Recognizing Interval Bigraphs by Forbidden Patterns

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Abstract

Let H be a connected bipartite graph with n vertices and m edges. We give an $\mathcal{O}(nm)$ time algorithm to decide whether H is an interval bigraph. The best known algorithm has time complexity $\mathcal{O}(nm^6(m+n)\log n)$ and it was developed by Muller in 1997 [16]. Our approach is based on an ordering characterization of interval bigraphs introduced by Hell and Huang in 2003 [11]. We transform the problem of finding the desired ordering to choosing strong components of a pair-digraph without creating conflicts. We make use of the structure of the pair-digraph as well as decomposition of bigraph H based on the special components of the pair-digraph. This way we make explicit what the difficult cases are and gain efficiency by isolating such situations.

1 Introduction

The vertex set of a graph H is denoted by $V(H)$ and the edge set of H is denoted by $E(H)$. A bigraph H is a bipartite graph with a fixed bipartition into *black* and *white* vertices. We sometimes denote these sets as B and W, and view the vertex set of H as partitioned into (B, W). A bigraph H is called an *interval bigraph* if there exists a family I_v, $v \in B \cup W$, of intervals (from the real line) such that, for all $x \in B$ and $y \in W$, the vertices x and y are adjacent in H if and only if I_x and I_y intersect. Then, this family of intervals is called an *interval representation* of bigraph H.

Interval bigraphs were introduced in [9] and have been studied in [2, 11, 16]. They are closely related to interval digraphs introduced by Sen et al. [6]. In particular, our algorithm can be used to recognize interval digraphs (in time $\mathcal{O}(mn)$), as well.

Interval bigraphs and interval digraphs have become of interest in such new areas as graph homomorphisms, e.g. [8].

A *co-circular arc bigraph* is a bipartite graph whose complement is a circular arc graph. The class of interval bigraphs is a subclass of co-circular arc bigraphs. Indeed, the former class consists exactly of those bigraphs whose complement is the intersection of a family of circular arcs no two of which cover the circle [11]. There is a linear-time recognition algorithm for co-circular arc bigraphs [15]. On the other hand, the class of interval bigraphs is a super-class of proper interval bigraphs (also known as bipartite permutation graphs), for which there is also a linear-time recognition algorithm [11, 17].

Interval bigraphs can be recognized in polynomial time using the algorithm developed by Muller [16]. Muller's algorithm runs in time $\mathcal{O}(nm^6(n+m)\log n)$. This is in sharp contrast with the recognition of *interval graphs*, for which several linear time algorithms are known, e.g., [1, 3, 4, 10, 14].

In [11, 16], the authors attempted to give a forbidden structure characterization of interval bigraphs, but fell short of the target. In this paper, some light is shed on these attempts, as we clarify which situations are not covered by the existing forbidden structures. We believe our algorithm can be used as a tool for producing the interval bigraph obstructions. For the time being, there are infinitely many obstructions, which still lack a description that fit them into a finite collection of nicely defined families. However, the main purpose of this paper is to devise an efficient algorithm for recognizing interval bigraphs.

We use the ordering characterization of interval bigraphs in [11]. A bigraph H is an interval bigraph if and only if its vertices admit a linear ordering $<$ without any of the forbidden patterns in Figure 1. Hence,

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we will rely on the existence of a linear ordering $<$ such that if $v_a < v_b < v_c$ (not necessarily consecutively) and v_a, v_b have the same color and opposite to the color of v_c then $v_a v_c \in E(H)$ implies that $v_b v_c \in E(H)$.

Figure 1: Forbidden Patterns

There are several graph classes that can be characterized by the existence of an ordering without a number of forbidden patterns. One such class is the class of interval graphs. A graph G is an interval graph if and only if there exists an ordering $<$ of $V(G)$ such that none of the following patterns appears [5, 7].

- $v_a < v_b < v_c$, $v_a v_c, v_b v_c \in E(G)$ and $v_a v_b \notin E(G)$
- $v_a < v_b < v_c$, $v_a v_c \in E(G)$ and $v_b v_c, v_a v_b \notin E(G)$

Some of the other classes of graphs that have ordering characterizations without forbidden patterns are proper interval graphs, comparability graphs, co-comparability graphs, chordal graphs, convex bipartite graphs, co-circular arc bigraphs, and proper interval bigraphs [13]. It is possible to view the ordering problem for some of these classes in some cases (e.g. interval bigraphs and interval graphs) as an instance of the 2-SAT problem together with transitivity clauses as described below. For every pair (u, v) of vertices of H, we define a Boolean variable X_{uv} which takes values zero or one only such that $X_{uv} \equiv \neg X_{vu}$. We introduce appropriate clauses with two literals expressing the forbidden patterns. Finally, we add all transitivity clauses, which are clauses of the from $(X_{uv} \vee X_{vw} \vee X_{wu})$ where u, v, and w are distinct. If $X_{uv} = 1$ then we put u before v; otherwise v comes before u in the ordering. However, we would like to consider a different approach proven to be more structural and successful in other ordering problems.

2 Basic definitions and properties

Note that a bigraph is an interval bigraph if and only if each connected component of it is an interval bigraph. In the remainder of this paper, we shall assume that H is a connected bigraph with a fixed bipartition (B, W).

We define the *pair-digraph* H^+ of H, corresponding to the forbidden patterns in Figure 1, as follows. The vertex set of H^+ consists of all pairs (u, v) such that $u, v \in V(H)$ and $u \neq v$— for clarity, we will often refer to vertices of H^+ as *pairs* (in H^+). Then, the arcs in H^+ are of one of the following two types:

- $(u, v)(u', v)$ is an arc of H^+ when u and v have the same color with $uu' \in E(H)$, and $vu' \notin E(H)$.
- $(u, v)(u, v')$ is an arc of $H+$ when u and v have different colors with $vv' \in E(H)$, and $uv \notin E(H)$.

Observe that if there is an arc from (u, v) to (u', v'), then both uv and $u'v'$ are non-edges of H. For two pairs $(x, y), (x', y') \in V(H^+)$ we say (x, y) *dominates* (x', y') (or (x', y') is *dominated by* (x, y)) and write $(x, y) \to (x', y')$ if there exists an arc (directed edge) from (x, y) to (x', y') in H^+. One should note that if $(x, y) \to (x', y')$ in H^+ then $(y', x') \to (y, x)$, to which property we will refer to as *skew-symmetry*.

Lemma 2.1. *Let $<$ be an ordering of H without the forbidden patterns in Figure 1, and let $(u, v) \to (u', v')$ with $u < v$. Then, $u' < v'$.*

Proof. According to the definition of H^+, we either have

Case (1) u, v have the same color, $v = v'$, $uu' \in E(H)$, and $vu' \notin E(H)$; or

Case (2) u, v have different colors, $u = u'$, $vv' \in E(H)$, and $uv \notin E(H)$

In Case (1) (resp. Case (2)), if $v' < u'$, then vertices v', u, v (resp. u, v, u')— in that order— would induce a forbidden pattern in H, a contradiction. Hence, in both cases we will have $u' < v'$, as desired. □

We shall generally refer to a strong component of H^+ simply as a *component* of H^+. We shall also identify a component by its vertex (pair) set. A component in H^+ is called *non-trivial* if it contains more than one pair. For any component S of H^+, we define its *couple component*, denoted S', to be $S' = \{(u, v) : (v, u) \in S\}$.

The skew-symmetry property of H^+ implies the following fact.

Lemma 2.2. *If S is a component of H^+ then so is S'.*

In light of Lemma 2.2, for each component S of H^+, S and S' are couple components of each other and we shall collectively refer to them as *coupled components*. It can be easily shown that coupled components S and S' are either disjoint or equal— in the latter case, we say component S is *self-coupled*.

Definition 2.3 (circuit)**.** A sequence $(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, x_0)$ of pairs in a set $D \subseteq V(H^+)$ is called a *circuit* in D.

Lemma 2.4. *If a component of H^+ contains a circuit then H is not an interval bigraph.*

Proof. Let $(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, x_0)$ be a circuit in a component S of H^+. Since S is strongly connected, for all non-negative integers i and j there exists a directed walk $W_{i,j}$ in H^+ from (x_i, x_{i+1}) to (x_j, x_{j+1}), where indices are mod $n + 1$. Now, for all $i, j \geq 0$, following the sequence of pairs on $W_{i,j}$ and using Lemma 2.1, we conclude that $x_j < x_{j+1}$ whenever $x_i < x_{i+1}$. Hence, we must either have $x_i < x_{i+1}$ for all i, or $x_i > x_{i+1}$ for all i. However, since $x_{n+1} = x_0$, either case implies $x_0 \neq x_0$; a contradiction. □

If H^+ contains a self-coupled component then H is not an interval bigraph. This is because a self-coupled component of H^+ contains two such pairs as (u, v) and (v, u), which comprise a circuit of length 2 (corresponding to $n = 1$ in the definition of a circuit). We remark that a similar result to Lemma 2.4 exists for co-circular arc bigraphs [12]. A *tournament* is a complete digraph with no directed cycle of length two and no self-loop. A tournament is called *transitive* if it is acyclic; i.e., if it does not contain a directed cycle.

Lemma 2.5. *Suppose that H^+ contains no self-coupled components, and let D be any subset of $V(H^+)$ containing exactly one component from each pair of coupled components. Then, D is the set of arcs of a tournament on $V(H)$. Moreover, such a D can be chosen to be a transitive tournament if and only if H is an interval bigraph.*

Proof. Suppose D is a transitive tournament. Then we obtain the ordering $<$, by letting $x < y$ when $(x, y) \in D$. It is clear that $<$ is a total ordering because D is transitive, and when $(x, y) \in D$, $(y, x) \notin D$. Observe that $<$ does not contain any of the forbidden pattern in Figure 1, and hence, H is interval bigraph. Conversely, if H is an interval bigraph then there exits ordering $<$, without forbidden patterns in Figure 1. We add (x, y) into set D whenever $x < y$ in the ordering. It is easy to see that D is a transitive tournament. □

In what follows, by a *component* we mean a non-trivial (strong) component unless we specify otherwise. For simplicity, we shall use a set S of pairs in H^+ to also denote the sub-digraph of H^+ induced by S, when no confusion arises.

We shall say two edges ab and cd of H are *independent* if the subgraph of H induced by the vertices a, b, c, and d has just the two edges ab and cd. We shall say two disjoint induced subgraph H_1 and H_2 of H are *independent* if there is no edge of H with one endpoint in H_1 and another endpoint in H_2.

Lemma 2.6. *If uu' and vv' are independent edges in H then the pairs $(u, v), (u', v), (u', v')$, and (u, v') form a directed four-cycle of H^+ in the given order (resp. in the reversed order) when u and v have the same color (resp. have opposite colors). In particular, $(u, v), (u', v), (u', v')$, and (u, v') belong to the same component of H^+. Moreover, if S is a component of H^+ containing a pair (u, v) then, there exist two independent edges uu' and vv' of H and, as such, the four pairs $(u, v), (u, v'), (u', v)$, and (u', v') are contained in S.*

Proof. The first part follows from the definition of H^+ and independent edges. As for the seond part, note that since S is a component, (u, v) dominates some pair of S and is dominated by some pair of S. First, suppose u and v have the same color in H. Then (u, v) dominates some $(u', v) \in S$ and is dominated by some $(u, v') \in S$. Now uu' and vv' must be edges of H, and $uv, uv', u'v$, and $u'v'$ must be non-edges of H. Thus, uu' and vv' are independent edges in H. In this case, according to the first part of the lemma, S contains the directed cycle $(u, v) \to (u', v) \to (u', v') \to (u, v') \to (u, v)$.

Second, suppose u and v have different colors. We note that (u, v) dominates some $(u, v') \in S$, and hence, $uv \notin E(H)$ and vv' is an edge of H. Since (u, v') dominates some pair $(u', v') \in S$, $uu' \in E(H)$ and $u'v' \notin E(H)$. Now uu' and vv' are edges of H, and $uv, uv', u'v$, and $u'v'$ must be non-edges of H. Thus, uu' and vv' are independent edges in H. In this case, according to the first part of the lemma, S contains the directed cycle $(u, v) \to (u, v') \to (u', v') \to (u', v) \to (u, v)$. □

2.1 Structural properties of the (strong) components of H^+

The structure of components of H^+ is quite special, and the trivial components interact with them in simple ways. A trivial component will be called a *source* if its unique vertex has in-degree zero, and a *sink* if its unique vertex has out-degree zero. Herein, we further explore these properties through establishing several lemmas. To this end, we need the following definition on reachability of pairs in H^+.

Definition 2.7 (reachability closure)**.** Let R be a subset of the pairs of H^+. Let $N^+[R]$ denote the set of all pairs in H^+ that are reachable (via a directed path in H^+) from a pair in R. (Notice that $N^+[R]$ contains R.) We call $N^+[R]$ the *reachability closure* of R. We say a pair (u, v) is *implied* by R if $(u, v) \in N^+[R] \setminus R$. If $R = N^+[R]$, we say that R is *closed* under reachability.

Lemma 2.8. *A pair (a, c) is implied by a component S of H^+ if and only if H contains an induced path a, b, c, d, e, such that $N(a) \subseteq N(c)$ and S contains all of the pairs $(a, d), (a, e), (b, d)$, and (b, e).*

Proof. If such a path exists, then ab, de are independent edges and so the pairs (a, d), (a, e), (b, d), and (b, e) lie in a component by the remarks preceding Lemma 2.6. Moreover, $(a, d) \to (a, c)$ is in H^+; hence (a, c) is indeed implied by this component.

Conversely, suppose (a, c) is implied by a component S. We first observe that the colors of a and c must be the same. Otherwise, say a is black and c is white, and there exists a white vertex u such that the pair (u, c) is in S and dominates (a, c). By Lemma 2.6, there would exist two independent edges uz and cy. Looking at the edges and non-edges between u, c and a, z, y, we see that H^+ contains the arcs $(u, c) \to (a, c) \to (a, y) \to (u, y)$. Since both (u, c) and (u, y) are in S, the pair (a, c) must also be in S, contrary to what we assumed. Therefore, a and c must have the same color in H, say black. In this case there exists a white vertex $d \in V(H)$ such that $(a, d) \in S$ and $(a, d) \to (a, c)$. Hence $dc \in E(H)$ and $da \notin E(H)$. If there was also a vertex t adjacent to a but not to c, then at and cd would be independent edges of H, placing (a, c) in S. Thus, every neighbor of a in H is also a neighbor of c in H. Finally, since (a, d) is in component S, Lemma 2.6 yields vertices b and e such that ab and de are independent edges in H. It follows that a, b, c, d, e is an induced path in H. □

We emphasize that ab and de from Lemma 2.8 are independent edges. The inclusion $N(a) \subseteq N(c)$ implies the following corollary.

Corollary 2.9. *If there is an arc from a component S of H^+ to a pair $(x, y) \notin S$ then (x, y) forms a trivial component of S that is a sink component. If there is an arc to a component S of H^+ from a pair $(x, y) \notin S$ then (x, y) forms a trivial component of H^+ that is a source. In particular, if there is a directed path in H^+ from component S_1 to component S_2, then $S_1 = S_2$.*

To give even more structure to the components of H^+, we recall the following definition. The *condensation* of a digraph G is a digraph obtained from G by identifying the vertices in each component and deleting loops and multiple edges.

Lemma 2.10. *Every directed path in the condensation of H^+ has at most three vertices.*

Proof. If a directed path P in the condensation of H^+ goes through a vertex corresponding to a component S in H^+, then P has at most three vertices by Corollary 2.9. Now suppose P contains only vertices in trivial components and let (x, y) be a vertex on P which has both a predecessor and a successor on P otherwise we are done. First suppose x and y have the same color in H. Then the successor is some pair (x', y) and the predecessor is some pair (x, y'), and hence, xx' and yy' are independent edges of H, and hence, by Lemma 2.6 $(x, y), (x', y)$, and (x, y') belong to the same component of H^+, contradicting that P goes through trivial components only. Thus, we continue by assuming that x and y have opposite colors in H, the successor of (x, y) in P is some (x, y'), and the predecessor of (x, y) in P is some (x', y). Thus, $xy \notin E(H)$, and hence, $x'y' \in E(H)$, otherwise, we would have independent edges xx' and yy' and conclude as above. By the same reasoning, every vertex adjacent to x is also adjacent to y', and every vertex adjacent to y is also adjacent to x'. Therefore, (x', y) has in-degree zero, and (x, y') has out-degree zero, and P has only three vertices. □

Lemma 2.11. *Suppose that H^+ has no self-coupled components. Let u, v, and w be three vertices of H such that S_{uv}, S_{vw} are components of H^+ where $S_{uv} \neq S_{wv}$. Then, S_{uw} is also a component of H^+. Moreover, suppose $S_{uv} \neq S_{uw}, S_{wu}$, and $S_{vw} \neq S_{uw}, S_{wu}$. Then, there exist maximal subgraphs H_1, H_2, and H_3 of H such that :*

- *H_1, H_2, and H_3 are pairwise independent (no edge between H_i and H_j, $1 \leq i < j \leq 3$).*
- *Let $X \subseteq H \setminus H'$ ($H' = H_1 \cup H_2 \cup H_3$) be the vertices with at least one neighbor in H'. Then every $x \in X$ is adjacent to all the vertices with the opposite color in $X \cup H'$.*

Proof. We assume u, v, w have the same color. The argument for other cases is similar. Since S_{uv}, S_{vw} are components of H^+, by Lemma 2.6, there are independent edges ua_1, vb_1 of H and independent edges va_2, wb_2 of H. Notice that by Lemma 2.6, $(u, v), (u, b_1), (a_1, b_1), (a_1, v) \in S_{uv}$ and $(v, w), (a_2, w), (a_2, b_2), (v, b_2) \in S_{vw}$. Now $a_1w \notin E(H)$, otherwise, $(a_1, v) \to (a_1, b_2) \to (w, a_2)$, and hence, by Corollary 2.9, $S_{uv} = S_{wv}$. Similarly, $ub_2 \notin E(H)$, otherwise, $S_{vw} = S_{vu}$, and by skew-symmetry, $S_{uv} = S_{wv}$. Now ua_1, wb_2 are independent edges, and hence, S_{uw} is a component. Note that $a_2u \notin E(H)$, otherwise, $(a_2, b_2) \to (u, b_2) \to (u, w)$, implying a directed path from S_{vw} to S_{uw}, and hence, $S_{vw} = S_{uw}$. Similarly $b_1w \notin E(H)$.

Let H_1, H_2, H_3 be maximal subgraphs of H such that $ua_1 \in E(H_1)$, $vb_1, va_2 \in E(H_2)$, and $wb_2 \in E(H_3)$ and H_1, H_2, H_3 are pairwise independent. It is easy to see that for every $a \in H_1$, $b \in H_2$, $c \in H_3$ we have $(a, b) \in S_{uv}$, $(a, c) \in S_{uw}$, and $(b, c) \in S_{vw}$. Let $x \in H \setminus H'$ where $H' = H_1 \cup H_2 \cup H_3$. W.l.o.g suppose x is adjacent to b_2. Since $x \notin H_3$, x must be adjacent to a vertex in H_1 or H_2. First suppose $xa_2 \in E(H)$. Now a_1x must be an edge of H, otherwise, $(u, a_2) \to (u, x) \to (a_1, x) \to (a_1, b_2)$ implying a directed path from S_{uv} to S_{uw}, and consequently $S_{uv} = S_{uw}$; a contradiction to our assumption. Second, suppose $xa_1 \in E(H)$. Now $a_2x \in E(H)$, otherwise, $(a_1, b_1) \to (x, a_2) \to (a_2, b_2)$, and hence, there is a directed path from S_{uv} to S_{vw}, and consequently, $S_{uv} = S_{vw}$, a contradiction. Suppose $xb_1, xb_2, yv, yw \in E(H)$. Then $xy \in E(H)$, otherwise, $(v, w) \to (b_1, w) \to (b_1, y) \to (x, y) \to (x, v) \to (b_2, v) \to (w, v)$, implying $S_{vw} = S_{wv}$, a contradiction. □

3 Recognition algorithm

In this section, we present our algorithm for the recognition of interval bigraphs. Firstly, to describe the algorithm, we introduce some technical definitions.

Definition 3.1 (envelope). Let R be a set of pairs of H^+. The *envelope* of R, denoted $N^*[R]$, is the smallest set of pairs that contains R and is closed under both reachability and transitivity (if $(u, v), (v, w) \in N^*[R]$ then $(u, w) \in N^*[R]$).

Remark: For the purposes of the proofs, we visualize taking the envelope of R as divided into consecutive *levels*, where in the zero-th level we just replace R by its reachability closure, and in each subsequent level we replace R by the rechability closure of its transitive closure. The pairs in the envelope of R can be thought of as forming the arc of a digraph on $V(H)$, and each pair can be thought of as having a label corresponding to its level. The pairs (arcs of the digraph) in R, and those implied by R have label 0, arcs obtained by transitivity from the arcs labeled 0, as well as all arcs implied by them have label 1, and so on. More precisely, $N^*[R]$

is the disjoint union of $R^0, R^1, \ldots, R^k$, where $R^0 = N^+[R]$ (level zero), and each R^i (level $i \geq 1$) consists of every pair (u, v) such that either (u, v) is obtainable by transitivity in R^{i-1} (meaning that there is some sequence $(u, u_1), (u_1, u_2), \ldots, (u_{r-1}, u_r), (u_r, v)$ in R^{i-1}), or (u, v) is dominated by a pair (u', v') obtainable by transitivity in R^{i-1}. Note that $R \subseteq N^+[R] \subseteq N^*[R]$.

Definition 3.2 (dictator component)**.** Let $\mathcal{R} = \{R_1, R_2, ..., R_k, S\}$ be the set of components of H^+ such that $N^*[\bigcup_{A \in \mathcal{R}} A]$ contains a circuit. We say S is a *dictator* if for every subset W of $\mathcal{R} \setminus \{S\}$, there exist a circuit in the envelope of $(\bigcup_{A \in W'} A) \cup (\bigcup_{B \in \mathcal{R} \setminus W} B)$, where $W' = \{R'_i \mid R_i \in W\}$. In other words, S is a dictator if by replacing some of the R_is with R'_is in $\mathcal{R}$ and taking the envelope of the union of elements we still get a circuit.

Definition 3.3 (complete set)**.** A set $D_1 \subseteq V(H^+)$ is called *complete* if for every pair of coupled components R, R' of H^+, exactly one of $R \subseteq D_1$ and $R' \subseteq D_1$ holds.

A component S is a dictator if and only if the envelope of every complete set D_1 containing S has a circuit.

Definition 3.4 (simple pair, complex pair)**.** A pair $(x, y) \in H^+$ is *simple* if it belongs to $N^+[S]$ for some component S, otherwise, we call it *complex*.

Before describing the algorithm, we establish the following counterpart of Lemma 2.4.

Lemma 3.5. *Let S, S' be coupled components in H^+, so that both $N^*[S]$ and $N^*[S']$ contain a circuit. Then, H is not an interval bigraph.*

Proof. According to Lemma 2.5 the final set D must be a total ordering with transitivity property. Therefore, one of the S and S' must be in D. In order to find a total ordering avoiding the patterns in Figure 1, one of the $N^*[S], N^*[S']$ must be in D, which is impossible. $\square$

An overview of the algorithm: The algorithm constructs H^+ and then considers its coupled components (recall that we mean strong components that are not trivial). In the preliminary stage, if there is a self-coupled component, then the algorithm reports H is not an interval bigraph. Otherwise, the algorithm takes four main stages. During the algorithm, we maintain a sub-digraph D of H^+. Initially, D is empty. At each subsequent step of the algorithm, a set of pairs from H^+ are added to D. The goal is to choose from each couple components (trivial and non-trivials) one and place into D without creating a circuit. Thus, we need to add into D the pairs that are reachable from the current pairs in D as well as the pairs that are obtained by applying transitivity on the existing pairs in D. So each pair is placed in D either by reachability or transitivity. When we say a pair (x, y) is *by transitivity*, we mean (x, y) is placed into D by applying transitivity on the existing pairs in D. Likewise, we say a pair is *by reachability* when (x, y) is implied by the existing pairs in D. Finally, at successful termination, D will be a transitive tournament as described in Lemma 2.5.

For the purpose of the algorithm once a pair (x, y) is added into D we assign a time (level) to (x, y), that is the level in which (x, y) is added into D. Each pair (x, y) carries a dictator code, say $Dic(x, y)$; that shows the dictator component involved in placing (x, y) into D. The four main stages of the algorithm are as follows.

In Stage 1, an empty set D is initialized. Then, from each pair S, S' of coupled components we select one, say S. If $D \cup N^+[S]$ does not have a circuit then add $N^+[S]$ (all the pairs in $N^+[S]$) into D and discard $N^+[S']$ from further consideration in this stage. Otherwise, we discard $N^+[S]$ in this stage and add $N^+[S']$ into D instead. Again, if D has a circuit then H is not an interval bigraph and the algorithm terminates. If we succeed in selecting exactly one of the coupled components S, S' of H^+ then we proceed to the next stage. Theorem 5.4 implies the correctness of this stage, and Corollary 5.2 provides the first set of obstructions if we fails to finish this stage.

In Stage 2, $N^*[D]$ is computed level by level, and is placed into D. If by adding a pair (x, y) into D a circuit C appears for the first time, then the length of C is exactly 4 and we can identify a dictator component S associated with C by using function Dictator(x, y), (i.e. $(Dic(x, y))$ where (x, y) is a complex pair in C. Furthermore, in that case, C has to be of the form $C = (x_0, x_1), (x_1, x_2), (x_2, x_3), (x_3, x_0)$, where x_0, x_3 belong to the same color class while x_1, x_2 are contained in the opposite one; moreover, no pair (x_i, x_{i+1}), $0 \leq i \leq 3$

has placed in D by transitivity (the sum is taken module 4). It turns out that if we keep S in D, then regardless of the selection of other components, we still will end up having a circuit in computation of $N^*[D]$. These facts and, hence, the correctness of Stage 2 will be established in Lemmas 6.1,6.2, and 6.3.

In Stage 3, we initialize D_1 to be the empty set. Then, for every dictator component $S \in \mathcal{DT}$ we add $N^+[S']$ into D_1 and discard $N^+[S]$ (since we will encounter a circuit). Moreover, for every (non-trivial strong) component $S_1 \in D \setminus \mathcal{DT}$ we add $N^+[S_1]$ into D_1 and discard $N^+[S_1']$. We then set $D = N^*[D_1]$. If there is a circuit in D, the algorithm reports H is not an interval bigraph and exit, otherwise, it proceeds to the next stage. The correctness of this stage is established in Lemma 7.1.

In Stage 4, one by one, we add into D the remaining (trivial strong components) components of H^+ that are outside D. At each step we add a sink component $S_1 \subseteq V(H^+) \setminus D$ and discard its coupled component S' from further consideration. Lemma 7.2 establishes the correctness of this step.

Algorithm 1 Algorithm for recognition of interval bigraphs

function IntervalBigraph(H)
  Input: A connected bigraph H with a bipartition (B, W)
  Output: An ordering of the vertices of H without patterns in Figure 1 or return false
  Construct the pair-digraph H^+ of H, and compute its components; if any component is self-coupled report false
 Stage1 : Adding (non-trivial strong) components
  Initialize D to be the empty set
  for all coupled components $S, S' \subseteq V(H^+)$ **do**
    if $D \cup N^+[S]$ does not have a circuit **then**
      add $N^+[S]$ into D and delete $N^+[S']$ from further consideration in this step
        ▷ add X to D means add all the pairs of X into D
      for all $(x, y) \in N^+[S]$ **do** $Dic(x, y) = S$
    else
      if $D \cup N^+[S']$ does not have a circuit **then**
        add $N^+[S']$ into D and delete $N^+[S]$ from further consideration in this step
        for all $(x, y) \in N^+[S']$ **do** set $Dic(x, y) = S'$
      else report that H is not an interval bigraph
 Stage2 : Computing the envelope of D and detecting dictator components
  Set $En = N^*[D]$, and $\mathcal{DT} = \varnothing$ ▷ $\mathcal{DT}$ is a set of components
  while $\exists (x, y) \in En \setminus D$ **do** ▷ we consider the pairs in En level by level
    Move (x, y) into D and set $Dic(x, y) =$ Dictator(x, y, D)
    if $D \cup \{(x, y)\}$ contains a circuit **then** add $Dic(x, y)$ into $\mathcal{DT}$ ▷ (x, y) is a complex pair
 Stage3 : Adding dual of dictator components, and other chosen components
  Let $D_1 = \varnothing$
  for all components $S \in \mathcal{DT}$ **do** add $N^+[S']$ into D_1
  for all components $R \in D \setminus \mathcal{DT}$ **do** add $N^+[R]$ into D_1
  Set $D = N^*[D_1]$
  if there is a circuit in D **then** report H is not an interval bigraph
 Stage4 : Adding other remaining trivial components and returning an ordering
  while $\exists$ trivial component S outside D, and S is a sink component **do**
    Add S into D and remove S' from further consideration
 Outputting the final ordering
  for all $(u, v) \in D$ **do** set $u \prec v$ ▷ yielding an ordering of $V(H)$ without the patterns from Figure 1
  Return the ordering $v_1 \prec v_2 < \cdots \prec v_n$ of $V(H)$

function DICTATOR(x, y, D)
  if $(x, y) \in N^{+}[S]$ for some component S in D **then** return S
  if x, y have different colors and $(u, y) \in D$ dominates (x, y) **then** return DICTATOR(u, y, D) ▷ we mean the earliest pair (u, y)
  if x, y have the same color and $(x, w) \in D$ dominates (x, y) **then** return DICTATOR(x, w, D)
  if x, y have the same color and (x, y) is by transitivity on $(x, w), (w, y) \in D$ **then** return DICTATOR(w, y, D)
  if x, y have different colors and (x, y) is by transitivity on $(x, w), (w, y) \in D$ **then** return DICTATOR(x, w, D)

In Section 8, we discuss the implementation of the algorithm and argue that the running time of Algorithm 1 is $\mathcal{O}(mn)$ where m is the number of edges and n is the number of vertices of the input bigraph H.

Theorem 3.6 (Correctness of Algorithm 1)**.** Let H be a bigraph with n vertices and m edges. If H is an interval bigraph then Algorithm 1 produces an ordering without forbidden patterns in Figure 1, otherwise, it outputs NOT. Moreover, the running time of Algorithm 1 is $\mathcal{O}(mn)$.

Proof. Theorem 5.4 validates Stage 1. Lemmas 6.1,6.2, 6.3 shows the correctness of Stage 2. Lemma 7.1 proves Stage 3 is valid, and Lemma 7.2 validates Stages 4. Lemma 8.1 shows the algorithm runs in $\mathcal{O}(mn)$. □

4 Example:

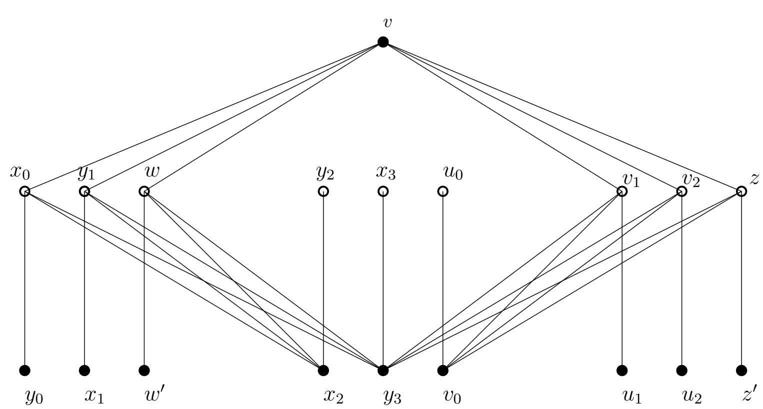

Figure 2: Bigraph H is not interval bigraph

We apply Algorithm 1 on the bigraph H depicted in Figure 2 whereby show that H does not admit an ordering without the forbidden patterns in Figure 1 and, hence, is not an interval bigraph. In fact, we encounter a circuit at Stage 2 as well as at Stage 3. Note that since x_0y_0, x_1y_1, and ww' are independent edges of H, both $S_{x_0x_1}$ and S_{x_1w} are components of H^{+}. Likewise, since $u_1v_1, u_2v_2, z'z$ are independent edges of H, $S_{v_1u_2}$ and S_{u_2z} are component of H^{+}. Finally, since x_2y_2, x_3y_3, v_0u_0 are independent edges of H, $S_{x_2x_3}$ and $S_{x_3v_0}$ are component of H^{+} (recall that by a component we mean a non-trivial strong component). Note that $(x_2, x_3), (x_3, v_0)$ are in the same component since x_2, y_3 are adjacent to w while v_0 is not adjacent to w; and y_3, v_0 are adjacent to v_1 while $x_2v_1 \notin E(H)$. Therefore, $(x_2, x_3) \rightarrow (x_2, y_3) \rightarrow (y_2, y_3) \rightarrow (y_2, v_1) \rightarrow (x_2, v_1) \rightarrow (x_2, v_0) \rightarrow (w, v_0) \rightarrow (w, u_0) \rightarrow (y_3, u_0) \rightarrow (y_3, v_0) \rightarrow (x_3, v_0)$, and hence, $S_{x_2x_3} = S_{x_3v_0}$ by Corollary 2.9.

Suppose at Stage 1 the algorithm selects components $S_{x_0x_1}$, S_{x_1w}, $S_{x_2x_3}$, alongside components $S_{v_1u_2} = S_{u_1v_2}$; $S_{u_2z} = S_{v_2z'}$, and adds their pairs into D. Then, $(x_0,x_1),(x_1,w),(x_1,x_2),(u_2,z),(x_3,v_0),(v_1,u_2) \in D$. In addition, note that we have $(x_1,w) \to (x_1,x_2)$; $(u_2,z) \to (u_2,v)$; and $(x_3,v_0) \to (x_3,v_1)$ in H^+. Therefore, $(u_2,v),(x_3,v_1) \in N^+[D]$. Since the pairs $(v_1,u_2),(u_2,v)$ are in $N^+[D]$, we have $(v_1,v) \in N^*[D]$. Then, since $(x_3,v_1),(v_1,v) \in N^*[D]$, we also have $(x_3,v) \in N^*[D]$. Moreover, $(x_3,v) \to (x_3,x_0) \in N^*[D]$ and, hence, we have the circuit $C = (x_0,x_1),(x_1,x_2),(x_2,x_3),(x_3,x_0)$ in $N^*[D]$.

Note that since y_3,v,v_0 are all adjacent to v_1,v_2,z, selecting $S_{v_2u_1}$ instead of $S_{u_1v_2}$ or selecting S_{zu_2} instead of S_{u_2z} would yield a circuit in $N^*[D]$ as long as we select $S_{x_2x_3}$ to be placed in D. Moreover, selecting one of $S_{x_0x_1},S_{x_1x_0}$ alongside one of S_{x_1w},S_{wx_1} at Stage 1 also yields a circuit in $N^*[D]$ as long as we select $S_{x_2x_3}$ at Stage 1. Note that by adding (x_3,v) into $N^*[D]$ we close circuit C. Now, in order to obtain $Dic(x_3,x_0)$ we need to find $Dic(x_3,v)$. According to the rules of the algorithm, since (x_3,v) is by transitivity on $(x_3,v_1),(v_1,v)$ where x_3,v_1 are white and v is black, we have $Dic(x_3,v) = Dic(x_3,v_1) = S_{x_3v_0} = S_{x_2x_3}$ (dictator component). Therefore, in order to avoid a circuit at Stage 2 of the algorithm we must select $S_{x_3x_2}$ and place it into D_1 at line 20 of the algorithm.

Suppose the algorithm selects $S_{v_1u_2},S_{u_2z},S_{x_0x_1},S_{x_1w}$ at line 20. This will place the pairs $(u_2,v_0),(x_3,x_0)$, (x_0,v), and (v_0,x_3) in $N^*[D_1]$, because $(u_2,z) \to (u_2,v_0)$; $(x_3,x_2) \to (x_3,x_0)$; $(x_0,x_1) \to (x_0,v)$, and $(v_0,x_3) \in S_{x_3x_2}$. Therefore, by applying transitivity, the algorithm places (x_0,v) into $N^*[D_1]$ (line 22). But then from $(x_3,x_0),(x_0,v) \in N^*[D_1]$ it follows that $(x_3,v) \to (x_3,v_1)$. This leads to the circuit $(v_1,u_2),(u_2,v_0),(v_0,x_3),(x_3,v_1)$ in D (line 22). Notice that selecting any two components from $S_{v_1u_2},S_{u_2v_1},S_{u_2z},S_{zu_2}$ instead of $S_{u_1v_2},S_{u_2z}$ also yields a circuit. Therefore, in any case, the algorithm reports that H is not an interval bigraph.

5 Correctness of Stage 1: Adding the (strong) components

We start this section by defining the first set of obstructions so-called exobiclique. We say bigraph $H=(B,W)$ is an *exobiclique* if the following hold.

- B contains a nonempty part B_1 and W contains a nonempty part W_1 such that $B_1 \cup W_1$ induces a biclique in H;
- $B \setminus B_1$ contains three vertices with incomparable neighborhood in W_1 and $W \setminus W_1$ contains three vertices with incomparable neighborhoods in B_1 (an examples given in Figure 3).

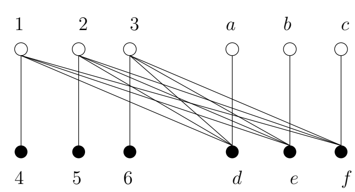

Figure 3: Exobicliques: Here, $B = \{4,5,6,d,e,f\}$, $W = \{1,2,3,a,b,c\}$ and $B_1 = \{d,e,f\}$, $W_1 = \{1,2,3\}$ and $B \setminus B_1 = \{4,5,6\}$, $W \setminus W_1 = \{a,b,c\}$.

Theorem 5.1. If H has an induced exobiclique then H is not an interval bigraph [11].

Theorem 5.2. Suppose at Stage 1 we have so far constructed a D without circuits, and then for the next component S we find that $D \cup N^+[S]$ has circuits. Let $C = (x_0,x_1),(x_1,x_2),\ldots,(x_n,x_0)$ be a shortest circuit in $D \cup N^+[S]$. Then one of the following must occur.

(i) each pair (x_i,x_{i+1}) is in a component.

(ii) H contains an exobiclique as an induced subgraph.

Proof. Suppose (i) does not occur. Thus, at least one pair (x_i, x_{i+1}) is implied by a component S_i. By Lemma 2.8 there exists vertices a_i, b_i, and c_i of H such that $x_i a_i$ and $b_i c_i$ are independent edges and $a_i x_{i+1}, c_i x_{i+1} \in E(H)$. Note that x_i and x_{i+1} have the same color and $N(x_i) \subseteq N(x_{i+1})$ (see Figure 4).

Claim 5.3. *x_{i+1} and x_{i+2} have different colors, and (x_{i+1}, x_{i+2}) is in a component, say S_{i+1}.*

Proof. If x_{i+1} and x_{i+2} have different colors then (x_{i+1}, x_{i+2}) is in a component and we are done. Thus, we assume x_{i+2} have the same color as x_i and x_{i+1}. Now $c_i x_{i+2} \notin E(H)$, otherwise, $(x_i, c_i) \rightarrow (x_i, x_{i+2})$ and, hence, (x_i, x_{i+2}) is an implied pair by component S_i, leading to a shorter circuit. Moreover, $a_i x_{i+2} \notin E(H)$, otherwise, $(a_i, c_i) \rightarrow (x_{i+2}, c_i) \rightarrow (x_{i+2}, x_{i+1})$; a contradiction to C having minimum length. Since $(x_{i+1}, x_{i+2}) \in N^+[S]$ for some component $S \in D$, there exists some c_{i+1} such that $x_{i+2} c_{i+1} \in E(H)$ and $x_{i+1} c_{i+1} \notin E(H)$ ($(x_{i+1}, c_{i+1}) \in S$). Notice that $c_{i+1} x_i \notin E(H)$, otherwise, $(x_{i+1}, c_{i+1}) \rightarrow (x_{i+1}, x_i)$; a contradiction to C having minimum length. Now $(x_{i+1}, x_{i+2}) \rightarrow (a_i, x_{i+2}) \rightarrow (a_i, c_{i+1}) \rightarrow (x_i, c_{i+1}) \rightarrow (x_i, x_{i+2})$, leading to a shorter circuit. □

1. By Claim 5.3, there exists a_{i+1} and b_{i+1} such that $x_{i+1} a_{i+1}$ and $x_{i+2} b_{i+1}$ are independent edges of H.

2. Claim 5.3 also implies that (x_{i-1}, x_i) is in a component S_{i-1}, and vertices x_{i-1} and x_i have different colors.

3. $c_i a_{i-1} \notin E(H)$, otherwise, $(x_i, c_i) \in S_i$ dominates (x_i, a_{i_1}) and, hence, $S_i = S'_{i-1}$; a contradiction. Similarly, $x_{i-1} b_i \notin E(H)$.

4. There are independent edges $x_{i-1} a_{i-1}$ and $x_i c_{i-1}$ of H, with $(x_{i-1}, c_{i-1}) \in S_{i-1}$.

5. By Lemma 2.8, $N(x_i) \subseteq N(x_{i+1})$. Thus, $x_{i+1} c_{i-1}, x_{i+1} a_i \in E(H)$.

6. $x_{i-1} x_{i+1} \in E(H)$, otherwise, $x_{i-1} a_{i-1}$ and $c_{i-1} x_{i+1}$ would be independent edges and, hence, $(x_{i-1}, x_{i+1}) \in S_{i-1}$, implying a shorter circuit.

7. $x_{i-1} b_{i+1} \in E(H)$, otherwise, $x_{i+1} x_{i-1}$ and $b_{i+1} x_{i+2}$ would be independent edges and, hence, $(x_{i-1}, x_{i+2}) \in S_{i+1}$, implying a shorter circuit. A similar argument implies $N(x_{i+2}) \subseteq N(x_{i-1})$.

8. $d_i a_{i+2} \in E(H)$ for every $a_{i+2} \in N(x_{i+2})$ and every $d_i \in N(x_i)$, otherwise, $(x_{i+1}, x_{i+2}) \rightarrow (x_{i+1}, a_{i+2}) \rightarrow (d_i, a_{i+2}) \rightarrow (d_i, x_{i+2}) \rightarrow (x_i, x_{i+2})$, implying a shorter circuit in D.

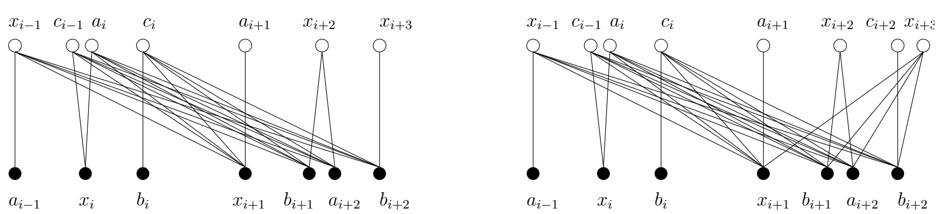

Figure 4: edges $a_{i-1} x_{i-1}, x_i c_{i-1}$, edges $x_i a_i, b_i c_i$, edges $x_{i+1} a_{i+1}, b_{i+1} x_{i+2}$, edges $x_{i+2} a_{i+2}, x_{i+3} b_{i+2}$ (left figure) are independent.

In what follows we show that H contains an exobiclique. First suppose (x_{i+2}, x_{i+3}) is in component S_{i+2} (Figure 4 left). Thus, $x_{i+2} a_{i+2}$ and $b_{i+2} x_{i+3}$ are independent edges of H. By (6), $x_{i-1} a_{i+2} \in E(H)$. By (7), $a_{i+2} c_{i-1}, a_{i+2} a_i \in E(H)$. Suppose x_{i+2} and x_{i+3} have different colors. Then, $x_{i+3} x_{i-1} \notin E(H)$, otherwise, $(x_{i+2}, x_{i+3}) \rightarrow (x_{i+2}, x_{i-1})$, a shorter circuit in D. But then, $(x_{i+2}, x_{i+3}) \rightarrow (x_{i+2}, x_{i-1})$; a shorter circuit in D. Therefore, x_{i-1} and x_{i+3} have to have the same color. Now, $b_{i+2} x_{i-1} \in E(H)$, otherwise, $(a_{i+2}, b_{i+2}) \rightarrow (x_{i-1}, b_{i+2}) \rightarrow (x_{i-1}, x_{i+3})$; a shorter circuit. Moreover, $b_{i+2} c_{i-1} \in E(H)$, otherwise, $(x_{i-1}, c_{i-1}) \rightarrow (b_{i+2}, c_{i-1}) \rightarrow (b_{i+2}, a_{i+2})$ and, hence, $S'_{i+2} \in D$; a contradiction. By a similar argument, we conclude that c_i is adjacent to b_{i+2}, a_{i+2} and b_{i+1}. Similar to (3), $x_{i+1} x_{i+3}$ and $a_{i+1} b_{i+2}$ are non-edges of H.

Now we get an exobiclique, i.e., $\{a_{i-1}, x_{i-1}, x_i, c_{i-1}, a_i, b_i, c_i, x_{i+1}, a_{i+1}, b_{i+1}, a_{i+2}, x_{i+2}, b_{i+2}, x_{i+3}\}$. Note that vertices a_{i-1}, x_i and b_i have incomparable neighborhoods in $N = \{x_{i-1}, c_{i-1}, a_i, c_i\}$, vertices a_{i+1}, x_{i+2}, and x_{i+3} have incomparable neighborhoods in $M = \{x_{i+1}, b_{i+1}, a_{i+2}, b_{i+2}\}$; and $M \cup N$ induces a biclique.

When (x_{i+2}, x_{i+3}) is implied, by a similar argument again we get an exobiclique (see Figure 4 right). □

Theorem 5.4. If at Stage 1 of the algorithm we encounter a component S such that we cannot add either of $N^+[S]$ and $N^+[S']$ to the current D, then H has an exobiclique.

Proof. We cannot add $N^+[S]$ and $N^+[S']$ because the additions create circuits in $D \cup N^+[S]$ respectively $D \cup N^+[S']$.

If either circuit leads to (ii) (in Theorem 5.2) we are done by Theorem 5.1. If both lead to (i) (in Theorem 5.2), we proceed as follows. Assume $C_1 = (x_0, x_1), \ldots, (x_n, x_0)$ is a shortest circuit created by adding $N^+[S]$ to the current D, and $C_2 = (y_0, y_1), \ldots, (y_m, y_0)$ is a shortest circuit created by adding $N^+[S']$ to the current D. We may assume that $N^+[S]$ contributes (x_n, x_0) to C_1 and $N^+[S']$ contributes (y_m, y_0) to C_2. By Theorem 5.2 each (x_i, x_{i+1}) is in a component S_i and each (y_j, y_{j+1}) is in a component. Since C_1 is a shortest circuit, $S_i \neq S'_{i+1}$, and hence, $S_{x_i x_{i+2}}$ is also a component. Thus, by Theorem 5.2 there exist maximal subgraphs H_i, H_{i+1}, and H_{i+2} containing x_i, x_{i+1}, and x_{i+2} respectively that are pairwise independent. By extending this idea we conclude, there exist pairwise independent maximal subgraphs $H_0, H_1, \ldots, H_n$, of H such that each H_i $(0 \leq i \leq n)$ contains x_i. By Theorem 5.2 (ii) it follows that for every $x \in X = H \setminus H'$, where $H' = H_0 \cup H_1 \cup \cdots \cup H_n$, and every $a \in H'$ with the same color as x, $N(a) \subseteq N(x)$. Now it is easy to see that there is no directed path from $(x_i, x_{i+1}) \in S_i$ to $(x_j, x_{j+1}) \in S_j$, $i \neq j$ because such a path must have a pair (x_j, x) for $x \in X$, but now (x_j, x) is an implied pair and by Corollary 2.9, $S_{x_j x}$ is a sink component since $N(x_j) \subseteq N(x)$. Similarly, there is no path from (y_m, y_0) to any of (y_j, y_{j+1}). We also observe that $S_{x_0 x_n} = S_{y_m y_0}$. Thus, we may assume that $(y_0, y_m) = (x_n, x_0)$. Therefore,

$$(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, y_0), (y_0, y_1), \ldots, (y_{m-2}, y_{m-1}), (y_{m-1}, x_0)$$

is a circuit in D, contrary to our assumption. □

6 Correctness of Stage 2 (finding dictator components)

We consider what happens when a circuit is formed during the execution of Stage 2 (lines 15–18) of the algorithm. In what follows, we specify the length and some other properties of a circuit in D, considering level by level construction of $N^*[D]$. This section is divided into three subsections. In Subsection 6.1 we define a minimal circuit and prove that such a circuit should have length four. In Subsection 6.2, we further analyze the pairs in D and identify its associated dictator component. We will show that for a pair (x, y) in D, $S = Dic(x, y)$ is the sole component responsible for placing pair (x, y) into D, regardless of the choice made at Stage 1 between any component not in $\{S, S'\}$ and its dual. Finally, in Subsection 6.3 we prove the following three lemmas which collectively show the correctness of Stage 2 of the algorithm.

Lemma 6.1. *Let $C = (x_0, x_1), (x_1, x_2), (x_2, x_3), (x_3, x_0)$ be a minimal circuit, form at Stage 2 of the algorithm. Let $S_0 = Dic(x_0, x_1)$, $S_1 = Dic(x_1, x_2)$, $S_2 = Dic(x_2, x_3)$, and $S_3 = Dic(x_3, x_0)$. Then the following hold.*

1. *If (x_1, x_2) is a complex pair and (x_2, x_3) is also a complex pair then $S_1 = S_2$.*
2. *If (x_1, x_2) is a complex pair and (x_0, x_1) is in a component S_0 then $(x_0, x_1) \in S_1$, and hence, $S_0 = S_1$.*
3. *If (x_2, x_3) is a complex pair and (x_3, x_0) is a simple pair implied by component S_3 then $S_3 = S_2$.*
4. *If (x_2, x_3) and (x_3, x_0) are complex pairs then $S_2 = S_3$.*
5. *If (x_1, x_2) and (x_3, x_0) are complex pairs and (x_0, x_1) and (x_2, x_3) are simple pairs then $S_1 = S_3$ and $(x_2, x_3), (x_0, x_1) \in S_1$.*

Lemma 6.2. *If we encounter a minimal circuit $C = (x_0, x_1), (x_1, x_2), (x_2, x_3), (x_3, x_0)$ at line 18 then there is a component S such that the envelope of every complete set D_1 where $S \subseteq D_1$ contains a circuit.*

Lemma 6.3. *The algorithm correctly computes $Dic(x, y)$.*

6.1 The length of a minimal circuit

We start this subsection by defining minimal chain and minimal circuit.

Definition 6.4. Let $(x, y) \in D$ by transitivity at (the earliest) level l. Then, by a *minimal chain* between x, y we mean a sequence $(x_0, x_1), (x_1, x_2), ..., (x_{n-1}, x_n)$ of minimum length (n) of pairs in D with $x_0 = x$ and $x_n = y$, such that each $(x_i, x_{i+1}) \in D$, $0 \leq i \leq n-1$, and at some level before l, and by reachability (and not by transitivity). We also say (x_0, x_n) is by transitivity on the minimal chain $(x_0, x_1), (x_1, x_2), ..., (x_{n-1}, x_n)$.

Definition 6.5. Let C be a circuit in $N^*[D]$. We say C is a *minimal circuit* if first, the latest pair in C is created as early as possible (the smallest possible level) during the execution of $N^*[D]$; second, C has the minimum length; third, no pair in C is by transitivity.

Lemma 6.6. *Let (x, y) be a pair in D after Stage 1 of the algorithm, and current D has no circuit. Suppose (x, y) is obtained by a minimal chain $CH = (x_0, x_1), (x_1, x_2), ..., (x_{n-1}, x_n), (x_n, x_{n+1})$ ($x_0 = x$ and $x_{n+1} = y$). Then the following hold.*

1. *x_i and x_{i+2} have always different colors.*

2. *If x and y have the same color then $n \leq 3$ and x_n, y have different colors.*

3. *If x and y have different colors then $n \leq 2$.*

   - *If $n = 2$ then x_n, y have the same color.*
   - *If $n = 1$ and xy is not an edge of H then x and x_1 have the same color.*
   - *If $n = 1$ and xy is an edge of H then x_1 and y have the same color.*

Proof of 1. First suppose all three vertices x_i, x_{i+1}, and x_{i+2} have the same color, say black. Since (x_i, x_{i+1}) is not obtained by transitivity, there exists a white vertex a of H such that the pair $(x_i, a) \in D$ dominates (x_i, x_{i+1}) in H^+, i.e. a is adjacent in H to x_{i+1} but not to x_i. For a similar reason, there exists a white vertex b of H adjacent to x_{i+1} but not to x_i, i.e. the pair $(x_{i+1}, b) \in D$ dominates (x_{i+1}, x_{i+2}) in H^+.

We now argue that a is not adjacent to x_{i+2}. Otherwise, $(x_i, a) \in D$ also dominates the pair (x_i, x_{i+2}), and hence, (x_i, x_{i+2}) is also in D (at the same level as (x_i, x_{i+1})), contradicting the minimality of CH.

Next we observe that (x_i, a) is not by transitivity. Otherwise, (x_i, x_{i+1}) and (x_{i+1}, x_{i+2}) can be replaced by a chain obtained from the pairs that implies (x_i, a) together with the pair (a, x_{i+2}). The pair (a, x_{i+2}) lies in the same component of H^+ as $(x_i, x_{i+2}) \in D$ since the edges $x_{i+1}a$ and $x_{i+2}b$ are independent. Since all pairs of a component are chosen or not chosen for D at the same time, this contradicts the minimality of CH. Thus, (x_i, a) is dominated in H^+ by some pair $(c, a) \in D$. Since a and x_i have different colors, this means c is a white vertex adjacent to x_i. Note that c is not adjacent to x_{i+2}, otherwise, $(c, a) \in D$ would dominate (x_{i+2}, a), placing (x_{i+2}, a) in D; and we get the circuit $(a, x_{i+2}), (x_{i+2}, a) \in D$ which is a contradiction.

Now, we claim that $bx_i \notin E(H)$. This is the case, otherwise, the pair $(x_{i+1}, b) \in D$ would dominates the pair (x_{i+1}, x_i), while $(x_i, x_{i+1}) \in D$, a circuit in D. Finally, $cx_{i+1} \notin E(H)$, otherwise, cx_{i+1} and bx_{i+2} would be independent edges in H, and cx_i and bx_{i+2} would also be independent edges in H; thus, the pairs (x_i, x_{i+2}) and (x_{i+1}, x_{i+2}) are in the same component, contradicting again the minimality of CH. Now (x_i, x_{i+1}), (x_{i+1}, x_{i+2}), and (x_i, x_{i+2}) are in components. Since there is no circuit in D, according to the rules of the algorithm we have $(x_i, x_{i+2}) \in D$, contradicting the minimality of CH.

We now consider the case where x_i and x_{i+2} are black and x_{i+1} is white. As before, there must exist a white vertex a and a black vertex b such that the pair (a, x_{i+1}) dominates (x_i, x_{i+1}) and the pair (b, x_{i+2}) dominates (x_{i+1}, x_{i+2}); thus, ax_i is an edge of H and so is bx_{i+1}. Note that the pair (a, x_{i+1}) dominates the pair (x_i, x_{i+1}), which dominates the pair (x_i, b). Therefore, we can replace x_{i+1} by b and obtain a chain which is also minimal. Now, (b, x_{i+2}) is by transitivity which contradict the minimality of CH.

Claim 6.7. $n \leq 4$.

Proof of the claim. Set $x_0 = x$ and $x_{n+1} = y$. Let i be the minimum number such that x_i and x_{i+1} have color, say, black; and x_{i+2} and x_{i+3} are white. Let x' be a vertex such that $(x_i, x') \in D$ dominates (x_i, x_{i+1}). Note that if x_{i+4} exists then it is black. If x_{i+4} exists and $n \geq 5$ then x_{i+4} is white, and $x'x_{i+4}$ is not an edge, otherwise, $(x_i, x') \to (x_i, x_{i+4})$ and we get a shorter chain. Now let y' be a vertex such that $(x_{i+4}, y') \in D$ dominates (x_{i+4}, x_{i+5}). Now $y'x_{i+1} \notin E(H)$, otherwise, $(x_{i+4}, y') \to (x_{i+4}, x_{i+1})$ and we get a circuit $(x_{i+1}, x_{i+2}), (x_{i+2}, x_{i+3}), (x_{i+3}, x_{i+4}), (x_{i+4}, x_{i+1})$ in D. Now $x'x_{i+1}$ and $y'x_{i+4}$ are independent edges, and hence, (x_{i+1}, x_{i+4}) is in a component. Note that each component or its coupled is in D. (x_{i+4}, x_{i+1}) is not in D, otherwise, we get a circuit in D, and hence, $(x_{i+1}, x_{i+4}) \in D$, and we get a shorter chain. Thus, we may assume that x_{i+4} does not exist, and hence, $x_{i+4} = y$. Now by minimality assumption for i, $x_{i-1} = x_0$, and hence, $n \leq 4$. □

Proof of 2. Suppose x and y have the same color. We show that $n \leq 3$. Toward a contradiction, suppose $n = 4$. Now according to (1) x, x_1, x_4, and y have the same color which is opposite to the color of x_2 and x_3. Let y' be a vertex such that (x_4, y') dominates (x_4, y), and let x' be a vertex such that $(x_0, x') \in D$ dominates (x_0, x_1). Note that $y'x \notin E(H)$, otherwise, $(x_4, y') \to (x_4, x_0)$, implying a circuit in D. Similarly, x_1y is not an edge of H. Finally, $x'y$ is not an edge of H, otherwise, $(x, x') \to (x, y)$, contradiction to the minimality of CH. Now, x_1x' and $y'y$ are independent edges and, hence, (x_1, y) is in a component; thereby, $(x_1, y) \in D$, contradicting the minimality of CH. Therefore, $n \leq 3$.

We continue by assuming $n = 3$. We first show that x_3 and y have different colors. On the contrary, suppose x_3 and y have the same color. According to (1), x_1 and x_2 have the same color opposite to the color of x, y, and x_3. Let $(x_1, x') \in D$ be a pair that dominates (x_1, x_2), and y'' be a vertex such that (x_3, y'') dominates (x_3, y). $y''x \notin E(H)$, otherwise, $(x_3, y'') \to (x_3, x)$ and we would get a circuit. Let x'' be a vertex such that $(x'', x_1) \in D$ dominates (x, x_1). Now, $x'x'' \notin E(H)$, otherwise, (x_1, x') would dominate (x'', x_1) and we would get a circuit in D. We continue by having $x_2x \in E(H)$, otherwise, x_2x' and xx'' would be independent edges and, hence, (x, x_2) would be in a component that has already been placed in D, contradicting the minimality of CH. Then, the chain $(x_2, x_3), (x_3, y'')$ would imply the pair (x_2, y''), and that $(x_2, y'') \to (x, y'') \to (x, y)$. The latter is a contradiction to the minimality of CH.
Proof of 3. Suppose x and y have different colors. We show that $n \leq 3$. For contradiction suppose $n = 4$. Now, according to (1), x, x_3, and x_4 have the same color and opposite to the color of x_1, x_2, and y. We observe that $xy \notin E(H)$, otherwise, (x_4, y) would dominate (x_4, x) and, hence, we would get a circuit in D. Let x' be a vertex such that $(x_1, x') \in D$ dominates (x_1, x_2) and x'' be a vertex such that $(x'', x_1) \in D$ dominates (x, x_1). Now, $x'x''$ is not an edge, otherwise, (x_1, x') would dominate (x'', x_1) and we would get a circuit in D. Moreover, $x_2x \in E(H)$, otherwise, x_2x' and xx'' would be independent edges and, hence, (x, x_2) would be in a component that has already been placed in D; contradicting the minimality of CH. Now, the chain $(x_2, x_3), (x_3, x_4), (x_4, y)$ implies (x_2, y) and that (x_2, y) dominates (x, y). This is a contradiction to the minimality of CH. In fact, we would obtain (x, y) in fewer steps of transitivity. Therefore, $n \leq 3$. Now it is not difficult to see that either $n = 1$ or, otherwise, $n = 2$ and vertices x and x_1 have the same color opposite to the color of x_2 and y.

Suppose $n = 1$. First assume xy is an edge. Now, x_1 and y have the same color, otherwise, $(x_1, y) \to (x_1, x)$; a contradiction. Thus, we continue by assuming xy is not an edge. We show that x_1 and x have the same color. Toward a contradiction, suppose x_1 and y have the same color. Let $(x', x) \in D$ be a pair that dominates (x, x_1) and let $(x_1, y') \in D$ be a pair that dominates (x_1, y). Now, $x'y'$ is not an edge and, hence, yy' and xx' are independent edges. This shows that (x, y) is in a component, contradicting the minimality of CH. □

Corollary 6.8. *Let (x, y) be a pair in D after Stage 1 of the algorithm, and assume the current D has no circuit.*

- *Suppose x and y have the same color and $(x, w) \to (x, y)$ such that (x, w) is by transitivity with a minimal chain $(x, w_1), (w_1, w_2), \ldots, (w_m, w)$. Then $m = 2$ and vertices x and w_1 have the same color and opposite to the color of w_2 and w.*

- *Suppose x and y have different colors and $(w, y) \to (x, y)$ such that (w, y) is in a trivial component. Then (w, y) is by transitivity with a minimal chain $(w, w_1), (w_1, w_2), (w_2, y)$ where w_1 and w_2 have the same color opposite to the color of w and y.*

Proof. If x and y have the same color then by Lemma 6.6 we have $m = 2$ or $m = 1$. If $m = 2$ then x and x_1 have the same color and opposite to the color of x_2 and w. If $m = 1$ then, by Lemma 6.6 (3), w_1 and y have the same color. Note that (w_1, w) dominates (w_1, y) and (w_1, y) is in $N^*[D]$ at the same time (w_1, w) is placed in D. Therefore, we can use the chain $(x, w_1), (w_1, y)$ in order to obtain (x, y); a contradiction. If x and y have different colors then by Lemma 6.6 either $m = 2$ or $m = 3$. If $m = 3$ then w, w_1, and y have the same color and opposite to the color of w_2 ans w_3. Let w' be a vertex such that $(w, w') \in D$ dominates (w, w_1). We observe that $w_1, x \notin E(H)$, otherwise, $(w_1, y) \to (x, y)$ and, hence, we obtain (x, y) in an earlier level or in fewer steps of transitivity application because $(w_1, w_2), (w_2, w_3)$, and (w_3, y) are in $N^*[D]$. Now, wx, and w_1w' are independent edges and, hence, (x, w_1) is already in D. In this situation, we can use the chain $CH = (x, w_1), (w_1, w_2), (w_2, w_3), (w_3, y)$ to obtain (x, y) in some earlier step since w_1 and w_2 have different colors; a contradiction by Lemma 6.6 (1). Therefore, $n = 2$ and Lemma 6.6 is applied. $\square$

Now by Lemma 6.6 and Corollary 6.8 we have the following.

Corollary 6.9. *Let $C = (x_0, x_1), (x_1, x_2), ..., (x_{n-1}, x_n), (x_n, x_0)$ be a minimal circuit, formed at Stage 2 of the algorithm. Then $n = 3$. Moreover, x_0 and x_3 have the same color and opposite to the color of x_1 and x_2.*

Lemma 6.10. *Suppose the current D is circuit-free. Let $(x_1, x_3) \in D$ be by transitivity on a minimal chain $(x_1, x_2), (x_2, x_3)$ in D where x_1 and x_2 have the same color and different from the color of x_3, and (x_1, x_3) is not dominated by any other pair (y, x_3). Then there are u_1 and w_1 with the same color as x_3 such that :*

(1) If (x_1, x_2) is complex then there exists $(x_1, w_1) \in D$ such that $(x_1, w_1) \to (x_1, x_2)$, and (x_1, w_1) is place in D by transitivity.

(2) If (x_2, x_3) is complex then there exists $(u_1, x_3) \in D$ such that $(u_1, x_3) \to (x_2, x_3)$, and (u_1, x_3) is placed in D by transitivity.

Proof. Suppose (x_1, w_1) is not by transitivity and there is $(w', w_1) \in D$ such that $(w', w_1) \to (x_1, w_1)$. Notice that $x_1x_3 \notin E(H)$, otherwise, $(x_2, x_3) \to (x_2, x_1) \in D$, and, hence, we get a circuit in D.

Now, by Corollary 6.8, there are vertices w'_1 and w'_2 so that (w', w_1) is by transitivity on the minimal chain $\mathcal{M} = (w', w'_1), (w'_1, w'_2), (w'_2, w_1)$. Let $(w'_1, v) \in D$ where $(w'_1, v) \to (w'_1, w'_2)$. Note that $vx_2 \notin E(H)$, otherwise, $(w'_1, v) \to (w'_1, x_2) \in D$ and, hence, we would get the chain $(w', w'_1), (w'_1, x_2), (x_2, x_3)$ in D. In this situation, $(w', x_3) \to (x_1, x_3)$; contradicting that (x_1, x_3) is by transitivity. Hence, $vx_2 \notin E(H)$. Next, note that w'_2v and x_2w_1 are independent edges, and (w'_2, w_1) and (w'_2, x_2) are in the same component. Therefore, we have the chain $(w', w'_1), (w'_1, v), (v, x_2), (x_2, x_3)$ in D and, hence, $(w', x_3) \in D$. Now $(w', x_3) \to (x_1, x_3)$, contradicting that (x_1, x_3) is by transitivity. Number (2) follows from Corollary 6.8. $\square$

Lemma 6.11. *Let $(x_0, x_3) \in D$ where D is circuit-free. Suppose $(x_0, x_1), (x_1, x_2), (x_2, x_3)$ is a minimal chain in D between x_0, x_3 where x_0 and x_3 have the same color and opposite to the color of x_1 and x_2. Then $x_0x_2 \in E(H)$.*

Proof. For contrary, suppose $x_0x_2 \notin E(H)$. Let (p, x_1) be a pair in D that dominates (x_0, x_1) ((x_0, x_1) is not by transitivity). Let w be a vertex of H such that $(x_1, w) \to (x_1, x_2)$. Now $wp \notin E(H)$, otherwise, (x_1, w) would dominate (x_1, p), implying an earlier circuit in D. Now, px_0 and wx_2 are independent edges and, hence, (x_0, x_2) would be in a component; consequently, (x_0, x_2) would have been already placed in D (if (x_2, x_0) was in D then we would have an earlier circuit), implying a shorter chain. Therefore, $x_0x_2 \in E(H)$. $\square$

In what follows, we often use a similar argument to the one for Lemma 6.11 and, hence, we do not repeat the details of it again.

6.2 Relationship between dictator components of the pairs in D

In this subsection, we trace back the creation of a complex pair, say, (x_1, x_2). For pairs (x, y) and (x', y') in H^+, we say (x', y') is reachable from (x, y) and write $(x, y) \rightsquigarrow (x', y')$ when there is a directed path in H^+ from (x, y) to (x', y'). For a component S and pair (x, y) we write $S \rightsquigarrow (x, y)$ if (x, y) is reachable from a pair in S. Notice that if $(x, y) \rightsquigarrow (x', y')$, then $(y', x') \rightsquigarrow (y, x)$, due to the skew-symmetry property.

Remark : In all of the following lemmas in this subsection, we assume that the current D is circuit-free.

In the next two lemmas we consider the process of obtaining a complex pair. In other words, we unravel the consecutive the rechability and transitivity operations in placing a pair in D.

Lemma 6.12 (decomposition of same-color pairs)**.** *Let $(x_1, x_3) \in D$ be by transitivity on a minimal chain $(x_1, x_2), (x_2, x_3)$ in D where x_1 and x_2 have the same color and opposite to color of x_3. Suppose (x_1, x_2) is a complex pair. Then, there exists the smallest m, and vertices $y_1, z_1, w_1, v_1, \ldots, y_{m-1}, z_{m-1}, w_{m-1}, v_{m-1}, a, b, w_m \in V(H)$ such that for $1 \leq i \leq m-1$ the following hold :*

(1) $(x_1, w_1), (x_1, w_{i+1}) \in D$ where $(x_1, w_1) \rightarrow (x_1, x_2)$ and $(x_1, w_{i+1}) \rightarrow (x_1, y_i)$

(2) (x_1, w_i) is obtained by transitivity on $(x_1, y_i), (y_i, z_i), (z_i, w_i) \in D$ where w_i, z_i have the same color as x_1;

(3) $(z_i, v_i) \in D$, and $(z_i, v_i) \rightarrow (z_i, w_i)$ where x_1, v_i, z_i have the same color.

(4) $w_{i+1}y_{i-1} \notin E(H)$, $i \geq 2$;

(5) $y_i w_i \in E(H)$;

(6) $v_{i+1}w_i \notin E(H)$;

(7) $w_{i+1}v_i \in E(H)$;

(8) $ay_{m-1}, ax_2 \in E(H)$; and

(9) $x_1 a$ and $w_m b$ are independent edges of H.

Moreover, $(x_1, w_m) \rightsquigarrow (x_2, v_1)$, and $(x_1, w_m), (x_2, v_1) \in Dic(x_1, x_2)$.

Proof. Since $(x_1, x_2), (x_2, x_3)$ is a minimal chain, by Lemma 6.10 there exists $(x_1, w_1) \in D$ so that $(x_1, w_1) \rightarrow (x_1, x_2)$ and (x_1, w_1) is by transitivity. Now, by Corollary 6.8, there are y_1 and z_1 such that $(x_1, y_1), (y_1, z_1), (z_1, w_1) \in D$, and x_1 and y_1 have the same color and opposite to the color of z_1 and w_1. Notice that $x_1 w_1 \notin E(H)$. Let v_1 be a vertex such that $(z_1, v_1) \in D$ and $(z_1, v_1) \rightarrow (z_1, w_1)$. Observe that x_1, v_1, and v_2 have the same color. By applying the above argument for pair (x_1, y_1) (when (x_1, y_1) is a complex pair) we conclude that there exists a smallest m and vertices $w_1, y_1, z_1, v_1, w_1, \ldots, y_{m-1}, z_{m-1}, v_{m-1}, w_{m-1}, a, b, w_m \in V(H)$, satisfying (1,2,3).

Proof of (4) Otherwise, (x_1, w_{i+1})— which is in D— dominates (x_1, y_{i-1}) and, hence, we obtain the chain $(x_1, y_{i-1}), (y_{i-1}, z_{i-1}), (z_{i-1}, w_{i-1})$ in D. Consequently, $(x_1, w_{i-1}) \rightarrow (x_1, y_{i-2})$. The latter implies (x_1, w_1) was obtained at some earlier step; a contradiction.

Proof of (5) Otherwise, by (3,4), $y_i w_{i+1}$ and $y_{i-1} w_i$ are independent edges and, hence, (y_i, w_i) is in a component. Since $(y_i, z_i), (z_i, w_i) \in D$, we conclude that (y_i, w_i) is in D and, hence, so are (x_1, y_i) and (y_i, w_i). Therefore, by transitivity, $(x_1, w_i) \in D$; a contradiction to Corollary 6.8.

Proof of (6) Otherwise, $(z_{i+1}, v_{i+1}) \in D$ dominates (z_{i+1}, w_i) and, hence, we get the chain $(x_1, y_{i+1}), (z_{i+1}, w_i)$ in D, which implies (x_1, x_2) has been placed in D in fewer than m steps; a contradiction.

Proof of (7) Otherwise, by (6) $w_{i+1}v_{i+1}$ and $w_i v_i$ are independent edges and, hence, $(w_{i+1}, w_i), (y_i, v_i)$, and (w_{i+1}, v_i) are in the same component. Since $(y_i, z_i), (z_i, v_i) \in D$, we conclude that $(y_i, v_i) \in D$, and consequently, since $(y_i, v_i) \rightarrow (w_{i+1}, v_i)$, we have $(v_{i+1}, w_i) \in D$. Now the chain $(x_i, w_{i+1}), (w_{i+1}, w_i)$ in D places (x_1, x_2) in D in fewer than m steps; a contradiction.

Proof of (8) Suppose $ay_{m-1} \notin E(H)$. Then ax_1 and $w_{m-1}y_{m-2}$ are independent, thereby, (x_1, w_{m-1}) is in a component and (x_1, x_2) is placed in D in fewer steps than m; a contradiction. Notice that by the same logic we have $ax_2 \in E(H)$.

Proof of (9) Finally, since (x, w_m) is in a component, we have independent edges $x_1 a$ and $w_m b$.

Notice that (x_1, w_1) is by transitivity on $(x_1, y_1), (y_1, w_1)$ and, hence, by definition of a dictator, $Dic(x_1, x_2) = Dic(x_1, w_1) = Dic(x_1, y_1)$ (see Line 6 of Dictator function). Observe that (x_1, w_m) and (a, w_m) are in component S_1 and, by definition, $S_1 = Dic(x_1, x_2)$. First suppose $m > 2$. By (8,9) we have $(a, w_m) \rightarrow (y_{m-2}, w_m) \rightarrow (y_{m-2}, v_{m-1})$. Moreover, $(x_1, w_m) \rightarrow (a, w_m)$. Thus, $(x_1, w_m) \rightsquigarrow (y_{m-2}, v_{m-1})$. By (6), $(y_i, v_{i+1}) \rightarrow (w_i, v_{i+1})$ and, by (2), $(w_i, v_{i+1}) \rightarrow (w_i, w_{i+1})$. Therefore, $(y_i, v_{i+1}) \rightsquigarrow (w_i, w_{i+1})$. Moreover, by (6,5) $(w_i, w_{i+1}) \rightarrow (y_{i-1}, w_{i+1}) \rightarrow (y_{i-1}, v_i)$. Thus, $(w_i, w_{i+1}) \rightsquigarrow (y_{i-1}, v_i)$. Now, we have

$(x, w_m) \rightsquigarrow (y_{m-2}, w_{m-1}) \rightsquigarrow (w_{m-2}, w_{m-1}) \rightsquigarrow (y_{m-3}, v_{m-2}) \rightsquigarrow \cdots \rightsquigarrow (w_1, w_2)$. Notice that $w_2x_2 \notin E(H)$ and $v_1w_2 \in E(H)$. These imply that $(w_1, w_2) \rightsquigarrow (x_2, v_1)$ and, consequently, $(x, w_m) \rightsquigarrow (x_2, v_1)$.
When $m = 2$, we have $(a, w_2) \to (x_2, w_2) \to (x_2, v_1)$; hence, again we get $(x_1, w_2) \rightsquigarrow (x_2, v_1)$. $\square$

Analogous to Lemma 6.12 we have the following lemma.

Lemma 6.13 (decomposition of different-color pairs). *Let $(x_1, x_3) \in D$ be by transitivity on a minimal chain $(x_1, x_2), (x_2, x_3)$ in D where x_1 and x_2 have the same color, and opposite to the color of x_3. Suppose (x_2, x_3) is a complex pair. Then there is a minimum number t, and $p_1, q_1, u_1, s_1, \ldots, p_{t-1}, q_{t-1}, s_{t-1}, u_{t-1}, c, d, q_t \in V(H)$ such that for $1 \le i \le t-1$ the following hold:*

(1) $(u_1, x_3), (u_{i+1}, x_3) \in D$ where $(u_1, x_3) \to (x_2, x_3)$ and $(u_{i+1}, x_3) \to (q_i, x_3)$

(2) (u_i, x_3) is by transitivity on pairs $(u_i, p_i), (p_i, q_i), (q_i, x_3) \in D$ where u_i and q_i have the same color as x_3

(3) $(p_i, s_i) \in D$ and $(p_i, s_i) \to (p_i, q_i)$ where x_3 and s_i have the same color

(4) $u_{i+1}q_{i-1} \notin E(H)$, $2 \le i$

(5) $u_iq_i \in E(H)$

(6) $s_iq_{i+1} \notin E(H)$

(7) $q_is_{i+1} \in E(H)$

(8) $ds_{t-1}, du_1 \in E(H)$

(9) x_3d and q_tc are independent edges of H.

Moreover, $(q_1, x_2) \rightsquigarrow (q_t, x_3)$ and $(q_1, x_2), (q_t, x_3) \in Dic(x_2, x_3)$.

In the next five lemmas we investigate the relationships between the dictators of two consecutive pairs $(x, y), (y, z)$ in D.

Lemma 6.14. *Let $(x_1, x_3) \in D$ be by transitivity on a minimal chain $(x_1, x_2), (x_2, x_3)$ in D where x_1 and x_2 have the same color and different from x_3 color. Suppose $(x_1, x_2), (x_2, x_3)$ both are complex pairs. Then, $Dic(x_1, x_2) = Dic(x_2, x_3)$.*

Proof. Let y_1, z_1, w_1, v_1, and w_m be the vertices in the decomposition of (x_1, x_2) according to Lemma 6.12. It follows from the lemma that $(x_1, w_m) \rightsquigarrow (x_2, v_1)$. Let u_1, q_1, and q_t be the vertices in the decomposition of (x_2, x_3) according to Lemma 6.13. Then, we have $(x_2, q_1) \rightsquigarrow (q_t, x_3)$.

Notice that $v_1u_1 \notin E(H)$, otherwise, we would have $(z_1, v_1) \to (z_1, u_1)$ and, hence, there would exist a chain $(x_1, y_1), (y_1, z_1), (z_1, u_1), (u_1, x_3)$; contradicting the minimality of the chain $(x_1, x_2), (x_2, x_3)$. Now, $(x_2, v_1) \to (u_1, v_1) \to (u_1, w_1)$ and, hence, $(x_2, v_1) \rightsquigarrow (u_1, w_1)$. On the other hand, $w_1q_1 \notin E(H)$, otherwise, $(x_1, w_1) \to (x_1, q_1)$ and we would obtain the chain $(x_1, q_1), (q_1, x_3)$; a contradiction to minimality of the chain $(x_1, x_2), (x_2, x_3)$. Thus, $(u_1, w_1) \to (q_1, w_1) \to (q_1, x_2)$ and, hence, $(u_1, w_1) \rightsquigarrow (q_1, x_2)$. From above, we conclude that $(x_2, v_1) \rightsquigarrow (x_2, q_1)$. By Lemma 6.13 and the skew-symmetry property we have $(q_1, x_2) \rightsquigarrow (q_t, x_3)$. Therefore, $(x_1, w_m) \rightsquigarrow (x_2, v_1) \rightsquigarrow (q_1, x_2) \rightsquigarrow (q_t, x_3)$, and by Corollary 2.9 $Dic(x_1, x_2) = Dic(x_2, x_3)$. $\square$

Lemma 6.15. *Let $(x_0, x_2) \in D$ be by transitivity on a minimal chain $(x_0, x_1), (x_1, x_2)$ in D where x_1 and x_2 have the same color and different from x_0. Suppose (x_0, x_1) is a simple pair and (x_1, x_2) is a complex pair. Then, $Dic(x_0, x_1) = Dic(x_1, x_2)$.*

Proof. Since (x_0, x_1) is simple and x_0 and x_1 have different colors, by Lemma 2.6, there exist independent edges x_0e and x_1f of H. Let y_1, z_1, w_1, v_1, and w_m be the vertices in the decomposition of (x_1, x_2) according to Lemma 6.12. Note that, according to Lemma 6.12, $x_1w_1 \notin E(H)$. Then, $w_1e \notin E(H)$, otherwise, we would get $(x_0, x_1) \to (e, x_1) \to (w_1, x_1)$; contradicting $(x_1, w_1) \in D$. Furthermore, $x_0x_2 \in E(H)$, otherwise, x_0e and w_1x_2 would be independent edges; thereby, (x_0, x_2) would be in a component. The latter contradicts the assumption that (x_0, x_2) is by transitivity. Likewise, observe that $fx_2 \in E(H)$, otherwise, x_1f and x_1w_1 would be independent edges; a contradiction with the assumption that (x_1, x_2) is a complex pair. Finally, $x_0v_1 \notin E(H)$, otherwise, $(z_1, v_1) \to (z_1, x_0)$, resulting in a circuit $(x_0, x_1), (x_1, y_1), (y_1, z_1), (z_1, x_0)$ in D; a contradiction with the assumption that the current D is circuit-free. Now, we have $(x_2, v_1) \to (x_0, v_1) \to (x_0, f) \to (x_0, x_2)$ and, hence, $(x_2, v_1) \rightsquigarrow (x_0, x_1)$. Therefore, by Lemma 6.12, $(x_1, w_m) \rightsquigarrow (x_2, v_1)$. Thus, $(x_1, w_m) \rightsquigarrow (x_0, x_1)$, implying that $Dist(x_1, x_2) = Dist(x_0, x_1)$. $\square$

Analogous to Lemma 6.15 we have the following lemma.

Lemma 6.16. *Let $(x_2, x_4) \in D$ be by transitivity on a minimal chain $(x_2, x_3), (x_3, x_4)$ in D where x_2 and x_4 have the same color and opposite to the color of x_3. Suppose (x_2, x_3) is complex and (x_3, x_4) is implied by a component. Then, $Dic(x_2, x_3) = Dic(x_3, x_4)$.*

Lemma 6.17. *Let $(x_1, x_2), (x_2, x_3)$ be a minimal chain in D between x_1 and x_3. Let (x_1, x_2) be a complex pair in D where x_1 and x_2 have the same color. Let $(x_1, w_1) \in D$ where $(x_1, w_1) \to (x_1, x_2)$. Moreover, suppose (x_1, w_1) is by transitivity on the minimal chain $(x_1, y_1), (y_1, z_1), (z_1, w_1)$ where (z_1, w_1) is a complex pair. Then, $Dic(x_1, y_1) = Dic(y_1, z_1) = Dic(z_1, w_1)$.*

Proof. By Corollary 6.8, x_1 and y_1 have the same color and opposite to the color of z_1 and w_1. Let $(z_1, v_1) \in D$ such that $(z_1, v_1) \to (z_1, w_1)$. Let $(u_1, x_3) \in D$ so that $(u_1, x_3) \to (x_2, x_3)$. Notice that $v_1u_1 \notin E(H)$, otherwise, we would have $(z_1, v_1) \to (z_1, u_1)$, resulting in the chain $(x_1, y_1), (y_1, z_1), (z_1, u_1), (u_1, x_3)$ in D; contradicting the minimality of the chain $(x_1, x_2), (x_2, x_3)$.

By Lemma 6.12 we have $S_1 \rightsquigarrow (x_2, v_1)$, where $S_1 = Dic(x_1, w_1)$. According to the definition of dictator components, we have $Dis(x_1, w_1) = Dis(x_1, y_1)$. Now, since (z_1, w_1) is a complex pair, by Lemma 6.12 for pair (z_1, w_1), we conclude that there exists p_1, q_1, and s_1 such that z_1, p_1, and s_1 have the same color and opposite to the color q_1 and v_1; the pairs $(z_1, p_1), (p_1, q_1), (q_1, v_1), (q_1, s_1)$ are in D; and $(q_1, s_1) \rightsquigarrow (q_1, v_1)$. By Lemma 6.12 for (w_1, z_1), we have $S_2 \rightsquigarrow (w_1, s_1)$. Notice that $s_1x_2 \notin E(H)$, otherwise, we would have $(q_1, s_1) \to (q_1, x_2)$ resulting in the chain $(x_1, y_1), (y_1, z_1), (z_1, p_1), (p_1, q_1), (q_1, x_2), (x_2, x_3)$ with pairs in D; contradicting the minimality of the chain $(x_1, x_2), (x_2, x_3)$. Therefore, $(w_1, s_1) \to (x_2, s_1) \to (x_2, v_1)$, implying that $S_2 \rightsquigarrow (x_2, v_1)$. Since u_1x_2 and v_1s_1 are independent edges, (x_2, v_1) is in a component. We then have $S_1 \rightsquigarrow (x_2, v_1)$ and $S_2 \rightsquigarrow (x_2, v_1)$. Since (x_2, v_1) is in a component, by Corollary 2.9, we conclude that $S_1 = S_2 = S_{x_2v_1}$.

Now, it follows from lemmas 6.15 and 6.14 that $Dis(y_1, z_1) = S_2$ and, hence, $Dis(x_1, y_1) = Dis(y_1, z_1) = Dis(z_1, w_1)$. □

Analogous to Lemma 6.17 we have the following lemma.

Lemma 6.18. *Let $(x_1, x_2), (x_2, x_3)$ be a minimal chain in D between x_1 and x_3. Let (x_2, x_3) be a complex pair in D where x_1 and x_2 have different colors. Let $(u_1, x_2) \in D$, where $(u_1, x_3) \to (x_2, x_3)$. Moreover, suppose (u_1, x_3) is by transitivity on the minimal chain $(u_1, p_1), (p_1, q_1), (q_1, x_3)$ where (q_1, x_3) is a complex pair. Then $Dic(u_1, p_1) = Dic(q_1, x_3) = Dic(p_1, q_1)$.*

The following lemma shows the role of a dictator component in placing a pair in D, alongside its independence from selection of other components.

Lemma 6.19. *Let $(x_1, x_3) \in D$ be by transitivity on a minimal chain $(x_1, x_2), (x_2, x_3) \in D$ where x_1 and x_2 have the same color and opposite to the color of x_3. Suppose (x_1, x_2) is a complex pair, and let $S_1, S_2, \ldots, S_k$ be the distinct components involving in the creation of (x_1, x_2). Suppose $Dic(x_1, x_2) = S_1$. Let D_1 be a set of pairs which contains S_1 and exactly one of S_i, S'_i for every $2 \leq i \leq k$. Then, $(x_1, x_2) \in N^*[D_1]$.*

Proof. We use induction on the number of steps in the decomposition of (x_1, x_2) according to Lemmas 6.12 and 6.13. Since x_1 and x_2 have different colors, it follows by Lemma 6.12 that there exists $(x_1, w_1) \in D$ such that $(x_1, w_1) \to (x_1, x_2)$ and (x_1, w_1) is by transitivity on the minimal chain $CH = (x_1, y_1), (y_1, z_1), (z_1, w_1)$. By definition of a dictator, $Dic(x_1, y_1) = Dic(x_1, x_2)$. Let $(z_1, v_1) \in D$ such that $(z_1, v_1) \to (z_1, w_1)$. Observe that $v_1w_2 \in E(H)$, otherwise, we would have $(y_1, z_1), (z_1, v_1) \in D$, implying that $(y_1, v_1) \to (w_2, v_1) \to (w_2, w_1)$ and, hence, we get the earlier chain $(x_1, w_2), (w_2, w_1)$ in D— the latter contradicts the minimality of CH.

We will consider two possible cases. First, consider the case where (y_1, z_1) and (z_1, w_1) are simple. According to Lemma 2.6 there exist independent edges $y_1y'_1$ and $z_1z'_1$ and independent edges z_1e and v_1f so that $w_1e, w_1z'_1 \in E(H)$. According to the argument for Lemma 6.11, y_1w_1 is an edge of H. Note that $(y_1, z'_1), (y_1, e) \in N^+[S_{y_1z_1}]$. Also, note that $w_2z'_1 \in E(H)$, otherwise, we would have $(y_1, z'_1) \to (w'_1, z'_1) \to (w_2, w_1)$ and, consequently, (w_2, w_1) would be simple. But then we would get an earlier chain $(x_1, w_2), (w_2, w_1)$ with pairs in D; a contradiction to the minimality of CH. Likewise, we conclude that

$w_2e \in E(H)$. Notice that by definition, $Dic(x_1, w_2) = S_1$, and observe that (x_1, w_2) dominates every pair in $\{(x_1, y_1), (x_1, v_1), (x_1, z_1), (x_1, e)\}$. By induction hypothesis, if S_1 is in D then $Dic(x_1, w_2) = S_1$. If we place into D components $S_{y_1z_1}$ and $S_{z_1v_1}$ at Stage 1, then $(y_1, z_1), (z_1, w_1) \in D$. Thus, CH will have its pairs in D and, consequently, we get $(x_1, w_1), (x_1, x_2) \in D$. If we place into D components $S_{v_1z_1}$ and S_{z_1,y_1} then $(v_1, z_1), (z_1, w_1) \in D$ (since $(z_1, y_1) \to (z_1, w_1)$) and, hence, $(x_1, v_1), (v_1, z_1), (z_1, w_1) \in D$. Consequently, in this case we get $(x_1, w_1), (x_1, x_2) \in D$. So, we may assume that $S_{y_1z_1}$ and $S_{v_1z_1}$ are selected to be placed in D at Stage 1 of the algorithm. Now, $y_1'v_1 \notin E(H)$, otherwise, $(y_1', z_1) \to (v_1, z_1)$ and, hence, $S_{z_1v_1} \in D$; a contradiction. Similarly, we get $y_1f \notin E(H)$. Therefore, y_1y_1' and v_1f are independent edges and, hence, $S_{y_1v_1}, S_{v_1y_1}$ are components. Now, without loss of generality we may assume the algorithm selects $S_{v_1y_1}$ at Stage 1. Then, $(v_1, y_1') \in D$ and $(y_1', v_1) \to (y_1', w_1) \in D$. Moreover, $(x_1, w_2) \to (x_1, v_1)$. Therefore, $(x_1, v_1), (v_1, y_1'), (y_1', w_1) \in D$ and, hence, $(x_1, w_1) \in D$.

Finally, consider the case where (z_1, w_1) is complex. By Lemma 6.17, we conclude that $Dis(x_1, y_1) = Dis(y_1, z_1) = Dis(z_1, w_1)$ and, hence, by induction hypothesis, if $Dis(x_1, y_1)$ is selected at Stage 1 of the algorithm then each of the pair $(x_1, y_1), (y_1, z_1)$ and (z_1, w_1) is placed in D; hence, (x_1, x_2) is placed in D. □

6.3 Proofs of Lemmas 6.1, 6.2, and 6.3

Proof of Lemma 6.1: (1) follows from Lemma 6.14 on the minimal chain $(x_1, x_2), (x_2, x_3)$. (4) follows from Lemma 6.14 on the minimal chain $(x_2, x_3), (x_3, x_0)$. (2) follows from Lemma 6.15. (3) follows from Lemma 6.16. Finally, (5) follows from the arguments in Lemma 6.17(considering the $(x_1, x_2), (x_2, x_3), (x_3, x_0)$ instead of the chain $(x_1, y_1), (y_1, z_1), (z_1, w_1)$), and Lemma 6.18. □

Proof of Lemma 6.2: This follows from lemmas 6.1 and 6.19. □

Proof of Lemma 6.3: The purpose of computing $Dic(x, y)$ is to identify a component that is responsible for creating a circuit in D. Therefore, we may assume that a minimal circuit C in D is created once (x, y) is added into D. By Corollary 6.9, C is on four pairs. Suppose $C = (x_0, x_1), (x_1, x_2), (x_2, x_3), (x_3, x_0)$ and assume, without loss of generality, that x_0 and x_3 are white vertices, and x_1 and x_2 are black vertices. Recall that the following determine the dictator of a pair (x, y).

(a) If $(x, y) \in N^+[S]$ for some component S then $Dic(x, y) = S$.

(b) If x and y have different colors and $(u, y) \to (x, y)$ then $Dic(x, y) = Dic(u, y)$.

(c) If x and y have the same color and $(x, w) \to (x, y)$ then $Dic(x, y) = Dic(x, w)$.

(d) If x and y have the same color and (x, y) is by transitivity on $(x, w), (w, y)$ then $Dic(x, y) = Dic(w, y)$.

(e) If x and y have different colors and (x, y) is by transitivity on $(x, w), (w, y)$ then $Dic(x, y) = Dic(x, w)$.

In what follows, we assume (x, y) is one of the pairs on C.

Let (u_1, x_3) be a pair in (the current) D and $(u_1, x_3) \to (x_2, x_3)$. According to definition, $Dic(u_1, x_3) = Dic(x_2, x_3)$. By Corollary 6.8, (u_1, x_3) is by transitivity on a minimal chain $(u_1, p_1), (p_1, q_1), (q_1, x_3)$ in D.

When we compute $N^*[D]$, (u_1, x_3) appears in D at some earlier level,i.e., when pairs of the forms (u_1, f) and (f, x_3) appear in $N^*[D]$ at some earlier level. According to the minimality of the chain between u_1 and x_3 we must have either $f = q_1$ or $f = p_1$. First suppose $f = q_1$. Then, according to (d), we have $Dic(x_2, x_3) = Dic(q_1, x_3)$. By induction hypothesis, we also have $Dic(q_1, x_3) = S_2$, where S_2 is the component obtained after decomposing the pair (x_2, x_3) in accordance with Lemma 6.13. Therefore, $Dic(x_2, x_3) = Dic(u_1, x_3) = Dic(q_1, x_3)$. Now, consider the case where $f = p_1$. Then, according to (d), we have $Dic(u_1, x_3) = Dic(p_1, x_3)$. Thus, using (e), we obtain $Dic(p_1, x_3) = Dic(q_1, x_3)$ because the chain $(p_1, q_1), (q_1, x_3)$ implies (p_1, x_3) where p_1 and x_3 have different colors. A similar argument can be applied to the pair (x_1, x_2), where x_1 and x_2 have the same color. □

7 Correctness of Stages 3 and 4 (lines 19–25)

If we encounter a circuit C in D in Stage 2 then, according to Lemma 6.2, there is a component S that is a dictator for C. By Lemma 6.2, it is clear that we should not add S to D, otherwise, we would not get the desired ordering. Therefore, we must take the coupled component of every dictator component of a circuit appearing at Stage 2. With this consideration, we continue to show the correctness of Stages 3.

Lemma 7.1 (correctness of Stage 3)**.** *If the algorithm encounters a circuit at Stage 3 (line 20) then H is not an interval bigraph.*

Proof. According to line 22 of the algorithm , D_1 contains components $S_1, S_2, \dots, S_j$ chosen at Stage 1, alongside components $S'_{j+1}, \dots, S'_t$ where $S_{j+1}, \dots, S_t$ are dictator components. Suppose we encounter a minimal circuit $C = (x_0, x_1), (x_1, x_2), ..., (x_{n-1}, x_n), (x_n, x_0)$ in line 22. If all the pairs in C are simple then, according to Theorem 5.4, we find an exobiclique and, hence, H is not an interval bigraph. Therefore, we may assume at least one pair, say, (x_i, x_{i+1}) is a complex pair. Let $S = Dic(x_i, x_{i+1})$. Notice that S is not among $S_1, S_2, \dots, S_j$, otherwise, we would have detected S as a dictator component at Stage 2, according to Lemma 6.2. Thus, S belongs to $\{S'_{i+1}, S'_{i+2}, \dots, S'_t\}$. But the latter means H is not an interval bigraph because we can not select either of S, S' at Stage 1; a contradiction in light of Lemma 3.5. ☐

Lemma 7.2 (correctness of Stage 4)**.** *The algorithm does not create a circuit by choosing a sink component $S \in H^+ \setminus D$ (satisfying $N^+[S] = S$) and adding it to D after taking its transitive closure.*

Proof. Suppose adding— according to the algorithm— a sink trivial component $\{(x, y)\}$ into D creates a circuit. By definition, there is no arc from (x, y) to any pair in $H^+ \setminus D$— i.e., (x, y) is a terminal pair. According to the algorithm, neither of $(x, y), (y, x)$ is presently in D. Moreover, (x, y) is not by transitivity on any of the pairs presently in D (otherwise (x, y) would have been placed in D, since D is closed under transitivity).

Now, since (x, y) is a terminal pair at the current step of the algorithm, (x, y) can only dominate pairs in D. Therefore, the only way that adding (x, y) into D creates a circuit is when (x, y) dominates a pair (u, v) while there is a chain $(v, y_1), (y_1, y_2), ..., (y_k, u) \in D$; in which case we have $(v, u) \in D$. However, since D is closed under reachability and transitivity, by the skew-symmetry $(u, v) \to (y, x) \in D$; a contradiction. ☐

8 Implementation and complexity

In this section we prove the following lemma.

Lemma 8.1. *Let H be a bigraph with n vertices and m edges. Then, Algorithm 1 runs in $\mathcal{O}(mn)$ time and produces an interval vertex ordering when H is an interval bigraph; otherwise, reports H is not an interval bigraph.*

Proof. In this proof, we denote the degree of a vertex z of H by d_z. In order to construct digraph H^+, we need to list all the neighbors of each pair in H^+. If vertices x and y in H have different colors then the pair (x, y) of H^+ has d_y out-neighbors; and if x and y have the same color then the pair (x, y) has d_x out-neighbors in H^+. For simplicity— without affecting the generality of the argument— we assume that $|W| = |B| = n$. For a fixed black vertex x, the number of all pairs which are neighbors of all pairs (x, z), $z \in V(H)$, is $nd_x + d_{y_1} + d_{y_2} + \cdots + d_{y_n}$, where $y_1, y_2, ..., y_n$ are all of the white vertices. We can use a linked list structure to represent H^+, therefore, overall, it takes time $\mathcal{O}(mn)$ to construct H^+. Notice that in order to check whether a component S is self-coupled, it is enough to pick any pair (a, b) in S and check if (b, a) is in S, as well. The latter task can be done in time $\mathcal{O}(mn)$, using Tarjan's strongly-connected component algorithm. Since we maintain a partial order on D, once we add a new pair into D we can decide whether that pair closes a circuit or not. Computing $N^*[D]$ also takes time $\mathcal{O}(n(n+m)) = \mathcal{O}(mn)$ since there are $\mathcal{O}(mn)$ edges in H^+ and $\mathcal{O}(n^2)$ vertices in H^+. Note that the envelope of D is computed at most twice (at lines 15 and 22).

Once a pair (x, y) is added into D, we put an arc from x to y in the partial order and give the arc xy a time label (also called level). Once a circuit is formed at Stage 2, we can find a dictator component S by using Dictator function, and store S into set $\mathcal{DT}$. Therefore, we spend at most $\mathcal{O}(nm)$ time to find all the dictator components. Stage 4, in which we add the remaining pairs, takes time at most $\mathcal{O}(n^2)$. Therefore, the overall running time of the algorithm is $\mathcal{O}(nm)$. ☐

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