C-Perfect K-Uniform Hypergraphs

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Abstract

In this paper we define the concept of clique number of uniform hypergraph and study its relationship with circular chromatic number and clique number. For every positive integer k,p and q, $2q \leq p$ we construct a k-uniform hypergraph H with small clique number whose circular chromatic number equals $\frac{p}{q}$. We define the concept and study the properties of c-perfect k-uniform hypergraphs .

Keywords : Hypergraph, Circular Coloring, c-Perfect Hypergraph

1 Introduction

For the necessary definitions and notations, we refer the reader to standard texts of graph theory such as [6]. In this paper p, q, r, k, m, n, and l are positive integers such that $(2q \leq p)$ and $k > 2$. Also, we consider only finite hyperhgraphs. The hypergraph $H = (V, E)$ is called k-uniform whenever every edge e of H is of size k. The hypergraph $H' = (V', E')$ is a subhypergraph of $H = (V, E)$ whenever $V' \subset V$, and $E' \subset E$ and it is an induced subhypergraph of H if and only if for every edge e of H with $e \subset V'$, then $e \subset E'$. The complement of a k-uniform hypergraph H denoted by $\overline{H}$, is defined to be a k-uniform hypergraph with vertex set $V(\overline{H}) = V(H)$ and a k-subset of $V(H)$ is an edge of $\overline{H}$ if and only if it is not an edge of H. A subset S of the vertices 0f H is said to be independent if it does not contained any edges of H and the independent number of H, denoted by $\alpha(H)$, is defined to be the largest size of independent sets of H. Now for the sake of referencewe state some definition and result from [2].

A k-uniform hypergraph H is called complete if every k-subset of the vertices is an edge of H. A mapping $f : V(H) \longrightarrow V(K)$ is a homomorphism from hypergraphs H to hypergraphs K if for every edge e of H, there exists an edge e' of K such that $e' \subseteq f(e)$. And H is called vertex transitive if for every vertices x and y in H there is an bijection homomorphism $f : V(H) \longrightarrow V(H)$ such that $f(x) = y$.

A (p, q)-*coloring* of H is a mapping $c : V \longrightarrow \{0, 1, .., p-1\}$ such that for every

edge $e \in E$, there exist x and y in e satisfying $q \leq |c(x) - c(y)| \leq p - q$.
The *circular chromatic number* of H, denoted by $\chi_c(H)$, is defined as

$$\chi_c(H) = inf\{\frac{p}{q} \mid \ there \ exists \ a \ (p,q) - coloring \ of \ H\}$$

We have shown that [2] the infimum in the definition of circular chromatic number can be replaced by minimum.
It is obvious that if H' is an induced subhypergraph of H then $\chi_c(H') \leq \chi_c(H)$.
Let $H_q^p(k)$ denote the k-uniform hypergraph with vertex set $\{0, 1, .., p-1\}$ and a k-subset $\{x_1, x_2, .., x_k\}$ of $V(H)$ is an edge of $H_q^p(k)$ if and only if there exist $1 \leq i, j \leq k$ such that $q \leq |x_i - x_j| \leq p - q$.
If $k = 2$ then $H_q^p(k)$ is the graph G_q^p defined by Zhu [7] . It was shown in [2] that $\chi_c(H_q^p(k)) = \frac{p}{q}$, and it is vertex transitive.
A mapping c from the collection S of independent sets of a hypergraph H to the interval $[0, 1]$ is a fractional-coloring of H if for every vertex x of H we have $\Sigma_{s \in S, x \in s} c(s) = 1$. The value of fractional-coloring c is $\Sigma_{s \in S} c(s)$. The fractional coloring number of H, denoted by $\chi_f(H)$, is the infimum of the values of fractional-colorings of H.
It is easy to see that $\alpha(H_q^p(k)) = q$. Since $H_q^p(k)$ is vertex transitive, by Proposition 4.b in [2] $\chi_f(H_q^p(k)) = \frac{p}{q}$.

Theorem 1 *1) Let H be a hypergraph then:*
$\chi_c(H) = min\{\frac{p}{q} \mid \ there \ exists \ a \ homomorphism \ f : H \longrightarrow G_q^p\}$.

2) Let H and K be two hypergraphs and there exist a homomorphism from H to K then,

a) $\chi(H) \leq \chi(K)$,

b) $\chi_c(H) \leq \chi_c(K)$.

3) Let H be a k-uniform hypergraph and $\chi_c(H) = \frac{p}{q}$ then,

a) $\chi(H) - 1 < \chi_c(H) \leq \chi(H)$.

b) If $c : V(H) \longrightarrow \{0, 1, .., p-1\}$ be a (p, q)-coloring of H, then c is onto, and $|V(H)| \geq p$.

c) If $\frac{p}{q} \leq \frac{p'}{q'}$ then H has a (p', q')-coloring.

d) $\frac{|V(H)|}{\alpha(H)} \leq \chi_f(H) \leq \chi_c(H)$,

e) If H is vertex transitive then $\chi_f(H) = \frac{|V(H)|}{\alpha(H)}$.

2 Circular Coloring And Clique Number

In this section we only consider k-uniform hypergraphs.

Definition 1 *Let H be a hypergraph. A subset A of $V(H)$ is called a clique of H if every k-subset of A is an edge of H. The clique number of H, denoted by $\omega(H)$, is defined as*

$$\omega(H) = \frac{max\{|A| \mid A \ is \ a \ clique\}}{k-1}.$$

The above definition is a generalization of the concept of clique number of graph.

Theorem 2 *For every k-uniform hypergraph H, we have $\omega(H) \leq \chi_c(H)$.*

Proof : Let $\omega(H) = \frac{p}{k-1}$ and $A = \{0, 1, .., p-1\}$ be a clique of H. Let $V(H^p_{k-1}(k)) = A$. Define mapping $f : H^p_{k-1}(k) \longrightarrow H$ by $f(x) = x$. It is obvious that f is a homomorphism; therefore, by Theorem 1, $\chi_c(H^{k-1}_p(k)) = \omega(H) \leq \chi_c(H)$. $\square$

Theorem 3 *For every $\frac{p}{q} > 2$ and $k \geq 4$ there exists a k-uniform hypergraph H with $\chi_c(H) = \frac{p}{q}$ and $\omega(H) \leq \frac{k+1}{k-1}$.*

Proof : Let r be an integer such that $qr > \lceil \frac{p}{q} \rceil (2k-4)$. Consider the graph G^{pr}_{qr} and construct a k-uniform hypergraph H as follows:
$V(H) = V(G^{pr}_{qr})$ and a k-subset e of $V(H)$ is an edge if either e is a 1-edge or e is a $(k-1)$-edge,
where a set e is called an l-edge if the induced subgraph of G^{pr}_{qr} generated by e has exactly l edges.
We show that $\omega(H) \leq \frac{k+1}{k-1}$. On the contrary let $A = \{a_0, a_1, .., a_{k+1}\}$ be a clique in H.

Case 1) Suppose a k-subset of A is a 1-edge. Without Without loss of generality assume that this k-subset is $\{a_0, a_1, a_2, .., a_{k-1}\}$ and a_0a_1 is the only edge of G^{pr}_{qr} in this set. If $\{a_0, a_2, .., a_{k-1}, a_k\}$ is a $(k-1)$-edge, then a_k is incident to all vertices $\{a_0, a_2, .., a_{k-1}\}$ in G^{pr}_{qr}. Now since $k > 3$ the set $\{a_1, a_2, a_3, .., a_k\}$ is a $k-1$-edge too. Therefore a_1a_k is also an edge of G^{pr}_{qr}. Now the induce subgraph of G^{pr}_{qr} generated by $\{a_0, a_1, a_3, .., a_{k-1}, a_k\}$ has k edges which is a contradiction. Therefore $\{a_0, a_2, a_3, .., a_{k-1}, a_k\}$ is an 1-edge. Similarly $\{a_1, a_2, a_3, .., a_{k-1}, a_k\}$ is an 1-edge. Hence there exists $j \in \{2, 3, .., k-1\}$ such that a_ja_k is not an edge of G^{pr}_{qr}. Now $\{a_0, a_1, .., a_k\} - \{a_j\}$ is 2-edge, and since $k \geq 4$, we have a contradiction.

Case 2) Let every k-subsets of A be a $(k-1)$-edge. Let $T = G^{pr}_{qr}[A]$, and T has a vertex of degree at least 3. Let $deg^{a_0}_T \geq 3$ and a_1, a_2, and a_3 are adjacent to a_0. Since $\{a_1, a_2, a_3, .., a_k\}$ is a $(k-1)$-edge, there exists i, $1 \leq i \leq k$ such that a_i is of degree 0 or 1 in the induced subgraph of G^{pr}_{qr} generated by $\{a_1, .., a_k\}$. Now the set $\{a_0, a_1, .., a_k\} - \{a_i\}$ is a l-edge, $k \leq l$. Therefore $\Delta(T) \leq 2$, and every component of T is a path or cycle . Let a_i and a_j be two nonadjacent vertices of degree 2 in T (it is obvious that there exist such vertices). Now the induced subgraph of G^{pr}_{qr} generated by $A - \{a_i, a_j\}$ has at most $(k-2)$ edges, a contradiction. Therefore $\omega(H) \leq \frac{k+1}{k-1}$.

Now we show that $\alpha(H) \leq qr$. Suppose B is an independent set and $0 \in B$. Let $C_0 = \{0, 1, .., qr-1\}$, $C'_0 = \{pr-qr+1, pr-qr+2, .., 0\}$, $C = \{qr, qr+1, .., pr-qr\}$, and $C'_t = \{t, t+1, ..., t+qr\}$, $qr \leq t \leq pr-2qr$. For all t, $qr \leq t \leq pr-2qr$, B has at most $k-2$ vertices of C'_t. Otherwise $(B \bigcap C'_t) \bigcup \{0\}$ is not independent. Therefore and $|B \bigcap C| \leq \lceil \frac{p-2q+1}{q} \rceil (k-2)$. Let $B \bigcap C_0 = \{a_1 < a_2 < .. < a_{l_1}\}$ and $B \bigcap C'_0 = \{b_1 < b_2 < .. < b_{l_2}\}$. Hence $a_1 = b_1 = 0$.

Case 1) $|B \bigcap C| > 0$, and $c \in B \bigcap C$. Let a_i be the last element of C_0 and b_j be the first element of C_1 such that ca_i and cb_j are edges in G^{pr}_{qr}. In this case each of the sets $\{a_{i+1}, .., a_{l_1}\}$, $\{b_{j+1}, .., b_{l_2}\}$, $\{a_1, .., a_{i-1}\}$, and $\{b_1, .., b_{j-1}\}$ has at most $k-3$ elements. Otherwise, if for example $|\{a_{i+1}, .., a_{l_1}\}| > k-3$, then the set $\{c, a_{i+1}, .., a_{i+k-1}\}$ which is an edge of H is a subset of B, a contradiction. Therefore $|C_0 \bigcap B| \leq 2k-5$, $|C'_0 \bigcap B| \leq 2k-5$, and hence $|B| < qr$.

Case 2) $|B \bigcap C| = 0$. If there is no edge a_ib_j in G^{pr}_{qr} then $\{a_1, a_2, \ldots a_{l_1}\}$ and $\{b_1, b_2, \ldots, b_{l_2}\}$ are subset of an independent set of G^{pr}_{qr}, and since independence number of G^{pr}_{qr} is qr we have $|B| \leq qr$. Assume that a_ib_j be an edge of G^{pr}_{qr} such that $b_j - a_i = max\{b_t - a_s \mid b_t a_s \ is \ an \ edge \ of \ G^{pr}_{qr}\}$. Therefore each of the sets $\{a_{i+1}, .., a_{l_1}\}$and $\{a_1, .., a_{i-1}\}$ has at most $k-3$ elements. Therefore, $|C_0 \cap B| \leq 2k-5$. By the same way $|C'_0 \cap B| \leq 2k-5$ and therefore, $|B| \leq qr$. But since the set $\{0, 1, .., qr-1\}$ is an independent set, then $\alpha(H) = qr$.

By Theorem 1 we have $\frac{pr}{qr} = \frac{|V(H)|}{\alpha(H)} \leq \chi_f(H) \leq \chi_c(H)$. On the other hand since the map $f : V(H) \longrightarrow G^{pr}_{qr}$ define by $f(x) = x$ is a homomorphism then Theorem 1 support that $\chi_c(H) \leq \chi(G^{pr}_{qr}) = \frac{p}{q}$, therefore $\chi_c(H) = \frac{p}{q}$ and proof is completed. □

By a similar proof one can show that Theorem 3 follows for $k = 3$ and $\omega(H) \leq \frac{5}{2}$.

Definition 2 *A k-uniform hypergraph H is called c-perfect if for every induced subhypergraph H_1 of H provided $\chi_c(H_1) > 2$, we have $\chi_c(H_1) = \omega(H_1)$.*

In this definition the condition $\chi_c(H) > 2$ is necessary, because if H_1 is an edge then $\chi_c(H_1) = \chi(H_1) = 2$ but $\omega(H_1) = \frac{k}{k-1}$ and therefore we never have a c-perfect hypergraph.

An example of c-perfect hypergraph is a complete k-uniform hypergraph H because, $\omega(H) = \chi_c(H) = \frac{|V(H)|}{k-1}$ and since every induced subhypergraph of H is complete, H is c-perfect.

Now from every perfect graph we construct a k-uniform c-perfect hypergraph.

Theorem 4 *Let G be a perfect graph, and $H = (V(H), E(H))$ be a hypergraph* *such that $V(H) = V(G)$ and $e \subset V(H)$ is an edge of H if and only if $|e| = k$ and e is contained in a clique of G. Then H is c-perfect.*

Proof: Let H_1 be an induced subhypergraph of H, and G_1 be induced subgraph of G generated by $V(H_1)$. It is easy to see that $\omega(H_1) = \frac{\omega(G_1)}{k-1}$. Since G is perfect, $\chi(G_1) = \omega(G_1)$.

Case 1) Let $\omega(G_1) \geq 2(k-1)$ and c be a $\omega(G_1)$-coloring of G_1. Consider the coloring c' define by $c'(x) = c(x)$ of H_1. Since every edge e of H_1 is a subset of a clique of G_1, there exists two vertices $x, y \in e$ such that $k - 1 \leq |c(x)-c(y)| \leq \omega(G_1)-k+1$ thus c' is an $(\omega(G_1), k-1)$-coloring of H_1. Therefore $\chi_c(H_1) \leq \frac{\omega(G_1)}{k-1}$, and by Theorem 5 $\chi_c(H_1) \geq \omega(H_1)$.

Case 2) $k \leq \omega(G_1) < 2(k-1)$. We will show that $\chi_c(H_1) = 2$. Let c be a $\omega(G_1)$-coloring of G_1. Consider the mapping $c' : V(H_1) \longrightarrow \{0, 1\}$ by $c'(x) = \lfloor \frac{c(x)}{k} \rfloor$ and let e be an edge of H_1. Since e is a k-subset of a clique of G_1 then there exist at least two vertices $x, y \in e$ such that $c(x) < k$ and $c(y) \geq k$. Therefore $c'(x) = 0$ and $c'(y) = 1$, and c' is a 2-coloring of H_1.

In the other case H_1 has no edge. □

We know that the complement of every perfect graph is perfect but it is not true for c-perfect hypergraphs. In the following we construct a hypergraph H which is c-perfect, but $\overline{H}$ is not c-perfect.
Let $V(H) = \{0, 1, 2, .., 3m-1\}$, $m > 2k$, and $e \in E(H)$ if e is a k-subset of one the sets $\{0, 1, 2, .., m-1\}$, $\{m, m+1, m+2, .., 2m-1\}$ or $\{2m, 2m+1, 2m+2, .., 3m-1\}$. Since H is union of three copy of disjoint complete hypergraphs, H is c-perfect. Let H' be the induced subhypergraph of $\overline{H}$ generated by the set $\{0, m, m+1, .., m+k-2, 2m, 2m+1, .., 3m-1\}$. By definition of H, the union of $\{0, m, .., m+k-2\}$ and every $(k-1)$-subset of $D = \{2m, 2m+1, .., 3m-1\}$ is a clique of $\overline{H}$, and every clique of H' has at most $2k-1$ vertices. Therefore $\omega(H') = \frac{2k-1}{k-1}$. Now we show that H' has no any $(2k-1, k-1)$-coloring. Let c be a $(2k-1, k-1)$-coloring of H'. Since the vertices of a clique of H' of size at least k have different colors then, $|c^{-1}(\{0, m, m+1, .., m+k-2\}| = k$. On the other hand since every $k-1$-subset of D with $\{0, m, .., m+k-2\}$ make a clique then $|c^{-1}(D)| = k-1$. Therefore there exist vertices $x, y \in D$ such that $c(x) = c(y)$. Since x and y appear in a clique of size $2k-1$ they must have different colors and it is a contradiction. Thus $\chi_c(H') > \omega(H')$. Now since $\chi_c(H') > 2$, then $\overline{H}$ is not c-perfect.

By the above discussion it is natural to look for k-uniform c-perfect hypergraph H such that $\overline{H}$ is also c-perfect. Also we look for k-uniform c-perfect hypergraph H, such that $\overline{H}$ is c-perfect, and $\omega(H)$ and $\omega(\overline{H})$ are arbitrary large. First we prove that for every m and n there exists a 3-uniform c-perfect hypergraph H such that $\overline{H}$ is c-perfect, and $\omega(H) = \frac{m+1}{2}$, $\omega(\overline{H}) = \frac{n+1}{2}$.

Let $V(H) = \{1, 2, .., m+n\}$ and $e \in E(H)$ either $e \subset \{1, 2, .., m\}$ or $|e \cap \{m+1, .., m+n\}| = 1$. By the construction of H we have, $\omega(H) = \frac{m+1}{2}$ and $\omega(\overline{H}) = \frac{n+1}{2}$ and $\{1, 2, .., m, m+1\}$ is a maximum clique, and $\{m, m+1, .., m+$

$n\}$ is a maximum independent set of H. On the other hand every induced subhypergraph of H and $\overline{H}$ has the same structure and therefore it is enough to prove that $\chi_c(H) = \omega(H)$. If $\omega(H) \leq 2$ color the vertices $1, 2, .., m$ by 1 and the vertices $m+1, m+2, .., m+n$ by 2; it is a 2-coloring for H. Let $\omega(H) > 2$. Define the map c by:

$$c : V(H) \longrightarrow \{0, 1, .., m\}$$

$$c(i) = \begin{cases} i-1 & 1 \leq i \leq m \\ m & \text{otherwise} \end{cases}$$

Reader can check that c is a $(m+1, 2)$-coloring. Therefore $\omega(H) = \chi_c(H)$, and proof is complete.

Theorem 5 *For every $k > 3$ there exists a k-uniform hypergraphs H, such that H and $\overline{H}$ are c-perfect, $\omega(H)$ is arbitrary large and $\omega(\overline{H}) > 2$.*

At first we prove the following lemma:

Lemma 6 *Suppose G' be a graph with vertex set $V(G) = \{m, m+1, \ldots, m+2k-1\}$, and ij is an edge of G if one of the following occurs :*
1) $1 \leq |i-j| \leq k-2$.
2) $m+2 \leq i, j \leq m+2k-2$ and $|i-j| > k$.
Then G is perfect.

Proof: It is enough to prove that $\overline{G'}$ is perfect. By the construction of G', the sets $\{m+1, .., m+k-1\}$ and $\{m+k+1, .., m+2k-1\}$ are independent in $\overline{G'}$ and the vertices of the set $\{m+1, m+k, m+2k-1\}$ are adjacent in $\overline{G'}$ and $deg_{\bar{G'}}\ (m+k) = 2$. On the other hand every odd cycle of $\overline{G'}$ has vertices $m+1, m+k, m+2k-1$. Consider the mapping $c : V(\overline{G'}) \Rightarrow \{1, 2, 3\}$ defined by:

$$c(x) = \begin{cases} 1 & m+1 \leq x \leq m+k-1 \\ 2 & m+k+1 \leq x \leq m+2k-1 \\ 3 & x = m+k \end{cases}$$

c is a 3-coloring of $\overline{G'}$ and since $\omega(G') = 3$ we have $\omega(G') = \chi(G')$. Let G_1 be an induced subgraph of $\overline{G'}$. If one of the vertices $m+1, m+k$ or $m+2k-1$ is not in $V(G_1)$ then G_1 is bipartite graph and $\omega(G_1) = \chi(G_1)$, otherwise $\omega(G_1) = \chi(G_1) = 3$. Therefore $\overline{G'}$ is perfect.

Now since join of a complete graph to a perfect graph is perfect we have the following lemma.

Lemma 7 *Suppose G' is the graph in Lemma 6 and $G = K_m * G'$ then G is perfect..*

Proof of Theorem 5 Let H be the k-uniform c-perfect hypergraph constructed from the graph G of Lemma 7 by using Theorem 4. Let $V(G) = \{1, 2, \ldots, m + 2k - 1\}$ where the vertex set of K_m is label by $\{1, 2, \ldots, m\}$. We show that $\overline{H}$ is c-perfect too. First we prove that $A = \{m + 1, m + 2, .., m + 2k - 1\}$ is a clique of $\overline{H}$. Let $e = \{m + i_1, m + i_2, \ldots, m + i_k\}$, $1 \le i_j \le 2k - 1$, is a subset of A. If $i_1 = 1$ then since $m + 1$ is only adjecent to $m + k + 1$ in G' then e is not subset of a cliqe in G' and therefore it is not an edge of H. Similarly if $i_k = 2k - 1$ since m_{2k-1} is only adjecent to $m + k - 1$ then e is not an edge of H. Let for each $2 \le j \le k$, we have $2 \le i_j \le 2k - 2$. Since $k > 3$ there exist at least two integer i_{j_1} and i_{j_2} such that $|i_{j_1} - i_{j_2}| \le k$ therefore e is not a subset of a clique of G' and e is an edge of $\overline{H}$. On the other hand, A is a clique of $\overline{H}$ imply $\omega(\overline{H}) \ge \frac{2k-1}{k-1}$. Now consider the map

$$c : V(\overline{H}) \Rightarrow \{0, 1, \ldots, 2k - 1\}$$

$$c(x) = \begin{cases} k - 1 & x \le m \\ x - m - 1 & x > m \end{cases}$$

If we show that c is a $(2k-1, k-1)$- coloring of $\overline{H}$ then $\chi(\overline{H}) \le \frac{2k-1}{k-1}$ and since $\chi_c(\overline{H}) \ge \omega(\overline{H})$ we have $\chi_c(\overline{H}) = \omega(\overline{H}) = \frac{2k-1}{k-1}$. Let e be an edge of $\overline{H}$.

Case 1) e is a subset of A. Since A has $2k - 1$ elements then e has at least two vertices x and y such that $|x - y| = k$ and therefore $|c(x) - c(y)| = k$.

Case 2) Let $e \cap \{1, 2, \ldots, m\} \ne \emptyset$ and $x \in e \cap \{1, 2, \ldots, m\}$. If $m+1$ or $m+2k-1$ are in e then $c(x) - c(m + 1) = k - 1$ or $c(m + 2k - 1) - c(x) = k - 1$. Let $m + 1$ and $m + 2k - 1$ are not in e. Since e is not an edge of $\overline{H}$, there exist at least two vertices a and b, $m + 2 \le a, b \le m + 2k - 1$ such that $k - 1 \le |a - b| \le k$ and therefore $k - 1 \le |c(a) - c(b)| \le k$. Thus for every edge of $\overline{H}$ we find at least two vertices such that distance between their colors is between $k - 1$ and k. Hence c is a $(2k - 1, k - 1)$-coloring of $\overline{H}$.

Now suppose H' be an induced subhypergraph of $\overline{H}$. If $A \subset V(H')$ there is nothing to proof. Let there exists i, $0 \le i \le k - 1$ such that $m + k + i \notin v(H')$. Define map

$$c : V(H') \Rightarrow \{0, 1\}$$

$$c(i) = \begin{cases} 0 & 1 \le i \le m + k - 1 \\ 1 & \text{otherwise} \end{cases}$$

c is a 2- coloring of H'. Therefore $\overline{H}$ is c-perfect. Now since $\omega(H) \ge \frac{m}{k-1}$ and $\omega(\overline{H}) = \frac{2k-1}{k-1}$ proof is complete. □

Conjecture 1 *For every k,n and m there exists a k-uniform c-perfect hypergraph H such that $\omega(H) \ge m$, $\omega(\overline{H}) \ge n$ and $\overline{H}$ is c-perfect.*

References

[1] J.A. Bondy and P. Hell, *A Note on the Star Chromatic Number*, J. Graph Theory, Vol.**14**, No.**4**(1990),479-482.

[2] CH.Eslahchi and A.Rafiey, *Circular Chromatic Number Of Hypergraph*, Submitted

[3] R. Diestel, *Graph Theory*, Springer-Verlag New York, Inc.

[4] E. Steffen and X. Zhu, *On the star chromatic numbers of graphs*, Combinatorica, **16**(1996), 439-448.

[5] A. Vince, *Star chromatic number*, J. Graph Theory, **12**(1988), 551-559.

[6] B. West, *Introduction to Graph Theory*, Prentice Hall, 1998.

[7] X. Zhu, *Circular chromatic number, a survey*, Discrete Math, No.**1-3**, Vol.**229**(2001), 371-410.