_i or $(g_i, h_{i+1}) \rightsquigarrow (a_{i+2}, a_{i+1})$ via a constricted path (in H^+) with net value the same as net length A'_i or $(u, v) \rightsquigarrow (a_i, a_{i+2})$ via a constricted path with net value the same as net length A'_i or $(u, v) \rightsquigarrow (a_{i+1}, a_i)$ via a constricted path with net value the same as net length of A'_i. Thus, one of the following occurs.

- $(p_i, q_{i+1}) \rightsquigarrow (a_i, a_{i+2})$ (in H^+) is constricted from below and has net value zero.
- $(p_i, q_{i+1}) \rightsquigarrow (a_{i+2}, a_{i+1})$ is constricted from below and has net value zero.
- $(p_{i+1}, q_{i+2}) \rightsquigarrow (a_i, a_{i+2})$ is constricted from below and has net value zero.
- $(p_{i+1}, q_{i+2}) \rightsquigarrow (a_{i+1}, a_i)$ is constricted from below and has net value zero.

When $j \neq i+1$, we apply Corollary 7 to A, B, A'_i, B'_{i+1}. If the first item of Corollary 7 does not occur, then either $(g_i, h_{i+1}) \rightsquigarrow (a', b')$ for some $(a', b') \in \{(a_i, a_j), (a_i, a_{j+1}), (a_j, a_{i+1}), (a_{j+1}, a_{i+1}), (a_{j+1}, a_j)\}$ via a path which is constricted and has net value which is the same as length A'_i or $(u, v) \rightsquigarrow (c', d')$ for some $(c', d') \in \{(a_i, a_{j+1}), (a_i, a_j), (a_{i+1}, a_{j+1}), (a_j, a_i), (a_j, a_{i+1})$ via a constricted path with net value the same as net length A'_i.

From the above discussion, it is easy to conclude that if the circuit is not minimal, then there is a constricted from below path with net value zero, from (p_i, q_{i+1}) to some $(a_r, a_s) \neq (a_j, a_{j+1})$.

9.2 Strong Circuits and Closed Walks in H.

The following Lemma and Theorem consider the case when a strong circuit exists in H^+ and are used in Section 11.

Lemma 10. *Let S be a strong component in H^+ such that $\widehat{S}$ is dual-free. If S contains a circuit then S contains a minimal circuit with at least three pairs. Moreover, S is an unbalanced component.*

Proof. Suppose $C : (a_0, a_1), (a_1, a_2), \dots, (a_n, a_0)$ is a circuit in S. By the argument at the beginning of proof of Theorem 7 we assume $n > 1$. We first prove S cannot be balanced. For contradiction, suppose S is a balanced component. In this case, we prove S contains an invertible pair which contradicts the assumption that $\widehat{S}$ is dual-free. Let (x, y) be a pair on the lowest layer of S. Now, consider a path $X_i = (A_i, B_i) \in H^+$ from (x, y) to (a_i, a_{i+1}), $0 \le i \le n$; here A_i is from x to a_i, and B_i is from y to a_{i+1} where A_i avoids B_i. Let Z_i be the suffix of X_i which is constricted from below and has net value zero.

There is no path from a vertex on a path X_i to (a_{i+1}, a_i), as otherwise, this would imply that $(x, y) \rightsquigarrow (a_{i+1}, a_i)$ as well as $(x, y) \rightsquigarrow (a_i, a_{i+1})$, and hence, $(x, y) \rightsquigarrow (y, x)$, a contradiction to $\widehat{S}$ being dual-free. Considering the paths $X_0, X_1, \dots, X_n$, by Proposition 2, one of the following occurs for Z_i's.

1. either there is no path from a vertex on Z_i to some (a_r, a_s), $r \ne s-1, s$.
2. there is a path from a vertex on Z_i to some (a_r, a_s), $r \ne s-1, s, s+1$, which is constricted from below and has the same net value as Z_i.

Note that if item 1 occurs then the circuit C is minimal. If item 2 occurs, we get a shorter circuit of length greater than 1 in $\widehat{S}$. Therefore, we continue by assuming there exists a minimal circuit C' that has length $m > 1$, and each pair (a'_i, a'_{i+1}), $0 \le i \le m$ of C' is reachable from (x, y) via a path X'_i that is constricted from below and has non-negative net value. So, without loss of generality, we continue with minimal circuit C'. We also assume at least one X'_i has net value zero. Otherwise, by repeatedly applying Theorem 17, t times we obtain another minimal circuit $(a^t_0, a^t_1), (a^t_1, a^t_2), \dots, (a^t_n, a^t_0)$ where the net value of a path from (x, y) to (a^t_i, a^t_{i+1}) is t times less than the net value of the path from (x, y) to (a_i, a_{i+1}). Thus, we may assume at least one of the X'_i has net value zero.

Suppose X'_i has net value zero. We may assume that X'_i has a maximum height in component S which can be done by adding a prefix to X'_i, going from (x, y) to a vertex with max height in S and then back to (x, y).

We show that none of X'_{i-1} and X'_{i+1} has net value zero. For contradiction, suppose, X'_{i+1} has net value zero. Again we may assume that X'_{i+1} has maximum height and is the same as X'_i. Thus, $X'_i = (A_i + A'_i, B_{i+1} + B'_{i+1})$ and $X'_{i+1} = (A_{i+1} + A'_{i+1}, B_{i+2} + B'_{i+2})$ according to Notations 14. Moreover, $(p_i, q_{i+1}) = (x, y)$ and $(p_{i+1}, q_{i+2}) = (x, y)$. Note that $A_i, B_{i+1}, A_{i+1}, B_{i+2}$ all have the same net length. By Lemma 7 A_i and A_{i+1} avoid each other, which is a contradiction since they have the same start vertex x. Similarly, X'_{i-1} does not have net value zero. Therefore, we assume both (a_{i-1}, a_i) and (a_{i+1}, a_{i+2}) are UL-pairs.

Now, by Lemma 8 (3) (Figure 8) we conclude that (a_i, a_{i+1}) is an UL-pair; a contradiction to X'_i have net value zero.

We proceed by assuming S is an unbalanced component and prove it contains a minimal circuit with at least three pairs. Let $C' : (a'_0, a'_1), (a'_1, a'_2), \dots, (a'_m, a'_0)$ be a minimal circuit in $\widehat{S}$ with $m > 1$. Notice that $m > 1$ because $\widehat{S}$ is dual-free. Now, according to Theorem 7 (1) and (3), there exists another minimal circuit $(b'_0, b'_1), \ldots, (b'_m, b'_0)$ where all (b'_i, b'_{i+1}) belong to S. Therefore, $(b'_0, b'_1), \ldots, (b'_m, b'_0)$ is the desired circuit.

Theorem 19. *If there exists a circuit in a strong component of H^+ then either H^+ contains an invertible pair or there exists a closed walk W in H composed of walks $W[v_0, v_1]$, $W[v_1, v_2]$,..., $W[v_r, v_0]$ with the following properties:*

1. *each $W[v_i, v_{i+1}]$ is constricted from below,*
2. *each $W[v_i, v_{i+1}]$ has a positive net length ℓ (all have the same net length ℓ),*
3. *$W[v_i, v_{i+1}], W[v_j, v_{j+1}]$ avoid each other for every $0 \le i < j \le r$ ($v_{r+1} = v_0$).*

Proof. Suppose $C : (a_0, a_1), (a_1, a_2), \ldots, (a_n, a_0)$ is a circuit in a strong component S of H^+. We may assume $n > 1$, as otherwise, an invertible pair exists, and we are done. Furthermore, if $S \rightsquigarrow S'$ (recall S' is the dual of S) then we consider a circuit in S' and observe that in this case $S' \not\rightsquigarrow S$, as otherwise, $S = S'$, and hence, there exists an invertible pair in S. Thus, without loss of generality, we may assume $S \not\rightsquigarrow S'$. This also means that $\widehat{S}$ is dual-free.

By Lemma 10, S is an unbalanced component and we can assume C is a minimal circuit. According to Theorem 7, we may assume all (a_i, a_{i+1}) are extremal pairs. Since all of (a_i, a_{i+1})'s are in the same strong component, we may assume that all lie on one directed cycle $(X, Y) = D$ in S in which (a_{i+1}, a_{i+2}) is after (a_i, a_{i+1}), $0 \le i \le n$, if we traverse D in the clockwise direction.

Consider the walks P_i, $0 \le i \le n$, for circuit C according to Theorem 7 (2). Each walk P_i starts at a_i, is constricted, and has unbounded positive net length. Every $P_i, P_j, 0 \le i < j \le n$, avoid each other. Moreover, P_i is obtained by walking around the closed walk X. This can be assumed according to Proposition 1. Notice that the net value of D is positive since each P_i has positive net value. Thus, without loss of generality, assume the portion of D from (a_0, a_1) to (a_1, a_2) has positive net value ℓ.

Now we are going to cut the closed walk X into pieces, of net length ℓ. More precisely, consider walks $W_0 = X[a'_0, a'_1]$ where $a'_0 = a_0$ and $a'_1 = a_1$, and $W_j = X[a'_j, a'_{j+1}]$; $j = 1, 2, \ldots$ where a'_{j+1} is an extremal vertex on X and W_j has net length ℓ. Notice that each a'_j lies on the same closed walk X. Since H is finite, at some point we must have two subscripts r, k such that $a'_{r+1} = a'_k$. W.l.o.g. assume that $a'_0 = a'_{r+1}$. It remains to observe that $W_i, W_j, 0 \le i \le j \le r$ avoid each other since $P_0, P_1, P_2, \ldots, P_n$ avoid each other.

10 Correctness of Phase Two

In this section, we prove the lemmas used in the proof of Lemmas 1, 2. Recall that in Line 16 of Algorithm 1 we seek a vertex $p \in V(H)$ with some special properties, namely, no $(q', p) \in V_c \cap \mathcal{L}_k$ and $\exists (p, q) \in \mathcal{R} \cap \mathcal{L}_k$ with $(p, q) \not\rightsquigarrow (q, p)$. In line 18 for vertex p, we add a pair (p, r) where $(p, r) \not\rightsquigarrow (r, p)$, into V_c. Pair (p, r) is called an *initial pair*. In the sub-digraph of H^+ induced by vertices of V_c, a pair reachable in H^+ from an initial pair is called an *1-implied* pair. Note that the path from an initial pair to a 1-implied pair is constricted from below.

This is because all the pairs that are reachable from an initial pair via a path of negative net value are in lower layers, and our algorithm has already processed them.

The sequence of pairs $Ch : (y_0, y_1), (y_1, y_2), \dots, (y_{m-1}, y_m)$ is called a *chain* of pairs between y_0 and y_m. When all the pairs of Ch are in V_c, we say Ch is *minimal* if none of its pairs is by transitivity, and among the chains between (y_0, y_m) in V_c, Ch has the minimum length.

Lemma 11. *Let $Ch : (y_0, y_1), (y_1, y_2), \dots, (y_{m-1}, y_m)$, $m > 1$ be a minimal chain in V_c which is currently circuit free. Suppose for each $0 \leq i \leq m-1$ either (y_i, y_{i+1}) is an UL-pair (with respect to V_c, see Definition 8) or it is a 1-implied pair reachable from an initial pair by a constricted from below path of net value zero. For each pair (y_i, y_{i+1}), $0 \leq i \leq m-1$, let $Z_i = (A_i + A'_i, B_{i+1} + B'_{i+1})$ be a path in H^+ with net value zero from a pair $(p_i, q_{i+1}) \in V_c$ to (y_i, y_{i+1}), and let L_i be the height of Z_i. Here A_i is constricted and has net length L_i, A'_i is constricted and has net length $-L_i$. A_i avoids B_{i+1}, and A'_i avoids B'_{i+1}. Then the following hold.*

1. *Suppose (y_i, y_{i+1}) is an 1-implied pair reachable from an initial pair (p_i, q_{i+1}). Then $L_i > L_{i-1}, L_{i+1}$.*
2. *Suppose $(y_i, y_{i+1}), (y_{i+2}, y_{i+3})$ are UL-pairs (with respect to V_c). Then $\min\{L_i, L_{i+2}\} < L_{i+1} < \max\{L_i, L_{i+2}\}$, and (y_{i+1}, y_{i+2}) is also an UL-pair.*
3. *If (y_i, y_{i+1}) is an 1-implied pair then $i = 0$ or $i = m-1$.*
4. *Suppose $(y_i, y_{i+1}), (y_{i+1}, y_{i+2})$, $1 < i \leq m-2$ are UL-pairs with respect to V_c such that $L_i \leq L_{i+1}$. Then (y_{i-1}, y_i) is also an UL-pair, and $L_{i-1} < L_i$.*
5. *Suppose $(y_i, y_{i+1}), (y_{i+1}, y_{i+2})$, $1 < i \leq m-2$ are UL-pairs (with respect to V_c) such that $L_i \geq L_{i+1}$. Then (y_{i+2}, y_{i+3}) is also an UL-pair, and $L_{i+1} > L_{i+2}$.*

Proof. The assumptions in Lemma 7 and Lemma 8 (no shortcut in the circuit, i.e., minimal circuit) are also applied for a minimal chain.

Proof of 1. For contradiction, first, assume that $L_i < L_{i+1}$. Let h be a vertex on A'_{i+1} such that $A_1 = A'_{i+1}[h, y_{i+1}]$ is constricted and has the same net length as A'_i, and let g be the corresponding vertex to h on B'_{i+1}, and let $B_1 = B'_{i+1}[g, y_{i+2}]$. By applying Lemma 7 on A'_i, B'_i, A_1, B_1 we conclude that A'_i, B'_i, A_1, B_1 have embedded pre-images that pair-wise avoid each other. Moreover, A'_i, B'_i avoid each other and A_1, B_1 avoid each other. Again by applying 7 on $A_i + A'_i$, $B_i + B'_i$, $A_1^{-1} + A_1$, and $B_1^{-1} + B_1$, we conclude that $A'_i, A_1^{-1}, B'_i, B_1^{-1}$ have embedded pre-images that pair-wise avoid each other. Therefore, (p_i, y_{i+2}), (y_i, y_{i+2}) are in the same strong component of H^+. We note that $(y_{i+2}, y_i) \notin V_c$. Notice that $(y_{i+2}, p_i) \notin V_c$, as otherwise, since $(y_{i+2}, p_i) \rightsquigarrow (y_{i+2}, y_i)$, we would have $(y_{i+2}, y_i) \in V_c$; a circuit in V_c. Now according to the rules of the Algorithm 1 lines 16, 17, $(p_i, y_{i+2}) \in V_c$ should have been added into V_c before (p_{i+1}, q_{j+2}), and hence, $(y_i, y_{i+2}) \in V_c$ (because $(p_i, y_{i+2}) \rightsquigarrow (y_i, y_{i+2})$), contradicting the minimality of the chain Ch; unless (y_{i+2}, p_i) is already in V_c which would yield a circuit in V_c; impossible. Notice that when $p_i = p_{i+1}$ again (p_i, y_{i+2}) is a pair that should be added into V_c (because we consider any circuit after running entire line 17) which gives a shorter chain. The argument for the case $L_{i-1} < L_i$ is analogous. This proves the first premise of the lemma.

Proof of 2. For $j = i, i+2$, let $(p'_j, q'_{j+1}) \in V_c$ such that $(p'_j, q'_{j+1})(p_j, q_{j+1})$ is an arc of H^+. By Lemma 8 (3), we conclude that $(q'_{i+1}, p'_{i+2}), (y_{i+1}, y_{i+2})$ are in the same strong component, and hence, $(q'_{i+1}, p'_{i+2}) \in V_c$. Since $(p'_i, q'_{i+1}), (q'_{i+1}, p'_{i+2}) \in V_c$ and we apply transitivity, $(p'_i, p'_{i+2}) \in V_c$. However, because $(p'_i, p_{i+2}) \rightsquigarrow (p_i, p_{i+2}) \rightsquigarrow (y_i, y_{i+2})$ we get a shorter chain, a contradiction.

Proof of 3. According to (1), if (y_j, y_{j+1}) is an 1-implied pair, then (y_{j+1}, y_{j+2}) (if exists) is **not** an 1-implied pair. Moreover, if (y_j, y_{j+1}) is an 1-implied pair then (y_{j-1}, y_j) (if exists) is an UL-pair with $L_{j-1} < L_j$, and (y_{j+1}, y_{j+2}) (if exists) is an UL-pair with $L_{j+1} < L_j$, a contradiction to (2). Thus, the only possibility is when (y_0, y_1) is an 1-implied pair or (y_{m-1}, y_m) is an 1-implied pair.

Proof of 4. For $j = i, i+1$, let $(p'_j, q'_{j+1}) \in V_c$ such that $(p'_j, q'_{j+1})(p_j, q_{j+1})$ is an arc of H^+, and $p'_j p_j, q'_{j+1} q_{j+1} \in A(H)$ (see Figure 10). For contradiction first assume that (y_{i-1}, y_i) is an 1-implied pair. Now by (1) we have $L_{i-1} > L_i$. First assume $L_{i-1} > L_{i+1}$. By applying Lemma 8 (5), we conclude that $(q_{i+1}, q_{i+2}), (q'_{i+1}, q'_{i+2})$ are in the same strong component. We also have $(p'_{i+1}, q'_{i+2}) \rightsquigarrow (y_{i+1}, y_{i+2}) \rightsquigarrow (q_{i+1}, q_{i+2}) \rightsquigarrow (q'_{i+1}, q'_{i+2})$. Therefore, (q'_{i+1}, q'_{i+2}) is in V_c. However, by the transitivity rule of the algorithm, we should have $(p'_i, q'_{i+2}) \in V_c$, and hence, $(p_i, q_{i+2}) \in V_c$, and consequently $(y_i, y_{i+2}) \in V_c$. This is a contradiction to the minimality of the chain Ch. Thus, we conclude that (y_{i-1}, y_i) is an UL-pair.

Now consider the case $L_{i+1} > L_{i-1} > L_i$. If (y_{i-1}, y_i) is an UL-pair then by similar argument one can conclude that (y_{i-1}, y_{i+1}) is already in V_c, a contradiction to the minimality of Ch. If (y_{i-1}, y_i) is an 1-implied pair then again similar to the argument in (2) above, we conclude that according to the rules of the algorithm, (p_{i-1}, y_{i+1}) is in V_c, and hence, $(p_{i-1}, y_{i+1}) \rightsquigarrow (y_{i-1}, y_{i+1})$, a contradiction to minimality of the Ch.

Proof of 5. It is analogous to the proof of 4.

Lemma 12. *Suppose V_c is circuit-free. Let $(a_i, a_{i+1}) \in V_c$ be a pair reachable from $(y_0, y_m) \in V_c$ where (y_0, y_m) is by transitivity on a minimal chain $Ch : (y_0, y_1), (y_1, y_2), \dots, (y_{m-1}, y_m)$, $m > 1$. Moreover, under this assumption for (y_0, y_m), suppose there exists $Z : (y_0, y_m) \rightsquigarrow (a_i, a_{i+1})$ that is symmetric, constricted from below, and with net value zero. Then (a_i, a_{i+1}) is by transitivity on pairs $(a_i, b_1), (b_1, b_2), \dots, (b_{r-1}, b_r), (b_r, a_{i+1})$ in V_c where each of them is either an UL-pair or 1-implied pair reachable from an initial pair via a constricted from below path of net value zero.*

Proof. We use induction on the number of transitivity and reachability steps taken in order to place (a_i, a_{i+1}) into V_c. For each pair (y_i, y_{i+1}), $0 \le i \le m-1$, let Z_i be a path in H^+ from $(p_i, q_{i+1}) \in V_c$ to (y_i, y_{i+1}) which is constricted from below and has net value zero. Let L_i denote the height of Z_i. We denote the $Z_i = (E_i, F_{i+1})$ where E_i is a constricted from below walk from p_i to y_i, with net length zero, and avoiding F_{i+1}. Let L be the height of Z.

Base of induction: First assume that each pair in Ch is either an UL-pair or is 1-implied pair. When (y_i, y_{i+1}), $0 \le i \le m-1$, is an UL-pair then let $(p'_i, q'_{i+1}) \in V_c$ such that $(p'_i, q'_{i+1})(p_i, q_{i+1})$ is an arc in H^+ ($p'_i p_i, q'_{i+1} q_{i+1} \in A(H)$, $p'_i q_{i+1} \notin A(H)$). Now the following hold.

1. $(y_m, y_0) \not\rightsquigarrow (y_1, y_0)$, otherwise, $(y_0, y_1) \rightsquigarrow (y_0, y_m)$ and this contradicts the minimality of the chain Ch. Moreover, $(a_{i+1}, a_i) \not\rightsquigarrow (y_1, y_0)$, otherwise, $(y_0, y_1) \rightsquigarrow (a_i, a_{i+1})$, contradiction to the mininality of Ch.
2. $(y_m, y_0) \not\rightsquigarrow (y_m, y_1)$, as otherwise, $(y_1, y_m) \rightsquigarrow (y_0, y_m) \rightsquigarrow (a_i, a_{i+1})$; contradicting the minimality of Ch, and assumption about (a_i, a_{i+1}). Also $(a_{i+1}, a_i) \not\rightsquigarrow (y_m, y_1)$, otherwise, $(y_1, y_m) \rightsquigarrow (a_i, a_{i+1})$; contradicting the minimality of Ch.
3. $(p_0, q_1) \not\rightsquigarrow (y_m, y_1)$, otherwise, we get a circuit $(y_1, y_2), \dots, (y_{m-1}, y_m), (y_m, y_1)$ in V_c, a contradiction.
4. $(p_0, q_1) \not\rightsquigarrow (y_0, y_m)$, otherwise, it contradicts the minimality of the chain Ch.
5. $(y_{m-1}, y_m) \not\rightsquigarrow (y_{m-1}, y_0)$, otherwise, $(y_0, y_{m-1}) \rightsquigarrow (y_m, y_{m-1})$, and we get a circuit in V_c, a contradiction.
6. $(y_{m-1}, y_m) \not\rightsquigarrow (y_0, y_m)$, otherwise, $(y_m, y_0) \rightsquigarrow (y_m, y_{m-1})$, and we get a circuit in V_c , a contradiction.
7. $(p_{m-1}, q_m) \not\rightsquigarrow (y_0, y_m)$, otherwise, it contradicts the minimality of Ch. Moreover, $(a_{i+1}, a_i) \not\rightsquigarrow (y_0, y_{m-1})$, otherwise, $(y_{m-1}, y_m) \rightsquigarrow (a_i, a_{i+1})$, contradiction to minimality of Ch.
8. $(p_{m-1}, q_m) \not\rightsquigarrow (y_{m-1}, y_0)$, because of the minimality of the chain Ch. Moreover, $(a_{i+1}, a_i) \not\rightsquigarrow (y_{m-1}, y_0)$, as otherwise, $(y_0, y_{m-1}) \rightsquigarrow (a_i, a_{i+1})$, contradiction to minimality of Ch.

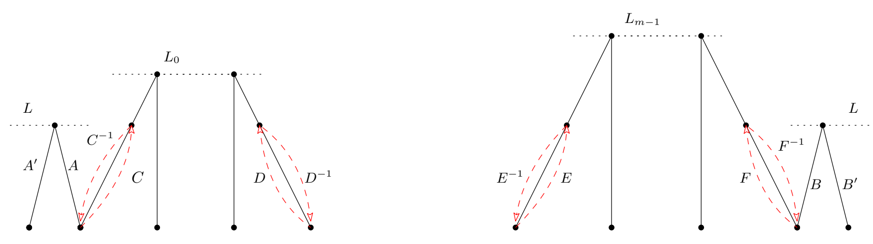

Fig. 11. In Lemma 12, assume $L_0, L_{m-1} > L$. $Z = (A + A', B + B')$ where $A + A'$ and $B + B'$ avoid each other. A, B, $(A')^{-1}$, $(B')^{-1}$, C, D, E, C^{-1}, D^{-1}, E^{-1}, F^{-1} constricted and have same net length. C avoids D, and E avoids F.

First suppose $L_{m-1} \geq L$ and $L_0 \geq L$. According to (1,2,3,4) by Lemma 6, and Lemma 7 on the four walks inside Z_0, Z (in Figure 11, $A, A', B, B', C, C^{-1}, D, D^{-1}$), we conclude that $(y_0, y_1) \rightsquigarrow (a_i, y_1)$. Similarly by considering (5,6,7,8) and applying Lemmas 6, 7 on the walks inside Z_{m-1}, Z (in Figure 11, $A, A', B, B', E, E^{-1}, F, F^{-1}$), we conclude that $(y_{m-1}, y_m) \rightsquigarrow (y_{m-1}, a_{i+1})$. Thus, we obtain the chain $(a_i, y_1), (y_1, y_2), \ldots, (y_{m-2}, y_{m-1}), (y_{m-}$ and the lemma holds.

Next we continue by assuming that $\min\{L_0, L_{m-1}\} < L$. We prove the lemma when $L_0 \leq L_{m-1}$ (the argument for the other case is similar). First assume that (y_0, y_1) is an 1-implied pair. Now since $L > L_0$, $(p_0, y_m) \rightsquigarrow (y_0, y_m) \rightsquigarrow (a_i, a_{i+1})$, and since (p_0, y_m) is also chosen as an initial pair or in line 17 of the algorithm, we get a contradiction to minimality of Ch. Thus, we may assume (y_0, y_1) is an UL-pair.

Claim. $m = 2$, and (y_1, y_2) is an 1-implied pair.

Proof. First suppose (y_1, y_2) is an 1-implied pair. Now according to Lemma 11(3) $m = 2$, and (y_1, y_2) is an 1-implied pair. Thus, we continue by assuming $m > 2$ and that (y_1, y_2) is an UL-pair. Let $1 < j \leq m-1$ be the smallest subscripts such that (y_j, y_{j+1}) be an 1-implied pair. Suppose such j exists. Thus, by Lemma 11 (3) $j = m-1$. Now according to Lemma 11 we conclude that $L_0 < L_1 < \cdots < L_{m-1}$ and all the pairs $(y_0, y_1), \ldots, (y_{m-2}, y_{m-1})$ are UL-pairs and (y_{m-1}, y_m) is an 1-implied pair. If j doesn't exist then all the pairs $(y_0, y_1), (y_1, y_2), \ldots, (y_{m-1}, y_m)$ are UL-pairs, and $L_0 < L_1 \cdots < L_{m-1}$ or $(y_0, y_1), (y_1, y_2), \ldots, (y_{m-1}, y_m)$ are UL-pairs, and $L_0 > L_1 \cdots > L_{m-1}$.

Suppose $L_0 > L_1 > \cdots > L_{m-1}$ and they are all UL-pair. Now one can use the argument in Lemma 11 (5) on the walks inside Z_{m-2}, Z_{m-1}, Z (since $L > L_{m-2}, L_{m-1}$) we get a contradiction to minimality of Ch. Similarly when $L_0 < L_1 \ldots L_{m-2} < L_{m-1}$ we get a contradiction.

We continue by assuming $m = 2$. First assume that $L_1 > L$. Now again similar to the argument in the proof of the Claim 10, by applying Lemma 6 and Lemma 7 on the walks inside Z_0, Z_1, Z, we conclude that $(y_0, y_1), (a_i, y_1)$ are in the same strong component of H^+, and $(y_1, y_2), (y_1, a_{i+1})$ are in the same strong component of H^+; a contradiction to minimality of the chain Ch. Therefore, $L > L_1$. Now in this case again using the same application of Lemma 6, 7, we conclude that (y_0, y_1) and (y_0, p_1) are in the same strong component of H^+. This would imply that (y_0, p_1) is an UL-pair because (y_0, y_1) is an UL-pair. However, this is a contradiction to the choice of p_1, as it implies that $(y_0, p_1) \in V_c \cap \mathcal{L}_k$.

Remark 2. By applying Lemma 7 on Z_i, Z_{i+1} when $L_i \leq L_{i+1}$, we conclude that Z_i is a symmetric path. Similarly if $L_i \geq L_{i+1}$ then Z_{i+1} is symmetric.

Induction step: Suppose some (y_i, y_{i+1}) is not an initial pair. If $L_i \leq L_{i+1}$ or $L_i < L_{i-1}$ then Z_i is symmetric, and hence, by induction hypothesis (y_i, y_{i+1}) is by transitivity on pairs where each is either an initial pair or is an UL-pair. Otherwise, suppose $L_i > L_{i+1}, L_{i-1}$. Note that Z_{i-1}, Z_{i+1} are symmetric, and by induction hypothesis, (y_{i-1}, y_i) is by transitivity on 1-implied pairs or UL-pairs.

So we may assume each of the (y_{i-1}, y_i) and (y_{i+1}, y_{i+2}) is either an UL-pair or is 1-implied pair. Note that by Lemma 11 none of the (y_{i-1}, y_i) and (y_{i+1}, y_{i+2}) is an 1-implied pair, and now this is a contradiction according to Lemma 11 (2).

Lemma 13. *Let X, Y be two paths that are constricted from below with net value zero in H^+. Suppose X starts from (p,q) and reaches (a,b) where $p = b$ or $q = a$. Suppose Y starts from (r,s) and reaches (b,c). Then one of the following occurs.*

1. *$(r,s) \rightsquigarrow (a,c)$; via a path that is constricted from below and has net value zero.*
2. *$(r,s) \rightsquigarrow (b,a)$; via a path that is constricted from below and has net value zero,*
3. *$(p,q) \rightsquigarrow (a,c)$; via a path that is constricted from below and has net value zero,*
4. *$(p,q) \rightsquigarrow (c,b)$; via a path that is constricted from below and has net value zero.*

Proof. Without loss of generality, suppose that $h(X) \leq h(Y)$ (here $h(X)$ is the height of X). Let (g_1, h_1), be a vertex on X with the maximum height, and let (g_2, h_2), be a vertex on Y with the maximum height. Let $X = (A + A', B + B')$ where A starts from p and ends at g_1, and A' starts from g_1 and ends at a (here g_1 is a vertex with the maximum height on $A + A'$). B starts from q and ends at h_1 (h_1 corresponding to g_1), and B' starts from h_1 and ends at b. Notice that $A + A'$ avoids $B + B'$. Let $Y = (C + C', D + D')$ where C starts from r and ends at g_2, and C' starts from g_2 and ends at b. D starts from s and ends at h_2 (h_2 corresponding to g_2), and D' starts from h_2 and ends at c. Note that A, C are constricted and $(A')^{-1}, (C')^{-1}$ are also constricted. By assumption of the lemma, $h(X) \leq h(Y)$. So, let (g,h) be a vertex on Y such that $E = C'[g,b]$ is constricted and have the same net length as A', and $F = D'[h,c]$ is constricted and have the same net length as B' (see Figure 12).

Suppose none of the 1,2,3,4 occurs. Now we can apply Lemma 6 on walks A', B', E, F, and hence, conclude that E, F have congruent embedded pre-images that avoid each other. Now Lemma 7 is applied to the walks $A, A', B, B', E, E^{-1}, F, F^{-1}$ (in Figure 12), and hence, we conclude that $(B)^{-1}$ and A' have (congruent) embedded pre-images that avoid each other, and $B', (A)^{-1}$ have (congruent) embedded pre-images that avoid each other. But, this is not possible because when $p = b$, the endpoint of B' and $(A)^{-1}$ are the same and they cannot avoid each other. Similarly, we get a contradiction when $q = a$. Therefore, one of the (1), (2), (3), (4) should occur.

Using Proposition 2 (2) we conclude that there is a constricted from below path with net value zero follows from (r,s) to (a,c) or from $(r,s) \rightsquigarrow (b,a)$ or from (p,q) to (a,c) or from (p,q) to (c,b).

Lemma 14. *Let X, Y be two constricted from below with net value zero in H^+. Suppose X starts from (p,q) and reaches (a,b). Suppose Y starts from (r,s) and reaches (b,c) where $r = c$ or $s = b$. Furthermore, assume that $h(Y) \leq h(X)$. Then one of the following occurs.*

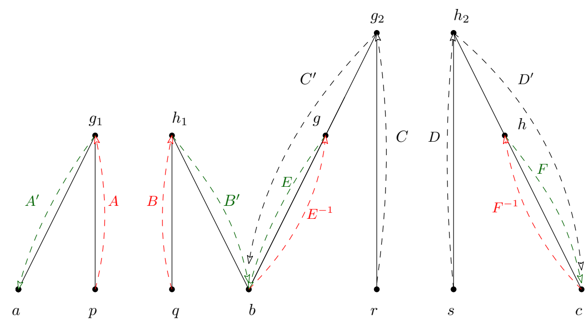

Fig. 12. In Lemma 13 where we assume $h(X) \leq h(Y)$; height of X is smaller than the height of Y which is the same as the height of A. Here $X = (A + A', B + B')$, where $A + A'$ avoids $B + B'$, and $Y = (C + C', D + D')$ where $C + C'$ avoids $D + D'$. $A', B', A^{-1}, B^{-1}, E, F, E^{-1}, F^{-1}$ are constricted, and E avoids F.

1. *$(r, s) \rightsquigarrow (a, c)$; via a path that is constricted from below and has net value zero.*
2. *$(r, s) \rightsquigarrow (b, a)$; via a path that is constricted from below and has net value zero.*
3. *$(p, q) \rightsquigarrow (a, c)$; via a path that is constricted from below and has net value zero.*
4. *$(p, q) \rightsquigarrow (c, b)$; via a path that is constricted from below and has net value zero.*

10.1 Proofs of Lemma 1 and Lemma 2

Recall that Phase One of the algorithm deals with the unbalanced components and after handling unbalanced components we are left with a subdigraph of H^+ which only contains balanced components. In this section we deal with the remaining balanced components in H^+ and prove the correctness of the Phase Two.

In Phase Two of the algorithm, we partition the remaining balanced subdigraph into layers and iterate on those layers. Having processed the pairs on layers $1, \ldots, k-1$, our goal here is to show there always exists a "good pair" on layer k to start the process with. The next lemma states the conditions of a such a "good pair" and its existence.

Lemma 15 (Lemma 1, paraphrased). *Suppose V_c does not contain a circuit and $\mathcal{L}_k \cap \mathcal{R}$ is not empty. Then there exists a vertex $p \in V(H)$ such that no $(q', p) \in V_c \cap L_k$ and there exists a vertex $q \in V(H)$ so that $(p, q) \in \mathcal{R} \cap \mathcal{L}_k$ and $(p, q) \not\rightsquigarrow (q, p)$.*

Proof. We construct digraph $G' = (V, A)$ as follow:

– $V(G') = \{p \in V(H) \mid (p, x) \in \mathcal{R} \cap \mathcal{L}_k\}$

– $A(G') = \{xp \mid (x,p) \in V_c \cap \mathcal{L}_k$ or $(p,x) \in \mathcal{L}_k$ with $(p,x) \rightsquigarrow (x,p)\}$.

If there exists a vertex p in G' with in-degree zero then p is the desired vertex. In what follows, we prove G' contains a vertex p with in-degree zero. Our proof is by contradiction. We prove if G' contains no vertex with in-degree zero then there must be a circuit in V_c which contradicts our assumption about V_c.

Suppose G' contains no vertex with in-degree zero then there exists a directed cycle $v_0, v_1, v_2, \ldots, v_n, v_0$ in G'. Now this means there exists a circuit $C_1 : (v_0, v_1), (v_1, v_2), \ldots, (v_n, v_0)$ so that each (v_i, v_{i+1}) is in $V_c \cap \mathcal{L}_k$ or $(v_{i+1}, v_i) \in \mathcal{L}_k$ with $(v_{i+1}, v_i) \rightsquigarrow (v_i, v_{i+1})$. Relax the conditions on the pairs of C_1 and assume there exists a circuit $C : (x_0, x_1), (x_1, x_2), \ldots, (x_n, x_0)$ such that each (x_i, x_{i+1}) is either a pair in V_c or $(x_i, x_{i+1}) \notin V_c$ and $(x_{i+1}, x_i) \rightsquigarrow (x_i, x_{i+1})$ and $(x_{i+1}, x_i) \in \mathcal{L}_k$. Suppose C has minimum length among such circuits i.e., n is the smallest possible. On one hand, Claim 10.1 states that we cannot have two consecutive pairs (x_i, x_{i+1}) and (x_{i+1}, x_{i+2}) on C such that neither of them is not in V_c i.e., $(x_i, x_{i+1}) \notin V_c$ and $(x_{i+1}, x_{i+2}) \notin V_c$. On the other hand, Claim 10.1 stats that we cannot have $(x_i, x_{i+1}) \notin V_c$ and $(x_{i+1}, x_{i+2}) \in V_c$ either. These two claims yield that all pairs on C must be in V_c, a contradiction to our assumption that V_c is circuit free.

Let us start off with some useful properties of paths $X_i : (x_{i+1}, x_i) \rightsquigarrow (x_i, x_{i+1})$ where $(x_i, x_{i+1}) \notin V_c$ and $(x_{i+1}, x_i) \in \mathcal{L}_k$. Note that if $(x_i, x_{i+1}) \in V_c$ then (x_{i+1}, x_i) in V_d, and hence, $(x_{i+1}, x_i) \notin \mathcal{R}$.

Claim. In the case where $(x_{i+1}, x_i) \notin V_c$, let X_i be a path in H^+ from (x_{i+1}, x_i) to (x_i, x_{i+1}). Such a path is constricted from below and has net value zero.

Proof. According to the definition of the layers and because $(x_{i+1}, x_i) \in \mathcal{L}_k$, X_i is constricted from below. Otherwise, consider a pair (a, b) on X_i, where $X_i[(x_{i+1}, x_i), (a, b)]$ has net value less than zero. Notice that $X_i : (x_{i+1}, x_i) \rightsquigarrow (a, b) \rightsquigarrow (x_i, x_{i+1})$, and by skew property, we have $(x_{i+1}, x_i) \rightsquigarrow (b, a) \rightsquigarrow (x_i, x_{i+1})$. Now, since (a, b) is on a lower layer, either (a, b) or (b, a) has been placed in V_c, and hence (x_i, x_{i+1}) is already in V_c, a contradiction to our assumption. Now suppose X_i has positive net value. Similarly, if X_i has net value greater than zero then by skew property, the reverse of X_i, X_i^{-1}, which is also a path from (x_{i+1}, x_i) to (x_i, x_{i+1}), has negative net value, and hence, there exists some $(a, b) \in X_i^{-1}$, so that (a, b) placed on a lower layer than (x_{i+1}, x_i). This means either $(a, b) \in V_c$ or $(b, a) \in V_c$. Now again this means (x_i, x_{i+1}) should be in V_c, and $(x_{i+1}, x_i) \in V_d$ and not in $\mathcal{R}$.

We further relax the conditions on C, and we may assume C has minimum length among all the circuits where each pair on the circuit is either a pair in V_c or is a pair (x, y) such that $(y, x) \rightsquigarrow (x, y)$ via a path that is constricted from below and has net value zero. In other words, we may assume we cannot short cut C.

Notice: In the rest of the proof, when we mention a path $X : (x, y) \rightsquigarrow (x', y')$, we assume X is constricted from below with net value zero, unless specified.

Moreover, $(x, y) \not\rightsquigarrow (x', y')$ means that (x', y') is not reachable from (x, y) by a constricted from below path with net value zero. Now we the following observation.

Observation 20 *Since C has a minimum length then the following occur.*

1. $(x_{i+2}, x_i) \not\rightsquigarrow (x_i, x_{i+2})$. *Otherwise, we replace C by* $(x_0, x_1), \dots, (x_{i-1}, x_i), (x_i, x_{i+2}), \dots, (x_n, x_0)$, *a contradiction to C having minimum length.*
2. *If* $(x_i, x_{i+1}) \notin V_c$ *then* $(x_{i+2}, x_i) \not\rightsquigarrow (x_{i+1}, x_i)$ *(i.e.* $(x_i, x_{i+1}) \not\rightsquigarrow (x_i, x_{i+2})$*). Otherwise, by skew property,* $(x_{i+2}, x_i) \rightsquigarrow (x_{i+1}, x_i) \rightsquigarrow (x_i, x_{i+1}) \rightsquigarrow (x_i, x_{i+2})$, *contradiction to (1).*
3. *Similar to (2), if* $(x_{i+1}, x_{i+2}) \notin V_c$ *then* $(x_{i+2}, x_i) \not\rightsquigarrow (x_{i+2}, x_{i+1})$ *(i.e.* $(x_{i+1}, x_{i+2}) \not\rightsquigarrow (x_i, x_{i+2})$ *).*

We proceed by showing all pairs on C are in V_c. Suppose there exists a pair $(x_i, x_{i+1}) \notin V_c$. Thus, according to our assumption, the path $X_i : (x_{i+1}, x_i) \rightsquigarrow (x_i, x_{i+1})$ is constricted from below and has net value zero. Note that $n > 1$ as otherwise, we must have $(x_0, x_1) \rightsquigarrow (x_1, x_0)$, and $(x_1, x_0) \rightsquigarrow (x_0, x_1)$, and hence, a strong circuit of length 2 in H^+, a contradiction. We consider two cases. First, we consider the case where $(x_i, x_{i+1}) \notin V_c$ and $(x_{i+1}, x_{i+2}) \notin V_c$. Second, we consider the case where $(x_i, x_{i+1}) \notin V_c$ and $(x_{i+1}, x_{i+2}) \in V_c$. In both cases we derive a contradiction. Before we proceed to the proof of Claim 10.1 we observe the following.

Observation 21 *Suppose* $(x_i, x_{i+1}) \notin V_c$ *and* $(x_{i+1}, x_{i+2}) \notin V_c$. *Since C has a minimum length, the followings hold.*

1. *At most one of the* $X : (x_{i+2}, x_i) \rightsquigarrow (x_i, x_{i+1})$, $Y : (x_{i+2}, x_i) \rightsquigarrow (x_{i+1}, x_{i+2})$ *exists. Otherwise, by Lemmas 13,14 for X, Y,* (x_{i+2}, x_i) *must reach one of the* $(x_i, x_{i+2}), (x_{i+1}, x_i), (x_{i+2}, x_{i+1})$. *However, by Observation 20 (1, 2, 3),* $(x_{i+2}, x_i) \not\rightsquigarrow \{(x_i, x_{i+2}), (x_{i+1}, x_i), (x_{i+2}, x_{i+1})\}$*; a contradiction.*
2. *At most one of the* $X : (x_{i+2}, x_i) \rightsquigarrow (x_i, x_{i+1})$, $Y : (x_i, x_{i+1}) \rightsquigarrow (x_{i+1}, x_{i+2})$ *exists. Otherwise, by Observation 20* $(x_i, x_{i+1}) \not\rightsquigarrow (x_i, x_{i+2})$. *Moreover,* $(x_i, x_{i+1}) \not\rightsquigarrow (x_{i+1}, x_i)$, *as otherwise, we have* $(x_{i+1}, x_i) \rightsquigarrow (x_i, x_{i+1}) \rightsquigarrow (x_{i+1}, x_i)$, *a strong circuit of length 2 in H^+. Finally, by Observation 20 (1, 3)* $(x_{i+2}, x_i) \not\rightsquigarrow \{(x_i, x_{i+1}), (x_{i+2}, x_{i+1})\}$. *However, this is a contradiction to Lemmas 13, 14 for X, Y.*
3. *At most one of the* $X : (x_{i+1}, x_{i+2}) \rightsquigarrow (x_i, x_{i+1})$, $Y : (x_i, x_{i+1}) \rightsquigarrow (x_{i+1}, x_{i+2})$ *exists. Otherwise, by Observation 20(2)* $(x_i, x_{i+1}) \not\rightsquigarrow (x_i, x_{i+2})$. *Moreover,* $(x_i, x_{i+1}) \not\rightsquigarrow (x_{i+1}, x_i)$, *as otherwise, a strong circuit of length 2 in H^+. Analogously,* $(x_{i+1}, x_{i+2}) \not\rightsquigarrow \{(x_i, x_{i+2}), (x_{i+2}, x_{i+1})\}$. *However, this is a contradiction to Lemmas 13,14 for X, Y.*
4. *Analogous to (2), at most one of the* $X : (x_{i+2}, x_i) \rightsquigarrow (x_{i+1}, x_{i+2})$, $Y : (x_{i+1}, x_{i+2}) \rightsquigarrow (x_i, x_{i+1})$ *exists.*

Claim. There does not exists two pairs (x_i, x_{i+1}) and (x_{i+1}, x_{i+2}) on C such that $(x_i, x_{i+1}) \notin V_c$ and $(x_{i+1}, x_{i+2}) \notin V_c$.

Proof. For contradiction suppose there are (x_i, x_{i+1}) and (x_{i+1}, x_{i+2}) on C such that $(x_i, x_{i+1}) \notin V_c$ and $(x_{i+1}, x_{i+2}) \notin V_c$. Recall that, this mean we have two paths $X_i : (x_{i+1}, x_i) \rightsquigarrow (x_i, x_{i+1})$ and $X_{i+1} : (x_{i+2}, x_{i+1}) \rightsquigarrow (x_{i+1}, x_{i+2})$. Having X_i and X_{i+1}, according to Lemma 13 (when $h(X_i) < h(X_{i+1})$; $h(X_i)$ is the height of X_i) or Lemma 14 (when $h(X_i) \geq h(X_{i+1})$) one of the following occurs.

1. $(x_{i+2}, x_{i+1}) \rightsquigarrow (x_i, x_{i+2})$
2. $(x_{i+2}, x_{i+1}) \rightsquigarrow (x_{i+1}, x_i)$
3. $(x_{i+1}, x_i) \rightsquigarrow (x_i, x_{i+2})$
4. $(x_{i+1}, x_i) \rightsquigarrow (x_{i+2}, x_{i+1})$.

In what follows, we show that none of the above can occur; yielding a contradiction.

For contradiction, suppose 1 occurs, i.e., $(x_{i+2}, x_{i+1}) \rightsquigarrow (x_i, x_{i+2})$. By skew property, $X'_{i+1} : (x_{i+2}, x_i) \rightsquigarrow (x_{i+1}, x_{i+2})$. Now consider the paths X_i, X'_{i+1} (see Figure 13).

1.1 By Observation 20 (1) $(x_{i+2}, x_i) \not\rightsquigarrow (x_i, x_{i+2})$.
1.2 By Observation 20 (2) $(x_{i+2}, x_i) \not\rightsquigarrow (x_{i+1}, x_i)$.
1.3 $(x_{i+1}, x_i) \not\rightsquigarrow (x_i, x_{i+2})$. Otherwise, $(x_{i+2}, x_i) \rightsquigarrow (x_i, x_{i+1})$ and because $(x_{i+2}, x_i) \rightsquigarrow (x_{i+1}, x_{i+2})$, we get a contradiction by Observation 21 (1).
1.4 $(x_{i+1}, x_i) \not\rightsquigarrow (x_{i+2}, x_{i+1})$. Otherwise, by skew property $(x_{i+1}, x_{i+2}) \rightsquigarrow (x_i, x_{i+2})$, and because $(x_{i+2}, x_i) \rightsquigarrow (x_{i+1}, x_{i+2})$, we get a contradiction by Observation 21 (2).

Since none of the 1.1, 1.2, 1.3, and 1.4 occurs, we get a contradiction by Lemmas 13, or 14 for walks X_i, X'_{i+1}. Therefore, 1 does not occur. By analogous argument, we can show that 3 does not occur. We show that 4 does not occur, and analogously, 2 cannot occur.

For contradiction, suppose 4 occurs i.e., $(x_{i+1}, x_i) \rightsquigarrow (x_{i+2}, x_{i+1})$. Thus, by skew property, we have $X'_i : (x_{i+1}, x_{i+2}) \rightsquigarrow (x_i, x_{i+1})$. Now for paths X'_i, X_{i+1} (Figure 14) we have the following :

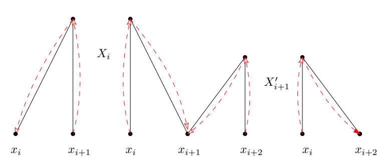

Fig. 13. Assuming $(x_{i+2}, x_i) \rightsquigarrow (x_{i+1}, x_{i+2})$

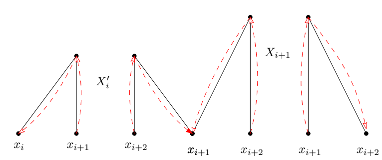

Fig. 14. Assuming $(x_{i+1}, x_{i+2}) \rightsquigarrow (x_i, x_{i+1})$

4.1 $(x_{i+1}, x_{i+2}) \not\rightsquigarrow (x_{i+2}, x_{i+1})$, as otherwise, since $(x_{i+2}, x_{i+1}) \rightsquigarrow (x_{i+1}, x_{i+2})$, we get a strong circuit of length 2; a contradiction.
4.2 $(x_{i+1}, x_{i+2}) \not\rightsquigarrow (x_i, x_{i+2})$. Otherwise, by skew property we have $(x_{i+2}, x_i) \rightsquigarrow (x_{i+2}, x_{i+1})$, a contradiction by Observation 20 (3).

4.3 $(x_{i+2}, x_{i+1}) \not\rightsquigarrow (x_i, x_{i+2})$. This follows from Observation 21 (4).
4.4 $(x_{i+2}, x_{i+1}) \not\rightsquigarrow (x_{i+1}, x_i)$. This follows from Observation 21 (3).

Since none of the 4.1, 4.2, 4.3, and 4.4 is satisfied for X'_i and X_{i+1}, we get a contradiction according to Lemmas 13, 14. Therefore, we conclude that 4 does not occur.

Finally, we conclude that none of the conditions 1, 2, 3, and 4 is satisfied for X_i, X_{i+1} which is a contradiction according to Lemma 13 or Lemma 14. This finishes the proof of Claim 10.1

Claim. There does not exists two pairs (x_i, x_{i+1}) and (x_{i+1}, x_{i+2}) on C such that $(x_i, x_{i+1}) \notin V_c$ and $(x_{i+1}, x_{i+2}) \in V_c$.

Proof. Since $(x_{i+1}, x_{i+2}) \in V_c$, there exists a path $X_{i+1} : (p_{i+1}, q_{i+2}) \rightsquigarrow (x_{i+1}, x_{i+2})$ with $(p_{i+1}, q_{i+2}) \in V_c$. Furthermore, recall that because $(x_i, x_{i+1}) \notin V_c$, we have the path $X_i : (x_{i+1}, x_i) \rightsquigarrow (x_i, x_{i+1})$, which is constricted from below and has net value zero. We observe the following .

1. $(p_{i+1}, q_{i+2}) \not\rightsquigarrow (x_i, x_{i+2})$, otherwise, (x_i, x_{i+2}) is a pair in V_c, yielding a shorter circuit than C.
2. $(p_{i+1}, q_{i+2}) \not\rightsquigarrow (x_{i+1}, x_i)$, otherwise, $(p_{i+1}, q_{i+2}) \rightsquigarrow (x_{i+1}, x_i) \rightsquigarrow (x_i, x_{i+1}) \rightsquigarrow (q_{i+2}, p_{i+1})$; circuit in V_c.
3. $(x_{i+1}, x_i) \not\rightsquigarrow (x_{i+2}, x_{i+1})$, otherwise, $(x_{i+1}, x_{i+2}) \rightsquigarrow (x_i, x_{i+1})$ implying that (x_i, x_{i+1}) is in V_c; a contradiction to our assumption about (x_i, x_{i+1}).
4. $(x_{i+1}, x_i) \not\rightsquigarrow (x_i, x_{i+2})$. Otherwise, let $X'_i = (x_{i+2}, x_i) \rightsquigarrow (x_i, x_{i+1})$. Now by 1,2, $(p_{i+1}, q_{i+2}) \not\rightsquigarrow (x_i, x_{i+2})$, and $(p_{i+1}, q_{i+2}) \not\rightsquigarrow (x_{i+1}, x_i)$. By Observation 20 (1) $(x_{i+2}, x_i) \not\rightsquigarrow (x_i, x_{i+2})$. Moreover, $(x_{i+2}, x_i) \not\rightsquigarrow (x_{i+2}, x_{i+1})$, as otherwise, by skew property $(x_{i+1}, x_{i+2}) \rightsquigarrow (x_i, x_{i+2})$, implying $(x_i, x_{i+2}) \in V_c$, a contradiction to the minimality of C. However, this is a contradiction according to Lemmas 13, 14 for paths X'_i, X_{i+1}.

The above four observations and the contrapositive of Lemma 13 imply that $h(X_i) > h(X_{i+1})$ and as a conclusion of Lemmas 13,14, and 7 it can be shown that X_{i+1} is a symmetric path. Note that by the argument in Claim 10.1, (x_{i-1}, x_i) must be a pair in V_c. Thus, again by a similar argument (using Lemma 14) we conclude that $h(X_{i-1}) < h(X_i)$, and X_{i-1} is symmetric.

Now if both $(x_{i-1}, x_i), (x_{i+1}, x_{i+2})$ are UL-pair with respect to V_c then according to the argument in Lemma 11 (5) we get a contradiction (i.e., (x_i, x_{i+1}) is in V_c). So, we may assume that at least one of the $(x_{i-1}, x_i), (x_{i+1}, x_{i+2})$ is not an UL-pair, and hence, one of $(p_{i-1}, q_i), (p_{i+1}, q_{i+2})$ is by transitivity.

Since, both X_{i-1}, X_{i+1} are symmetric, according to Lemma 11 and the rules of the algorithm, the pair (p_{i+1}, q_{i+2}) is replaced by the UL-pairs. This would allow us without loss of generality to assume (p_{i+1}, q_{i+2}) is an UL-pair, and similarly, (p_{i-1}, q_i) is also an UL-pair (with respect to V_c). Thus again, according to the argument in Lemma 11 (3) we get a contradiction (i.e. (x_i, x_{i+1}) is in V_c).

Lemma 16 (Lemma 2 repeated). *At the end of line 17 of the Algorithm 1, V_c is still circuit free.*

Proof. Suppose by adding (p,r) we close a minimal circuit $C : (a_0,a_1),\dots,(a_{n-1},a_n),(a_n,a_0)$. In the case where $n=1$ we have $(p,r) \rightsquigarrow (a_0,a_1)$, and also $(p,r) \rightsquigarrow (a_1,a_0)$. Now by skew property, we have $(p,r) \rightsquigarrow (a_1,a_0) \rightsquigarrow (r,p)$; contradiction to the choice of (p,r). Hence, we proceed by assuming $n>1$.

Notice that C is minimal if every sub-chain of it is minimal. Thus, by Lemma 12, we may assume that each (a_i,a_{i+1}) is either 1-implied or is UL-pair. Observe that if each pair (a_i,a_{i+1}) is a UL-pair then by Lemma 17 there is another circuit $C' : (a'_0,a'_1),\dots,(a'_n,a'_0)$, and hence, we consider C' instead. Moreover, we consider a circuit after adding an initial pair on layer k, so we may assume at least one pair is an 1-implied pair. Suppose for some $0 \le i \le n$, (a_i,a_{i+1}) is an 1-implied pair.

For each pair (a_i,a_{i+1}), $0 \le i \le n$, let Z_i be a constricted path of net value zero from $(p_i,q_{i+1}) \in V_c$ to (a_i,a_{i+1}). Let L_i denote the height of Z_i. Since (a_i,a_{i+1}) is an 1-implied pair, thus, by Lemma 11 (1) $(a_{i-1},a_i),(a_{i+1},a_{i+2})$ are both UL-pairs this is because when (a_{i-1},a_i) is 1-implied then by Lemma 11 $L_{i-1} > L_i$, on the other hand, since (a_i,a_{i+1}) is 1-implied, by Lemma 11 we must have $L_i > L_{i-1}$, a contradiction (the same argument applied for (a_{i+1},a_{i+2})). Therefore, we have that (a_{i-1},a_i) is 1-implied and both $(a_{i-1},a_i),(a_{i+1},a_{i+2})$ are UL-pairs. However, this is in contradiction to Lemma 11 (2).

11 Connection to other polymorphisms

We first draw a comparison between the obstruction to conservative majority and conservative Maltsev and conservative semilattice. A conservative *majority* polymorphism μ of H is a ternary polymorphism such that $\mu(x,x,y) = \mu(x,y,x) = \mu(y,x,x) = x$ for all $x,y \in V(H)$. A conservative *Maltsev* polymorphism h of H is a ternary polymorphism such that $h(x,y,y) = h(y,y,x) = x$ for all $x,y \in V(H)$.

Definition 9. *Let H be a digraph. Define H^{+k} to be the digraph with the vertex set $V(H^{+k}) = \{(a_1,a_2,\dots,a_k) \mid a_i \in V(H), 1 \le i \le k\}$ and the arc set :*
$A(H^{+k}) = \{(a_1,a_2,\dots,a_k)(b_1,b_2,\dots,b_k) \mid a_ib_i \in A(H), 1 \le i \le k, a_1b_j \notin A(H) \ \ \forall j, 2 \le j \le k\} \cup$
$\{(a_1,a_2,\dots,a_k)(b_1,b_2,\dots,b_k) \mid b_ia_i \in A(H), 1 \le i \le k, b_ja_1 \notin A(H) \ \ \forall j, 2 \le j \le k\}$

When $k=2$, then we get the usual H^+ defined in the previous sections.

Definition 10 (permutable triple). *A permutable triple is three vertices a,b,c together with vertices $\alpha_a,\alpha_b,\alpha_c,\beta_{ab}$,*
β_{bc},β_{ca} together with three directed paths P_1,P_2,P_3 in H^{+3} such that $P_1 : (a,b,c) \rightsquigarrow (\alpha_a,\beta_{bc},\beta_{bc})$, $P_2 : (b,c,a) \rightsquigarrow (\alpha_b,\beta_{ca},\beta_{ca})$ and finally $P_3 : (c,a,b) \rightsquigarrow (\alpha_c,\beta_{ab},\beta_{ab})$.

Theorem 22. *[19] A digraph H admits a conservative majority polymorphism if and only if H does not admit a permutable triple.*

We say $a, b, c \in V(H)$ is a *Maltsev triple* if there exist vertices $\alpha_a, \alpha_c, \beta_{ab}, \beta_{bc}$ in H such that $(\alpha_a, \beta_{bc}, \beta_{bc}) \rightsquigarrow (a, b, c)$ in H^{+3} and $(\alpha_c, \beta_{ab}, \beta_{ab}) \rightsquigarrow (c, b, a)$ in H^{+3}.

Theorem 23. *[19] A digraph H admits a conservative Maltsev polymorphism if and only if H does not admit a Maltsev triple.*

We continue to consider other interesting polymorphisms. In order to establish our results about other polymorphisms, it is required to carefully consider the situation where we have a circuit in a strong component of H^+. In this case, as we have shown in Subsection 9.2 either H^+ contains an invertible pair, and hence, H does not admit a CC polymorphism or there exists a closed walk with special properties described in Theorem 19

11.1 Collapse: CSL = CST = Conservative Cyclic of All Arities

A polymorphism f of H of arity k is *totally symmetric* if $f(x_1, x_2, \ldots, x_k) = f(y_1, y_2, \ldots, y_k)$ whenever the sets $\{y_1, y_2, \ldots, y_k\}$ and $\{x_1, x_2, \ldots, x_k\}$ are the same. A *set polymorphism* of H is a mapping f of the non-empty subsets of $V(H)$ to $V(H)$, such that $f(S)f(T) \in A(H)$ whenever S, T are non-empty subsets of $V(H)$ with the property that for each $s \in S$ there is a $t \in T$ with $st \in A(H)$ and also for every $t \in T$ there is an $s \in S$ with $st \in A(H)$. It is easy to see, cf. [10], that H has a conservative set polymorphism if and only if it has conservative totally symmetric (CTS) polymorphisms of all arities k. A polymorphism f of arity k on digraph H is called *cyclic* if $f(x_1, x_2, \ldots, x_k) = f(x_2, x_3, \ldots, x_k, x_1)$ for all $x_1, x_2, \ldots, x_k \in V(H)$.

We note that a digraph H that admits a CSL polymorphism also admits CTS polymorphisms of all arities: the conservative set function that assigns to each set S the minimum under the min ordering. Moreover, a CTS polymorphism applies to all arities, including arity two, whence it implies a CC polymorphism. Thus, the class of digraphs with a min ordering is included in the class of digraphs with a conservative set polymorphism, which is included in the class of digraphs with a CC polymorphism.

Theorem 24. *A digraph H admits a CSL polymorphism if and only if it admits a conservative set polymorphism.*

Proof. Since a min ordering allows to define a conservative set polymorphism as the minimum, it suffices to show that a digraph that does not have a min ordering also cannot have a conservative set polymorphism. We show this by showing that a circuit in one component of H^+ means that H does not have a conservative set polymorphism.

So suppose $(a_0, a_1), (a_1, a_2), \ldots, (a_n, a_0)$ is a circuit in a strong component S of H^+. Then, by Theorem 19, we have that either there exists an invertible pair in H^+, and hence, there is no CC polymorphism, or there exists a closed walk W composed of walks $W[v_0, v_1]$, $W[v_1, v_2], \ldots, W[v_r, v_0]$ with the following properties:

– each $W[v_i, v_{i+1}]$ is constricted from below,
– each $W[v_i, v_{i+1}]$ has a positive net length ℓ,
– $W[v_i, v_{i+1}]$ and $W[v_j, v_{j+1}]$ avoid each other for every $0 \leq i < j \leq r$ ($v_{r+1} = v_0$).

Now, for any conservative set polymorphism f, we must have $f(v_0, v_1, \dots, v_r) = f(v_1, v_2, \dots, v_r, v_0)$. Since the walks $W[v_i, v_{i+1}], W[v_j, v_{j+1}]$, $0 \leq i < j \leq r$ avoid each other(*avoidance definition requires being congruent*) they all have the same number of vertices. Let $W[v_j, v_{j+1}] = v_j, v_j^1, \dots, v_j^t, v_{j+1}$, $0 \leq j \leq r$. Now we apply the conservative polymorphism f on the vertices of the walks $W[v_0, v_1], W[v_1, v_2], \dots, W[v_r, v_0]$, and conclude that, if $f(v_0, v_1, \dots, v_r) = v_i$ then $f(v_0^1, v_1^1, \dots, v_r^1) = v_i^1 \in W[v_i, v_{i+1}]$, $f(v_0^2, v_1^2, \dots, v_r^2) = v_i^2$, consequently $f(v_0^t, v_1^t, \dots, v_r^t) = v_i^t$, and finally $f(v_1, v_2, \dots, v_r, v_0) = v_{i+1}$, a contradiction.

We remark that in the proof, we have only used the fact that H does not have a conservative cyclic polymorphism. Thus, we have actually proved the following.

Theorem 25. *The class of bi-arc digraphs coincides with each of the following classes of digraphs:*

1. *digraphs with a CSL polymorphism,*
2. *digraphs with a conservative set polymorphism,*
3. *digraphs with CTS polymorphisms of all arities, and*
4. *digraphs with conservative cyclic polymorphisms of all arities.*

Remark 3. It is proved in [22] that CSP($\mathbb{H}$) is decided by the so-called canonical LP relaxation if and only if $\mathbb{H}$ admits a symmetric polymorphisms of all arities. In the same paper, it is claimed that any relational structure has symmetric polymorphisms of all arities if and only if it has TS polymorphisms of all arities. However, this claim turned out to be wrong for general relational structures, see example 99 in [23].

Remark 4. A constraint language admits TS polymorphisms of all arities if and only if it has *width 1*. CSP($\mathbb{H}$), has width 1 if the so-called 1-minimality algorithm refutes every unsatisfiable instance of CSP($\mathbb{H}$).

12 NP-complete cases and a dichotomy classification

For NP-completeness, we use a reduction from the classical NP-complete problem *betweenness* [26]. Given a set $U = \{u_1, u_2, \dots, u_n\}$ and a list of triples (i, j, k) of distinct integers from $\{1, 2, \dots, n\}$, the betweenness problem asks if there is a linear ordering of $\{u_1, u_2, \dots, u_n\}$ such that for each triple (i, j, k) in the list, u_j is between u_i and u_k. In the following subsections, we use reductions from the betweenness problem.

12.1 NP-completeness for Two Binary Relations

This subsection discusses the case when a relational structure R contains two binary relations. The argument for the following lemma can be obtained from [20]. For the sake of completeness, we provide detailed proof here.

Lemma 17. *Let H_1 and H_2 be two digraphs (binary relations) on the same vertex set V. Then it is NP-complete to decide whether V admits an ordering that it is a min ordering with respect to both H_1 and H_2.*

Proof. For positive integer n, let $I_n = \{1, 2, \dots, n\}$. Let B be an instance of the betweenness problem. We are given a set U, and a subset S from set $\{(i, j, k) \mid i, j, k \in I_n\}$. The goal is to find an ordering $u_1 < u_2 < \cdots < u_n$ of the vertices in U such that for every $(i, j, k) \in S$, either $u_i < u_j < u_k$ or $u_k < u_j < u_i$.

Let $B = (U, S)$ be an instance of the betweenness problem and let $r = |S|+1$ where $|S|$ denotes the number of triples in S. Suppose $U = \{u_1, u_2, \dots, u_n\}$. H_1 and H_2 have the same vertex set which consists of $r = |S| + 1$ copies of U. Precise construction for H_1 and H_2 is as follows (an example for this construction is given in Figure 15).

1. $V(H_1) = V(H_2) = \{a_{i,j} \mid 1 \le i \le |S|+1, \text{ and } 1 \le j \le n, \ a_{i,j}$ corresponds to u_j in copy i of $U\}$,
2. $A(H_1) = \{a_{t,i}a_{t,j}, a_{t,j}a_{t,k} \mid (i, j, k)$ is the t-th element of $S\}$.
3. $A(H_2) = \{a_{i,j}a_{i+1,j} \mid 1 \le i \le r, \text{ and } 1 \le j \le n\}$.

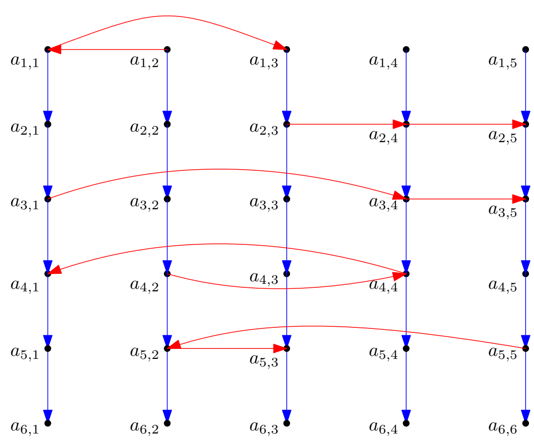

Fig. 15. An example for the construction in Lemma 17. Here, $S = \{(2, 1, 3), (3, 4, 5), (1, 4, 5), (2, 4, 1), (5, 2, 3)\}$ and $A(H_1)$ is in red, and $A(H_2)$ is in blue.

Claim. There is an ordering of U satisfying the betweenness condition if and only if there is an ordering of $V(H_1)$ that is a min ordering with respect to both H_1 and H_2.

Proof. If U has an ordering $<$ consistent with all the triples, then we can order the vertices of H_1 by taking this ordering on all copies of U, and put all the vertices of the i-th copy before all the vertices of the $(i+1)$-st copy. It is easy to see that the resulting ordering is a min ordering. Conversely, if $<$ is a min ordering of H_1 and H_2 simultaneously, then the arcs in $A(H_2)$ ensure that all copies are ordered in the same way, i.e., if x precedes y in some copy, then it also precedes it in next copy, and hence in all the copies of U. This means there is an ordering $<$ of U corresponding to all of them. The arcs in $A(H_1)$ ensure that each triple is consistent with respect to $<$. This is because when ab and bc are arcs in $A(H_1)$, and H_1 has a min ordering, then either $a < b < c$ or $c < b < a$ in the ordering.

This finishes the proof of the lemma. An example for Claim 12.1 is given in Figure 16.

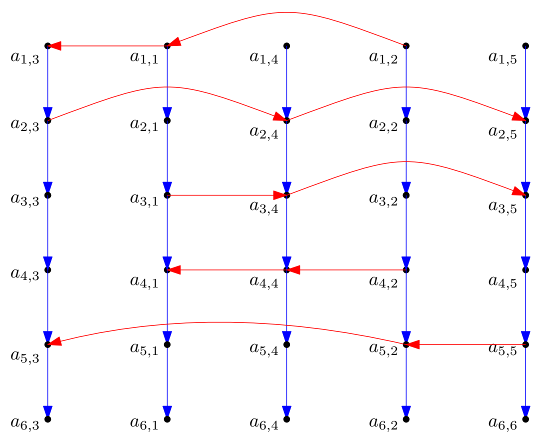

Fig. 16. An example for Claim 12.1. An ordering for $V(H_1)$, according to the discussion in Claim 12.1, which is a min ordering with respect to both $A(H_1)$ (red arcs) and $A(H_2)$ (blue arcs). The betweenness ordering for set $S = \{(2,1,3),(3,4,5),(1,4,5),(2,4,1),(5,2,3)\}$ is $3,1,4,2,5$.

12.2 NP-completeness for Arity 3

In this subsection, we focus on the case where a relational structure consists of a ternary relation. We prove to decide if a ternary relation admits a CSL polymorphism is NP-complete.

Theorem 26. *Deciding if a ternary relation admits a CSL polymorphism is NP-complete.*

Proof. We start from an instance of the betweenness $B = (U, S)$, and construct two digraphs H_1 and H_2 and from these two digraphs we construct a ternary relation. Let $U = \{u_1, u_2, \ldots, u_n\}$. Construct digraphs H_1 and H_2 in the same way discussed in Lemma 17. Recall that, by Claim 12.1, there is an ordering of U satisfying the betweenness condition if and only if there is an ordering of $V(H_1)$ that is a min ordering with respect to both H_1 and H_2. Let R be a ternary relation constructed as follows. The ground set of R is $V(H_1)$. The tuples of R are $(a_{t,i}, a_{t,j}, a_{t+1,j+1})$ where $a_{t,i}a_{t,j} \in A(H_1)$. In other words, the tuples of R are (x, y, z) where xy is an arc of H_1 and yz is an arc of H_2. In what follows, we prove R admits a CSL polymorphism if and only if there is an ordering of the vertices of H_1 that is a min ordering for both H_1 and H_2.

$\Rightarrow$) Suppose there is an ordering $<$ of the vertices of H_1 that is a min ordering for both H_1 and H_2. According to the proof of Claim 12.1, in this ordering, the vertices of each copy appear together and according to the ordering of the elements of U in the betweenness instance (when all the triples are satisfied). More precisely, suppose $u_1 < u_2 < \cdots < u_n$ is the ordering of U, where for each triple $(i, j, k) \in S$ either $u_i < u_j < u_k$ or $u_k < u_j < u_i$. Then $a_{t,1} < a_{t,2} < \cdots < a_{t,n} < a_{t+1,1} < a_{t+1,2} < \cdots < a_{t+1,n}$, for $1 \le t \le r+1$, is an ordering of $V(H_1)$ which is a min ordering for both H_1, H_2.

Now define $f(a, b) = f(b, a) = a$ if $a < b$ in the ordering, i.e., $f(a, b) = \min\{a, b\}$. We show that f is a semilattice polymorphism for R. We need to show that R is closed under f. Suppose $(a, b, c) \in R$, and $(a', b', c') \in R$. Thus, $ab, a'b' \in A(H_1)$, and $bc, b'c' \in A(H_2)$. Since $<$ is a min ordering for both H_1, H_2, $\min\{a, a'\}\min\{b, b'\}$ is an arc of H_1 and $\min\{b, b'\}\min\{c, c'\}$ is an arc of H_2. This means $f(a, a')f(b, b') \in A(H_1)$, and $f(b, b')f(c, c') \in A(H_2)$, and hence, by definition of R, $(f(a, a'), f(b, b'), f(c, c'))$ is in R. Note that since min ordering is commutative and associative, f is a semilattice.

$\Leftarrow$) Conversely, suppose f is a CSL polymorphism of R. We will define an ordering on $V(H_1)$ that is a min ordering on both H_1, H_2. First, we will show that f can be modified in a way that it is still a CSL polymorphism of R and, furthermore, it satisfies the following properties:

1. for every $a_{t,i}, a_{s,j}$, $t < s$, $f(a_{t,i}, a_{s,j}) = f(a_{s,j}, a_{t,i}) = a_{t,i}$,
2. $f(a_{t,i}, a_{t,i'}) = a_{t,i}$ if and only if $f(a_{s,i}, a_{s,i'}) = a_{s,i}$.

We first obtain f_1 from f as follows. For every $a_{t,i}, a_{s,j}$, $t < s$, $f_1(a_{t,i}, a_{s,j}) = f_1(a_{s,j}, a_{t,i}) = a_{t,i}$. In any other case $f_1(x, y) = f(x, y)$. In other words, f_1 and f have the same outcome on the vertices inside each copy of U. Clearly by definition $f_1(x, y) = f_1(y, x) \in \{x, y\}$. Thus, f_1 is a CC operation. Now, we show that $f_1(x, f_1(y, z)) = f_1(f_1(x, y), z)$ for every x, y, z in the ground set of R (i.e., for all $x, y, z \in V(H_1) = V(H_2)$).

If all x, y, z belong to the same copy of U, then because f has associative property, f_1 would be associative. Suppose $x = a_{t,i}$, $y = a_{s,j}$, $z = a_{p,k}$. We may assume $|\{t, s, p\}| > 1$, otherwise, since f is associative, f_1 is also associative. First, suppose $t < s, p$. Then, by definition, $f_1(a_{t,i}, f_1(a_{s,j}, a_{p,k})) = a_{t,i}$, and $f_1(a_{t,i}, a_{s,j}) = a_{t,i}$ and since $f_1(a_{t,i}, a_{p,k}) = f_1(a_{t,i}, a_{s,j}) = a_{t,i}$, we have $f_1(a_{t,i}, f_1(a_{s,j}, a_{p,k})) = f_1(f_1(a_{t,i}, a_{s,j}), a_{p,k})$. Second, suppose $t \ge \min\{s, p\}$. If

$t > s, p$ then $f_1(a_{t,i}, f_1(a_{s,j}, a_{p,k})) = f_1(a_{s,j}, a_{p,k})$, and $f_1(f_1(a_{t,i}, a_{s,j}), a_{p,k}) = f_1(a_{s,j}, a_{p,k})$, and $f_1(a_{t,i}, f_1(a_{s,j}, a_{p,k})) = f_1(f_1(a_{t,i}, a_{s,j}), a_{p,k})$. Third, assume $t = s < p$ (or $t = p < s$). In this case, $f_1(a_{t,i}, f_1(a_{s,j}, a_{p,k})) = f_1(a_{t,i}, a_{s,j}) = f(a_{t,i}, a_{s,j})$, and $f_1(f_1(a_{t,i}, a_{s,j}), a_{p,k}) = f_1(f(a_{t,i}, a_{s,j}), a_{p,k}) = f(a_{t,i}, a_{s,j})$. The other case can be treated similarly. Therefore, f_1 is associative. Next, we show that R is closed under f_1. Suppose $(a_{t,i}, a_{t,j}, a_{t+1,j}), (a_{s,i'}, a_{s,j'}, a_{s+1,j'}) \in R$, and let $\mu = (f_1(a_{t,i}, a_{s,i'}), f_1(a_{t,}, a_{s,j'}), f_1(a_{t+1,j}, a_{s+1,j'}))$. First, suppose $t < s$. Then $\mu = (a_{t,i}, a_{t,j}, a_{t+1,j}) \in R$. Similarly, if $t > s$ then $\mu \in R$. So we continue by assuming $t = s$. In this case, according to the construction of R, we have $i' = j$, $j' = \ell$ such that $a_{t,i}a_{t,j}, a_{t,j}a_{t,\ell} \in A(H_1)$. Thus, $\mu = (f(a_{t,i}, a_{t,j}), f(a_{t,j}, a_{t,\ell}), f(a_{t+1,j}, a_{t+1,\ell}))$ and hence, $\mu \in R$.

By the above discussion, without loss of generality, we proceed by assuming that f has the following property. For every $a_{t,i}, a_{s,j}$, $t < s$, $f(a_{t,i}, a_{s,j}) = a_{t,i}$.

Suppose $a_{t,i}a_{t,j}, a_{t,j}a_{t,\ell}$ are arcs of H_1. Then, $(a_{t,i}, a_{t,j}, a_{t+1,j}), (a_{t,j}, a_{t,\ell}, a_{t+1,\ell}) \in R$, and since f is a CSL we have $(f(a_{t,i}, a_{t,j}), f(a_{t,j}, a_{t,\ell}), f(a_{t+1,j}, a_{t+1,\ell})) \in R$. Moreover, because the arcs of H_1 are among each copy of U, one of the following must hold.

1. $f(a_{t,i}, a_{t,j}) = a_{t,i}, f(a_{t,j}, a_{t,\ell}) = a_{t,j}, f(a_{t+1,j}, a_{t+1,\ell}) = a_{t+1,j}$, and $f(a_{t,i}, a_{t,\ell}) = a_{t,i}$ (because f is associative)
2. $f(a_{t,i}, a_{t,j}) = a_{t,j}, f(a_{t,j}, a_{t,\ell}) = a_{t,\ell}, f(a_{t+1,j}, a_{t+1,\ell}) = a_{t+1,\ell}$, and $f(a_{t,i}, a_{t,\ell}) = a_{t,\ell}$ (because f is associative).

Thus, we would have the following observation.

Observation 27 *At this point, the restriction of f on the arcs of H_1, is a CSL. In other words, if $a_{t,i}a_{t,j}, a_{t,j}a_{t,\ell}$ are arcs of H_1, then $f(a_{t,i}, a_{t,j})f(a_{t,j}, a_{t,\ell})$ is also an arc of H_1.*

In order to obtain a min ordering for H_1, using f, we further modify f so that $f(a_{t,i}, a_{t,i'}) = a_{t,i}$ if and only if $f(a_{s,i}, a_{s,i'}) = a_{s,i}$, and keeping f being a CSL with respect to the arcs of H_1 (inside each copy of U), as well as with respect to the arcs of H_2. Notice that f defines a min ordering on each copy of U; that is for every $1 \le t \le r$, we obtain ordering $\prec_t$, by setting $a_{t,i} \prec_t a_{t,j}$ if and only if $f(a_{t,i}, a_{t,j}) = a_{t,i}$.

Let G be a graph constructed as follows. $V(G) = \{(x, y) \mid x, y \in V(H_1)\}$. The edge set of G consists of the union of the following,

$$\begin{aligned}
E(G) = \{&(a_{t,i}, a_{t,j})(a_{s,i}, a_{s,j}) \mid \\
&\quad f(a_{t,i}, a_{t,j}) = a_{t,i}\ , f(a_{s,i}, a_{s,j}) = a_{s,i} \text{ and } a_{t,i}a_{t,j}, a_{s,i}a_{s,j} \in A(H_1)\} \\
\cup \{&(a_{t,i}, a_{t,j})(a_{s,i}, a_{s,j}) \mid \\
&\quad f(a_{t,i}, a_{t,j}) = a_{t,j}\ , f(a_{s,i}, a_{s,j}) = a_{s,j} \text{ and } a_{t,i}a_{t,j}, a_{s,i}a_{s,j} \in A(H_1)\} \\
\cup \{&(a_{t,i}, a_{t,j})(a_{t,j}, a_{t,\ell}), (a_{t,i}, a_{t,j})(a_{t,i}, a_{t,\ell}), (a_{t,j}, a_{t,\ell})(a_{t,i}, a_{t,\ell}) \mid \\
&\quad [f(a_{t,i}, a_{t,j}) = a_{t,i}, f(a_{t,j}a_{t,\ell}) = a_{t,j}] \textbf{ or } [f(a_{t,i}, a_{t,j}) = a_{t,j}, f(a_{t,j}a_{t,\ell}) = a_{t,\ell}]\}
\end{aligned}$$

Suppose for some arc $a_{t,i}a_{t,j}$ of H_1, $f(a_{t,i}, a_{t,j}) = a_{t,i}$ while for some arc $a_{s,i}a_{s,j}$ of H_1 with $t < s$, we have $f(a_{s,i}, a_{s,j}) = a_{s,j}$. We may assume t is the smallest subscript, and secondly, s is the smallest subscript. Let G_1 be the set of vertices in G that are reachable from $(a_{s,i}, a_{s,j})$ in G, i.e., a connected component of G containing $(a_{s,i}, a_{s,j})$. Now, for every $(x, y) \in G_1$, set $f(x, y) = f(y, x) = x$. Notice that by the construction of G, and since f is also a CC polymorphism, there is no path from $(x, y) \in G_1$ to $(y, x) \in G_1$. Therefore, the changes to f would be consistent.

Notice that after this modification for a fixed pair of indices i, j, $(a_{t,i}, a_{t,j})$, $1 \leq t \leq r+1$, $f(a_{t,i}, a_{t,j}) = a_{t,i}$ when $a_{t,i}a_{t,j} \in A(H_1)$ (note that direction of the arcs is not according to f). Next we consider another arc $a_{t',i'}a_{t',j'} \in A(H_1)$, with $f(a_{t',i'}, a_{t',j'}) = a_{t',i'}$ while for some arc $a_{s',i'}, a_{s',j'}$ with $t' < s'$, we have $f(a_{s',i'}, a_{s',j'}) = a_{s',j'}$. Let G_2 be the vertices that are reachable from $(a_{s',i'}, a_{s',j'})$. Again for every pair $(x, y) \in G_2$, we set $f(x, y) = x$. Since G is a graph, there is no vertex in G_1 that is reachable from a vertex in G_2. Thus, the f value for the vertices in G_1 is not going to change anymore. In other words, the changes of f on G_2 would be consistent with the changes of f on G_1. Notice that since f is a CC polymorphism, there is no path from $(x, y) \in G_2$ to $(y, x) \in G_2 \cup G_1$. We repeat the above procedure until no such pairs $(a_{t',i'}, a_{t',j'}), (a_{s',i'}, a_{s',j'})$ where $a_{t',i'}a_{t',j'}, a_{s',i'}a_{s',j'}$ are arcs of H_1, and $f(a_{t',i'}, a_{t',j'}) = a_{t',i'}$, $f(a_{s',i'}, a_{s',j'}) = a_{s',j'}$, can be found.

In the next step we look for some $(a_{t,i}, a_{t,j})$, where t is the smallest index so that there exists a pair $(a_{s,i}, a_{s,j})$, $t < s$ where $(a_{s,i}, a_{s,j})$ is not an isolated vertex in G, and $f(a_{t,i}, a_{t,j}) = a_{t,i}$ while $f(a_{s,i}, a_{s,j}) = a_{s,j}$. Let G_3 be connected component of G, containing $(a_{s,i}, a_{s,j})$. We further modify f on the vertices of G_3, by setting $f(x, y) = x$ for every $(x, y) \in G_3$. Notice that as we argued above the changes are consistent with the previous changes on f.

At the final stage, we consider pairs $(a_{t',i'}, a_{t',j'})$, and $(a_{s',i'}, a_{s',j'})$ so that both are isolated vertices in G, $t' < s'$, and $f(a_{t',i'}, a_{t',j'}) = a_{t',i'}$ while $f(a_{s',i'}, a_{s',j'}) = a_{s',j'}$ (assuming t' is the smallest index, and then s' is then smallest index). In this case we set $f(a_{s',i'}, a_{s',j'}) = a_{s',i'}$.

Finally, we define an ordering $<$ on the vertices H_1 by setting $x < y$ if and only if $f(x, y) = x$. This means that we would have $a_{t,1} < a_{t,2} < \cdots < a_{t,n} < a_{t+1,1} < a_{t+1,2} < \cdots < a_{t+1,n}$, $1 \leq t \leq r$. Notice that this ordering is a min ordering for H_1, and it is easy to see that is also a min ordering for H_2.

12.3 Higher Arities and a Dichotomy Theorem

Theorem 28. *Let R be a relation of arity $r > 3$. Then deciding whether R admits a CSL is NP-complete.*

Proof. We use reduction from deciding whether a ternary relation has a CLS. Let R_1 be an arbitrary ternary relation on set A. Let R be a relation of arity r, with the tuples $(\overbrace{a, a, \ldots, a}^{r-3}, a_1, a_2, a_3)$ for every $a \in A$, and every $(a_1, a_2, a_3) \in R_1$. Suppose f is a CSL on R_1. Then we show that f is also a CSL for R. We need to show that for every two tuples $t_1 = (a, a, \ldots, a, a_1, a_2, a_3), t_2 =$

$(b, b, \ldots, b, b_1, b_2, b_3)$,

$$(f(a,b), f(a,b), \ldots, f(a,b), f(a_1,b_1), f(a_2,b_2), f(a_3,b_3)) \in R.$$

Since R_1 is closed under f, we have $(f(a_1,b_1), f(a_2,b_2), f(a_3,b_3)) \in R_1$, and hence, $f(t_1,t_2) \in R$. Therefore, f is a CLS for R. Conversely, suppose f is a CSL on R. Then the projection of f on the last three coordinates of R is a CSL on R_1.

Theorems 26, 28, and Lemma 17 together with Theorem 6 provide a full complexity classification of Problem 1 and yield us the following dichotomy theorem.

Theorem 29 (Dichotomy theorem). *Deciding if a relational structure* $\mathbb{H} = \langle V, R_1, \ldots, R_k \rangle$ *admits a CSL polymorphism is polynomial-time solvable if all relations* R_i *are unary, except possibly one binary relation. In all other cases, the problem is NP-complete.*

13 Conclusions

We have provided polynomial time algorithm, obstruction characterizations, for digraphs admitting a min ordering, i.e., a CSL polymorphism. We believe they are a useful generalization of interval graphs, encompassing adjusted interval digraphs, monotone proper interval digraphs, complements of circular arcs of clique covering number two, two-directional orthogonal ray graphs, and other well-known classes. We have also similarly characterized digraphs admitting a CC polymorphism. We showed that the class of digraphs admitting a set polymorphism, i.e., CTS polymorphisms of all arities, coincides with the class of digraphs with a min ordering, and so is equal to the class of bi-arc digraphs. Our algorithm can be adapted to recognize the digraphs that admit extension of min ordering, the so-called k-min ordering ($k \geq 2$).

Open Problem 30 *What is the complexity of deciding whether a digraph admits a (not necessarily conservative) semilattice polymorphism?*