Fixed Parameter Algorithms for Interval Vertex Deletion and Interval Completion Problems

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Abstract

We consider two classical problems related to interval graphs. Let G be an input graph with n vertices and m edges and let k be a fixed parameter. We provide a fixed parameter algorithm that decides whether it is possible to turn G into an interval graph by deleting at most k vertices from G. This solves an open problem posed by Marx [19]. The running time of the algorithm is $O(c^k n(n+m))$, $c = min\{18, k\}$.

We also provide an algorithm with running time $O(c^k n(n+m))$, $c = min\{17, k\}$ that transforms G into an interval graph by adding at most k edges to G if such a transformation is possible. Our algorithm improves the previous algorithm with running time $O(k^{2k} n^3 m)$ appeared in [24].

The algorithms are based on a structural decomposition of G into smaller subgraphs when G is free from small interval graph obstructions. The decomposition allows us to manage the search tree more efficiently.

1 Introduction

An interval graph is a graph G which admits an interval representation, i.e., a family of intervals I_v , $v \in V(G)$, such that $uv \in E(G)$ if and only if I_u and I_v intersect. Interval graphs have been characterized in many different ways [8, 9, 12, 18].

The following theorem is the best known characterization.

Theorem 1.1 *[18] G is an interval graph if and only if it contains no asteroidal triple and no induced cycle C_ℓ , $\ell \geq 4$.*

An asteroidal triple, AT, is an induced subgraph of G with three none adjacent vertices a, b, c such that for every permutation x, y, z of a, b, c there is a path between x, y outside the neighborhood of z. A graph G is chordal if it does not contain an induced cycle C_ℓ, $\ell \geq 4$. Cycle C_ℓ in G is induced if it does not have any chord; an edge in G joining two non-adjacent vertices

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of the cycle. In the rest of the paper for simplicity we assume the cycles are induced and instead of an induced cycle we say a cycle.

The k-interval deletion problem is : Given a graph G and integer k, one asks whether there is a way of deleting at most k vertices from G such that the resulting graph is interval.

The k-interval completion (minimum interval completion) problem is : Given a graph G and integer k, one asks whether there is a way of adding at most k edges to G such that the resulting graph is interval.

Both k-interval deletion problem and k-interval completion problem are known to be NP-hard [10, 16] when k is part of the input. The k-chordal completion and k-proper interval completion problem are defined respectively. These problems arise in area such as sparse matrix computations [11], database management [1, 23], computer vision [3], and physical mapping of DNA [11, 13]. Due to their practical applications they have been extensively studied.

A parameterized problem with parameter k and input size x that can be solved by an algorithm with runtime $f(k) \cdot x^{O(1)}$ is called a fixed parameter tractable (FPT) where $f(k)$ is a computable function of k (see [6] for an introduction to fixed parameter tractability and bounded search tree algorithms). An early result related to k-interval completion problem is due to Kaplan, Shamir and Tarjan [15]. They gave an FPT algorithm for k-chordal completion, k-strongly chordal completion, and k-proper interval completion problem. The first FPT algorithm with runtime $O(k^{2k}n^3m)$ for the k-interval completion problem was developed by Villanger, Heggernes, Paul and Telle [24].

The k-interval deletion problem was posed by D.Marx [19]. He considered the k-chordal deletion problem as follows. Given an input graph G and a parameter k, one asks whether there is a way of deleting at most k vertices from G such that the resulting graph becomes chordal. Marx deployed a heavy machinery to obtain an FPT algorithm for k-chordal deletion problem.

In the approximation world, there is no constant approximation algorithm for minimum interval completion problem. The first $O(log^2 n)$-approximation algorithm for minimum interval completion was obtained by Ravi, Agrawal and Klein [21] and then it was improved to an $O(logn \ \ loglogn)$- approximation by Even, Naor, Rao and Schieber [7] and finally to an $O(logn)$-approximation algorithm by Rao and Richa [20]. There are polynomial time algorithms for minimum interval completion on special classes of graphs. The minimum interval completion is polynomial time solvable on trees. Kuo and Wang [17] gave an $O(n^{1.77})$ algorithm minimum interval completion on trees and then it was improved to $O(n)$ algorithm by Diaz, Gibbons, Paterson and Torn [4].

We use deep structural graph theory analysis to obtain a single exponential FPT algorithms for the k-interval deletion problem and k-interval completion problem.

We consider simple, finite, and undirected graphs. For a graph G, $V(G)$ is the *vertex set* of G and $E(G)$ is the *edge set* of G. For every edge $uv \in E(G)$, vertices u and v are *adjacent* or *neighbors*. The *neighborhood* of a vertex u in G is $N_G(u) = \{v \mid uv \in E(G)\}$, and the *closed neighborhood* of u is $N_G[u] = N_G(u) \cup \{u\}$. When the context will be clear we will omit the subscript. A set $X \subseteq V(G)$ is called *clique* of G if the vertices in X are pairwise adjacent. A *maximal* clique is a clique that is not a proper subset of any other clique. For $U \subseteq V$, the *subgraph of G induced by U* is denoted by $G[U]$ and it is the graph with vertex set U and edge set equal

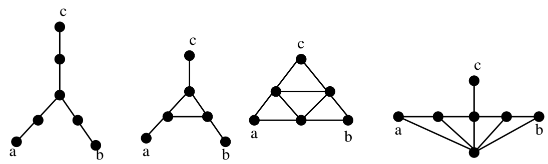

Figure 1: Some small ATs

to the set of edges $uv \in E$ with $u, v \in U$. For every $U \subseteq V$, $G' = G[U]$ is an *induced subgraph* of G. By $G \setminus X$ for $X \subseteq V$, we denote the graph $G[V \setminus X]$. For two disjoint subsets X, Y of $V(G)$, $S \subset G - (X \cup Y)$ is a (X, Y)-separator if there is no path from any vertex of X to any vertex in Y in $G \setminus S$. Let G_1, G_2 be two subgraph of graph G. For simplicity, we denotes $V(G_1) \cap V(G_2)$ by $G_1 \cap G_2$ and for subset X of $V(G)$, $X \cap G_1$ denotes $X \cap V(G_1)$. For a subset P of vertices in G, let $N(P)$ denote the neighborhood of P and $N[P]$ be the closed neighborhood of P. We do not use many non-standard terminologies and definitions and we refer to a standard text book in graph theory such as [5].

2 Outline

In the rest of this paper for simplicity we assume the cycles are induced and instead of an induced cycle we say a cycle. *We always refer to the cycles of length at least four unless we specify the length.* Note that according to the definition of AT, every cycle C_ℓ, $\ell \geq 6$ contains an AT. However for our purpose we distinguish the AT's and cycles. Suppose G contains a small induced subgraph Z which is either a small AT (See Figure 1) or a cycle of length at most nine. Then we consider all the possible ways of deleting (adding) one vertex (a few edges when Z is a cycle) from (to) Z and hence we can follow a search tree with at most 9 branches and obtain a FPT algorithm with run time $O(9^k n(m+n))$. This is a standard technique in developing FPT algorithms (For example see [2]). Thus in what follows we may assume that :
Every cycle in G has length at least 10 and G does not contain a small AT as an induced subgraph, i.e., G does not contain small obstructions.

Under this assumption if G does not contain any cycle then either G is an interval graph or it contains only two types of AT; so called *big AT*, depicted in Figure 2.

Let $S_{a,b,c}$ denote an AT over the vertices a, b, c. $S_{a,b,c}$ with the vertex set $a, b, c, u, v_1, v_2, \ldots, v_p$ and the edge set

$$E(S_{a,b,c}) = \{av_1, v_1v_2, \ldots, v_{p-1}v_p, v_pb, uv_1, uv_2, \ldots, uv_p, uc\}$$

is called type 1 AT. $S_{a,b,c}$ with vertex set $a, b, c, u, w, v_1, v_2, \ldots, v_p$ and the edge set

$$E(S_{a,b,c}) = \{au, bw, cu, cw, av_1, bv_p, uv_1, wv_1\} \cup \{v_iv_{i+1}, uv_{i+1}, wv_{i+1} | 1 \leq i \leq p-1\}$$

is called type 2 AT. Here $p \geq 6$.

Definition 2.1 *For AT $S_{a,b,c}$, let $G[a,b,c]$ be the induced subgraph of G on the vertices outside the neighborhood of c and adjacent to some vertices in $\{v_3, v_4, \ldots, v_{p-2}\}$. We say $S_{a,b,c}$ is* ripe *if $G[a,b,c]$ is an interval graph.*

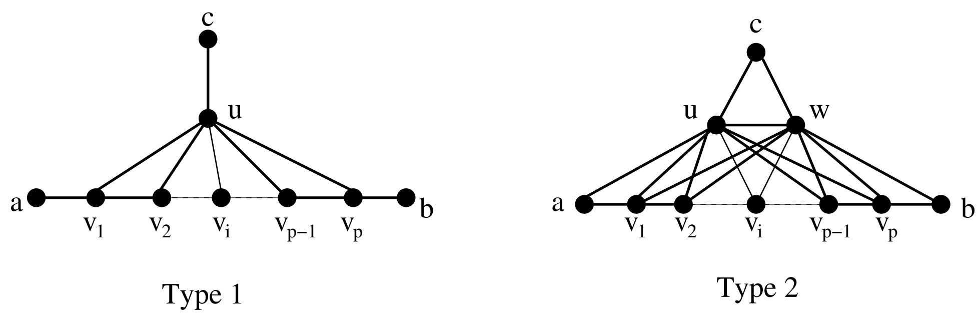

Figure 2: Big ATs

An overview of the k-interval deletion algorithm:

There are two main steps in the algorithm.

Step 1) *G is a chordal graph.*

If G is not interval then according to our assumption it contains a big AT. We show that there exists a *ripe AT* in G. The algorithm starts with a ripe AT $S_{a,b,c}$, and it proceeds as follows.

- Branch by deleting one of the vertices $\{a, b, u, c, v_1, v_2, v_3, v_4, v_5, v_6, v_{p-5}, v_{p-4}, v_{p-3}, v_{p-2}, v_{p-1}, v_p\}$
- Branch by deleting all the vertices in X, where X is a minimum set of vertices outside the neighborhood of c that separates v_6 from v_{p-5} outside the neighborhood of c

For the correctness we show the following lemma.

Lemma 2.2 *Let G be a chordal graph without small AT's and let $S_{a,b,c}$ be a ripe AT. Let X be a minimum separator in $G - N(c)$ that separates v_6 from v_{p-5} and X contains a v_j, $7 \leq j \leq p-6$. Then there is a minimum set of deleting vertices F such that $G - F$ is an interval graph and at least one of the following holds:*

(i) F contains at least one vertex from

$$\{a, b, u, c, v_1, v_2, v_3, v_4, v_5, v_6, v_{p-5}, v_{p-4}, v_{p-3}, v_{p-2}, v_{p-1}, v_p\}$$

(iii) F contains all the vertices in X.

Briefly speaking we start with a chordal graph G and find a minimum big AT, S_{a_0,b_0,c_0} (the length of the path between a_0, b_0 is minimum among all other AT's). The interaction of S_{a_0,b_0,c_0} with the rest of the graph is special. Now we look at $G[a_0, b_0, c_0]$ and if there is an AT in $G[a_0, b_0, c_0]$ then we continue our search with that AT. Eventually we find a ripe AT $S_{a,b,c}$ (the details are in section 3).

The none trivial part of the algorithm is the decision for deleting a vertex v_j, $7 \leq j \leq p-6$. We show that if v_j belongs to a minimum separator X that separate v_6 from v_{p-5} in $G \setminus N(c)$ then deleting v_j is a right a decision. The intuition behind the separator is, in order to obtain an interval graph we must break all the paths from a to b outside the neighborhood of c.

In order to prove the Lemma 2.2 we need to investigate the vertex intersection of some other AT $S_{x,y,z}$ with $S_{a,b,c}$. We show that if $S_{x,y,z}$ has intersection with $\{v_7, v_8, \ldots, v_{p-6}\}$ then X also contains a deleting vertex (a possible solution) for $S_{x,y,z}$. The ripeness of AT $S_{a,b,c}$ allows us to show that there are three possible nice configurations (see subsection 3.1 AT and AT interaction and Figures 4,5,6). The most important vertex intersection is shown in Figure 4 that facilitates the proof of the Lemma 2.2. The most important vertex intersection is shown in Figure 5 that facilitates the proof of the Lemma 2.2. The bound $6, \ldots, p-5$ helps to prove that if AT $S_{x,y,z}$ has a vertex in $\{v_7, v_8, \ldots, v_{p-6}\}$ then we must have one of the configurations depicted in Figures 4,5,6 (see Lemma 3.18) otherwise the entire $S_{x,y,z}$ lies in $G[a, b, c]$ and it contradict the ripeness of $S_{a,b,c}$.

Step 2) *G is not chordal.* Let C be a shortest cycle in G. The absence of the small obstructions allows us to partition the vertices in $N[C]$ into two sets $D(C)$ and $N[C] \setminus D(C)$, such that every vertex x in $D(C)$ is adjacent to every vertex in $N[C] - x$. We start with the following definition.

Definition 2.3 *We say a shortest cycle C ($10 \leq |C|$) of G is* **clean** *if for every cycle C_1 ($10 \leq |C_1|$) in $N[C] - D(C)$, every vertex in C_1 is adjacent to at most three consecutive vertices in C and $N[C_1]$ contains C, i.e., C_1 goes around C. We say C is* **ripe** *if it is clean and it does not contain any AT in its closed neighborhood.*

Statement 1. If G is not chordal then there exists a clean cycle C in G.

In order to obtain a clean cycle we start with an arbitrary shortest cycle C_0 and then we show that for every other cycle C_1 in $N[C_0]$, either $V(C_0) \subseteq N[C_1]$ or $V(C_1)$ is contained in the neighborhood of at most two consecutive vertices of C_0. If the first case happens then C_0 is the desired cycle otherwise the search for a clean cycle is continued in the subgraph of G induced by $N[C_1] - V(C_0)$.

Statement 2. Consider a clean cycle C that is not ripe. Let $S_{a,b,c}$ be a big AT such that $V(S_{a,b,c}) \subseteq N[C]$ ($N[C]$ is the closed neighborhood of C). Then we can assume that $S_{a,b,c}$ lies in the union of the neighborhood of at most three consecutive vertices of C.

Step 2.1) Start with a clean cycle C. If C is not ripe then consider big AT $S_{a,b,c}$ in $N[C]$ and let u, v, w be three consecutive vertices of C such that $N[\{u, v, w\}]$ contains the vertex set $S_{a,b,c}$. Apply the algorithm for the chordal case on the subgraph of G induced by $N[\{u, v, w\}]$. This procedure is repeated as long as C is not ripe.

Step 2.2) Start with a ripe cycle C. Find a minimum set X of vertices in $N[C] - D(C)$

whose deletion break all the cycles in $N[C] - D(C)$.

Set X is called a minimum *cycle-separator*. At this point the algorithm either deletes all the vertices in C, or it deletes all the vertices in X at once. We show that the choice of set X is arbitrary. For the correctness of Step 2 we show the following lemma.

Lemma 2.4 *Let C be a ripe cycle and let X be a minimum cycle-separator in $N[C]$. Then there is a minimum set of deleting vertices F such that $G - F$ is an interval graph and at least one of the following holds:*

(i) F contains all the vertices of the cycle C.

(ii) F contains all the vertices in X.

Let $C = v_0, v_1, \ldots, v_{p-1}, v_0$. In order to find set X, for every $0 \leq i \leq p-1$ we find a minimum set of vertices X_i that separates v_i from v_{i+3} in $W_i = N[\{v_{i+1}, v_{i+2}\}] - D(C)$. X is the smallest set X_i. Note that W_i is an interval graph since C is ripe.

In order to prove the lemma we need to analyze the interaction of ripe cycle C with the others big AT's. We show that if a vertex of big AT $S_{x,y,z}$ belongs to C then for every vertex v in C one of the vertices of $S_{x,y,z}$ can be replaced by v to obtain a new AT (For example $S_{x,y,v}$ is an AT). This justifies the first item of the lemma (See Figure 8). If no such $S_{x,y,z}$ exists then item (ii) of the lemma is justified.

The overall complexity of the algorithm for k-interval deletion is $O(c^k n(m+n))$ where $c = \min\{18, k\}$. By using slightly more restricted definition for ripe AT we can get a better running time $O(12^k n(n+m))$.

An overview of the k-interval completion algorithm :

Suppose input graph G contains cycle C of length at least 4. In order to obtain an interval graph we must add a set of at least $|C| - 3$ edges into vertices of C, or equivalently we need to triangulate cycle C. It is not difficult to see that there are at most $O(4^{|C|-3})$ different ways of triangulating cycle C. Thus we branch on all different ways of triangulating cycle C, and after each of them the parameter k decreases by $|C| - 3$. As explained before we handle the small AT's by branching on possible add edges (at most 8 possible ways).

For the sake of clarification and simplicity we just explain what we do when dealing with AT of type 1. The algorithm treats the type 2 AT very similar to the type 1.

We need to add at least one edge e to $S_{a,b,c}$ such that $S_{a,b,c} \cup \{e\}$ is no longer induces an AT in G. We add one of the edges cv_i, $1 \leq i \leq 6$ or one of the edges cv_i, $p-5 \leq i \leq p$ or we add one of the edges au, bu, av_p, bv_1 (and ab if of type 2). If we add edge av_p then we need to find a minimal triangulation of the cycle $a, v_1, \ldots, v_p, a$ (However we show that we can assume that this triangulation has a special form, but considering any minimal triangulation would be fine). The main non-trivial case is a decision for adding an edge cv_j for some $7 \leq j \leq p-6$. We show that we can add the edge cv_j when v_j belongs to a minimum (v_6, v_{p-5})-separator outside the neighborhood of c. This allows us to get a single exponential FPT algorithm (see Figure 3). We prove the following lemma.

Lemma 2.5 *Let G be a chordal graph without small ATs and let $S_{a,b,c}$ be a ripe AT with the path $P_{a,b} = a, v_1, v_2, \ldots, v_p, b$. Let X be a minimum separator in $G - N(c)$ that separates v_6 from v_{p-5} and it contains a vertex v_i, $7 \leq j \leq p-6$. Then there is a minimum set of edges F such that $G \cup F$ is an interval graph and at least one of the following holds:*

(i) F contains at least one edge from

$$\{bu, au, cv_1, cv_2, cv_3, cv_4, cv_5, cv_6, cv_{p-5}, cv_{p-4}, cv_{p-3}, cv_{p-2}, cv_{p-1}, cv_p\}$$

(ii) F contains all the edges av_i, $2 \leq i \leq p$ (and ab when $S_{a,b,c}$ is of type 2)

(iii) F contains all the edges bv_i, $1 \leq i \leq p-1$ (and ab when $S_{a,b,c}$ is of type 2)

(iv) F contains all edges $E_X = \{cx | x \in X\}$.

We need to take into account two issues. There might be a situation in which the edge cv_r $7 \leq r \leq p-6$, $r \neq j$ is also an add edge (part of a optimal solution) for some other AT $S_{x,y,z}$. We investigate the AT and AT (add) edge in common and we show that there are two possible configurations (see Figures 11 and 12). The most interesting configuration is when $z = c$ and v_r is a vertex of the path $P_{x,y}$. In this case we show that at least one edge of E_X is an add edge for $S_{x,y,z}$ and hence the item (iv) of the above lemma is justified. The second issue is when we add the edge cv_j to $S_{a,b,c}$ we might create new AT's and hence the choice of cv_j matters. Here again the ripeness of $S_{a,b,c}$ plays an important role. In fact the set X is a clique containing v_j and adding edges E_X would not yield a new AT with the vertex set $V(S_{a,b,c}) \cup V(G[a,b,c])$. We further show that the optimal solution also has to treat an AT $S_{a,b,c'}$ (where cc' is an edge of G, and path $P_{a,b}$ is the same in both AT's) similar to $S_{a,b,c}$ and hence we conclude the lemma.

Overall the running time of the algorithm is $O(c^k n(n+m))$, $c = \min\{17, k\}$. By using slightly more restricted definition for ripe AT we can get a better running time $O(11^k n(n+m))$.

The paper is organized as follows. In Section 3 we investigate the structure of a chordal graph G which does not contain small ATs as induced subgraphs. We start with a minimum AT, and then we obtain a ripe AT $S_{a,b,c}$. Next we consider the interaction (vertex intersection) of another minimum AT, $S_{x,y,z}$ with $S_{a,b,c}$. The $S_{a,b,c}$ and $S_{x,y,z}$ interact in a very particular way. In Section 4 we consider k-interval deletion problem. In Subsection 4.1 we consider the case when G is chordal and does not contain neither small ATs as induced subgraph. If G is chordal then the results in Section 3 with regard to the vertex interaction of $S_{a,b,c}$ and $S_{x,y,z}$ enable us to reduce the number of branches in a search tree into a constant number and hence we obtain an efficient FPT algorithm. In Subsection 4.2 we deal with the non-chordal case. We start with a shortest cycle C and we show that any other cycle (of length more than 8) in $N[C]$ interact with C in a special way due to absence of the small obstructions. The interaction between cycle C and AT $S_{a,b,c}$ is investigated and we show that either $S_{a,b,c}$ lies in the neighborhood of at most three consecutive vertices of C or the entire path $P_{a,b}$ of $S_{a,b,c}$ lies outside the neighborhood of C. Finally the main algorithm is presented at Section 4.3 and its correctness is proved. In Section 5 we consider the Interval completion problem. In Subsection 5.1 we further investigate the edge interaction of

$S_{a,b,c}$ and $S_{x,y,z}$, i.e., when $S_{a,b,c}$ and $S_{x,y,z}$ have an edge in common. The edge interaction occurs in a special way and we make a use of it to get a single exponential FPT for k-interval completion problem. In Subsection 5.2 we present the main algorithm for interval completion problem and we prove its correctness.

3 Structure when G is chordal and there are no small AT's

In this section we assume that G is chordal and it does not contain small AT's (See Figure 1). By the results of Lekkerkerker and Boland [18], every other possible minimal AT in G is one of two graphs depicted in Figure 2.

Let $S_{a,b,c}$ denote an AT with the vertices a, b, c, such that the path between a, c and the path between b, c are of length 2 and the path between a, b has length at least 7. Vertex c is called a *shallow vertex* .

Definition 3.1 *We say AT $S_{a,b,c}$ is of type 1 if $S_{a,b,c}$ has the vertex set $\{a, b, c, u, v_1, v_2, ..., v_p\}$ and the edge set*

$$\{av_1, cu, bv_p, uv_1\} \cup \{v_iv_{i+1}, uv_{i+1} | 1 \leq i \leq p-1\}.$$

Vertex u is called a center vertex. *We set $v_0 = a$ and $v_{p+1} = b$.*

Definition 3.2 *We say AT $S_{a,b,c}$ is of type 2 if $S_{a,b,c}$ has the vertex set $\{a, b, c, u, w, v_1, v_2, ..., v_p\}$ and the edge set*

$$\{au, bw, cu, cw, av_1, bv_p, uv_1, wv_1\} \cup \{v_iv_{i+1}, uv_{i+1}, wv_{i+1} | 1 \leq i \leq p-1\}.$$

The vertices u, w are called central vertices. *We set $v_0 = a$ and $v_{p+1} = b$.*

Let G' be an induced subgraph of G, and let $S_{a,b,c}$ be an AT in G'. We say $S_{a,b,c}$ is *minimum* if among all the AT, $S_{a',b',c'}$ in G' the path between a, b in $S_{a,b,c}$ has the minimum number of vertices and if there is a choice we assume that $S_{a,b,c}$ is of type 1. We denotes the path $a, v_1, v_2, \ldots, v_p, b$ by $P_{a,b}$.

Definition 3.3 *We say a vertex x is a dominating vertex for $S_{a,b,c}$ if x is adjacent to all the vertices $v_1, v_2, v_3, \ldots, v_{p-1}, v_p$.*

In the rest of this paper the set of dominating vertices for $S_{a,b,c}$ is denoted by $D(a, b, c)$. The following lemma shows the relationship of minimum $S_{a,b,c}$ with the other vertices of G.

Lemma 3.4 *Let G be a chordal graph without small ATs. Let $S_{a,b,c}$ be a minimum AT with a path $P_{a,b} = a, v_1, v_2, \ldots, v_p, b$. Let x be a vertex in $G \setminus S_{a,b,c}$. Then the following hold.*

(1) If cx is an edge of G then ux is an edge of G when $S_{a,b,c}$ is of type 1, and xu, xw are edges of G when $S_{a,b,c}$ is of type 2.

(2) If $S_{a,b,c}$ is of type 1 and x is adjacent to v_j, for some $2 \leq j \leq p-1$, then x is adjacent to u.

(3) If $S_{a,b,c}$ is of type 2 and x is adjacent to v_j, for some $1 \le j \le p$, then x is adjacent to both u and w.

(4) Let vertex x be adjacent to c. If x is adjacent to v_i, for some $0 \le i \le p+1$, then x is a dominating vertex for $S_{a,b,c}$.

(5) Every vertex $x \in G \setminus N(c)$ is adjacent to at most three vertices of the path $a, v_1, \ldots, v_p, b$. Moreover, the neighbors of x are consecutive vertices in the path $a, v_1, \ldots, v_p, b$.

(6) If $x \in G \setminus N(c)$ is adjacent to v_i, $3 \le i \le p-2$, then every vertex $y \in G \setminus N(c)$ adjacent to x, is also adjacent to at least one of the vertices v_j, $i-2 \le j \le i+2$.

(7) If x is adjacent to some v_i, $2 \le i \le p-1$, then x is adjacent to every dominating vertex y.

(8) If $x \in G \setminus N(c)$ is adjacent to some v_i, $2 \le i \le p-1$, and x is adjacent to some $y \in N(c)$, then y is a dominating vertex .

Proof: (1). Let us first suppose that $S_{a,b,c}$ is of type 1. If xu is not an edge of G, then x should be adjacent to at least one of the vertices v_1, v_p, a, and b, because otherwise vertices x, c, u, v_1, v_p, a, b induce a small AT in G. If xa is an edge, then because G is chordal, the cycle induced by $\{x, c, u, v_1, a\}$ should have chord xv_1. Similarly, if $\{x, b\}$ are adjacent, so should be $\{x, v_p\}$. But neither xv_1, nor xv_p can form an edge of G because otherwise we obtain an induced 4-cycle x, c, u, v_1 or x, c, v_p, b in chordal graph G.

Now suppose that $S_{a,b,c}$ is of type 2. Targeting towards a contradiction, let us assume that xw is not an edge. Then xb is not an edge because otherwise x, c, w, b would induce C_4 in G. Furthermore, xv_1 is not an edge because otherwise C_4 is induced by vertices x, c, w, and v_1. We also note that xa is not an edge as otherwise x, a, v_1, w, c would induce C_5 in G. Thus if x is not adjacent to w, then x cannot be adjacent to a, b and v_1. But then set $\{x, c, u, w, v_1, v_p, a, b\}$, even when x and u are adjacent, induces a small AT in G, which is a contradiction. Similar argument implies that xu is an edge.

(2). If xu is not an edge then by (1), vertices x and c are not adjacent. Then vertex x has at most three neighbors among the vertices of path $P_{a,b}$. This is because otherwise, there will be a shorter (a,b)-path in G passing through x and avoiding the closed neighborhood of c. But then vertices of this paths together with u and c induce an AT $S'_{a,b,c}$ of size smaller than the size of $S_{a,b,c}$. This is a contradiction to the choice of $S_{a,b,c}$. Thus x has at most three neighbors in $P_{a,b}$. Let v_i, $i \le j$, be the leftmost neighbor of x in $P_{a,b}$, and v_k, $k \le j$, be the rightmost neighbor. We observe that $k - i \le 2$, because otherwise we obtain cycle of length at least four in G. Because G has no small ATs and thus $n \ge 7$, we have that either $i \ge 2$, and in this case vertices $a, v_1, v_2, ..., v_i, x, c, u$ induce a smaller AT than $S_{a,b,c}$, or $k \le p-1$, and then $x, v_k, v_{k+1}, ..., v_p, b, u, c$ form a smaller AT.

(3). The proof here is similar to the proof of (2).

(4). We prove the statement when $S_{a,b,c}$ is of type 1. The argument for when $S_{a,b,c}$ is of type 2 is similar. By (1), xu is an edge. If x is adjacent to v_i for some $0 \le i \le p-1$, then xv_{i+2} is also an edge of G as otherwise the vertices $a, v_1, \ldots, v_{i+2}, c, x$ induce a smaller AT $S_{a,v_{i+2},c}$. In this case we note that xv_{i+1} is also an edge because vertices x, v_i, v_{i+1}, v_{i+2} would induce C_4 otherwise. Similarly if x is adjacent to v_j, $2 \le j \le p+1$, then xv_{j-2} is an edge as otherwise the vertices $b, v_p, v_{p-1}, \ldots, v_{i-2}, c, x$ induce smaller AT $S_{a,v_{i-2},c}$. In this case we note that xv_{i-1} is also an edge as otherwise there would be an induced C_4 on x, v_i, v_{i-1}, v_{i-2}. By applying these arguments inductively, we obtain that x is adjacent to every v_i, for $2 \leq j \leq p-1$. Now if none from the pairs xv_1, xv_p is an edge, then $v_1, v_2, \ldots, v_p, x, c$ induce smaller AT $S_{v_1,v_p,c}$, a contradiction. Therefore we may assume that x should be adjacent either to v_1, or to v_p. Let us assume, without loss of generality, that x is adjacent to v_1. Now if xv_p is not an edge, then $a, v_1, v_2, \ldots, v_{p-1}, c, x$ is a smaller AT when ax is not an edge. We conclude that if xv_p is not an edge, then xa, xv_1 are edges of G. However $c, x, u, a, v_1, v_2, \ldots, v_p$ induce an AT $S_{a,v_p,c}$ of type 2 and the path between a, v_p is shorter the path between a, b in $S_{a,b,c}$, this is a contradiction. Therefore xv_p is an edge.

(5). If there was a vertex $x \in G \setminus N(c)$ adjacent to more than three vertices in the path $P_{a,b}$ then there is a shorter path between a, b using vertex x avoiding neighborhood of c. Thus we construct a smaller AT. The neighbors are consecutive vertices of the path because otherwise we obtain cycle of length at least four.

(6). If y is adjacent to none of the vertices $v_{i-2}, v_{i-1}, \ldots, v_{i+2}$, then vertices $y, x, v_i, v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}$ induce a smaller AT unless x is adjacent to v_{i-2} or v_{i+2}. Suppose that x is adjacent to v_{i-2}. Now by (6), x is adjacent to v_{i-1}. By (5), x cannot be adjacent to more than 3 vertices of the path $v_1, \ldots, v_p$, and thus x is not adjacent to v_{i-3} and v_{i+1}. Vertex y is not adjacent to v_{i-3} because vertices v_{i-3}, v_{i-2}, x, y do not induce a cycle. In this case vertices $v_{i-3}, v_{i-2}, v_{i-1}, v_i, v_{i+1}, x, y$ induce a small AT.

(7). If x is adjacent to c then by (4), x is a dominating vertex and hence x is adjacent to y as otherwise x, y, v_1, v_3 induces a C_4. So we may assume that $x \notin N(c)$. By (5), y should be adjacent to c. In this case, if x is not adjacent to y, then either $c, u, x, y, v_i, v_{i+1}, \ldots, v_p, b$, or $a, v_1, v_2, \ldots, v_i, y, x, u, c$ induce a smaller AT.

(8). If y is adjacent to at least one vertex v_i for some $0 \leq i \leq p+1$, then by (4) y is a dominating vertex for $S_{a,b,c}$. Let us assume that y is non-adjacent to all vertices v_i, $0 \leq i \leq p+1$. Now $S_{a,b,y}$ has exactly the same number of vertices as $S_{a,b,c}$, and thus is also a minimum AT. By applying item (4) for $S_{a,b,y}$ we conclude that x is a dominating vertex for $S_{a,b,y}$ and hence x is adjacent to more than three vertices in the path $P_{a,b} = a, v_1, \ldots, v_p, b$. This is a contradiction to (5) because by assumption $x \in G \setminus N(c)$.

◇

The following Lemma follows from item (4) of Lemma 3.4.

Lemma 3.5 *Let $S_{a,b,c}$ be a minimum AT with a path $P_{a,b} = a, v_1, v_2, \ldots, v_p, b$ and center vertex u (central vertices u, w). Let Q be a chordless path from c to some v_i, $0 \leq i \leq p+1$. Then the second vertex of Q is a dominating vertex for $S_{a,b,c}$. Moreover if $1 \leq i \leq p$ then the length of Q is 2.*

Proof: Let $Q = c, c_1, c_2, \ldots, c_r, v_i$ be a chordless path from c to v_i, $0 \leq i \leq p+1$. Suppose c_1 is adjacent to some vertex v_j, $0 \leq j \leq p+1$. Then by Lemma 3.4(4), c_1 is a dominating vertex for $S_{a,b,c}$ and hence c_1v_j is an edge for every $1 \leq j \leq p$. If $j \neq 0, p+1$ then c_1v_i is an edge and hence $r = 1$ and lemma is proved. Thus we assume that c_1 is not adjacent to any v_j, $0 \leq j \leq p+1$. By Lemma 3.4 (1), uc_1 is an edge. This implies that S_{a,b,c_1} is an AT with the same number of vertices as $S_{a,b,c}$. Now by applying the same argument for S_{a,b,c_1} we conclude that c_2 is a dominating vertex for S_{a,b,c_1}. However by item (6) of Lemma 3.4 for $S_{a,b,c}$, c_2 is a dominating vertex for $S_{a,b,c}$ and hence by item (5) of Lemma 3.4 we conclude that c_2 is adjacent to c. This is a contradiction to Q being a chordless path. $\diamond$

Let G be a chordal graph without small ATs. Let $S_{a,b,c}$ be a minimum AT in G. Then by item (5) of Lemma 3.4, every vertex x of $G \setminus N(c)$ has at most three neighbors in the $P_{a,b}$ path, and moreover, these neighbors should be consecutive vertices of this path. Note that we assume $v_0 = a$ and $v_{p+1} = b$. We introduce the following notations. We define the following subsets of $G \setminus N(c)$

- S_i vertices adjacent to v_i and not adjacent to any other v_j, $j \neq i$, $1 \leq i \leq p$;
- D_i vertices adjacent to v_i, v_{i+1} and not adjacent to any other v_j, $j \neq i, i+1$, $0 \leq i \leq p$;
- T_i vertices adjacent to v_i, v_{i+1}, v_{i+2}, $0 \leq i \leq p-1$.

The following corollary is obtained from Lemma 3.4 (1,7,8).

Corollary 3.6 *Let $S_{a,b,c}$ be a minimum AT. Then the vertices in $D(a,b,c)$ form a clique. Every vertex adjacent to c is also adjacent to every dominating vertex. Moreover every vertex in $D(a,b,c)$ is also adjacent to c.*

Definition 3.7 *For minimum AT, $S_{a,b,c}$ let $B[a,b]$ be the set of vertices in $D_0 \cup T_0 \cup D_1 \cup S_1 \cup \{v_1\} \cup S_2$ and $E[a,b]$ be the set of the vertices in $S_{p-1} \cup D_{p-1} \cup T_{p-1} \cup D_p \cup S_p \cup \{v_p\}$.*

Definition 3.8 *For minimum AT, $S_{a,b,c}$ let $G[a,b,c] = G[\{x | x \in N[v_i] \setminus N(c); 3 \leq i \leq p-2\}]$.*

Since every vertex in $G[a,b,c]$ is adjacent to some v_i, $3 \leq i \leq p-2$ by Lemma 3.4(7) we have the following.

Corollary 3.9 *Every vertex in $G[a,b,c]$ is adjacent to every vertex in $D(a,b,c)$.*

Lemma 3.10 *Let x be a vertex adjacent to some vertex in $G[a,b,c]$. Then x is adjacent to every vertex in $D(a,b,c)$.*

Proof: If $x \in N(c)$ then by Corollary 3.6 the Lemma holds. Therefore we may assume that $x \notin N(c)$. Let xx' be an edge of G for some $x' \in G[a,b,c]$. By definition of $G[a,b,c]$, x' is adjacent to some v_i, $3 \leq i \leq p-2$. By Lemma 3.4 (6), x is adjacent to some v_j, $i-2 \leq j \leq i+2$. If x is adjacent to one of the v_{i-1}, v_i, v_{i+1} then by Lemma 3.4(7) x is adjacent to every vertex in $D(a,b,c)$.
Therefore w.l.o.g assume that x is adjacent to v_{i-2} and not adjacent to any of v_{i-1}, v_i. Now we observe that x' is adjacent to v_{i-2}, v_{i-1}, v_i as otherwise we obtain an induced C_4 or induced C_5 with the vertices $v_{i-2}, v_{i-1}, v_i, x', x$. Now by replacing v_{i-1} with x' we obtain a minimum AT $(S_{a,b,c})'$ with the same number of vertices as $S_{a,b,c}$, and path $P'_{a,b} = a, v_1, \ldots, v_{i-2}, x'$, $v_i, \ldots, v_p, b$. Note that $2 \leq i-1 \leq p-1$. Thus the set $D(a,b,c)$ is also the set of dominating vertices for $(S_{a,b,c})'$. Now because xx' is an edge Lemma 3.4(7) for $(S_{a,b,c})'$ implies that x is adjacent to every vertex in $D(a,b,c)$.

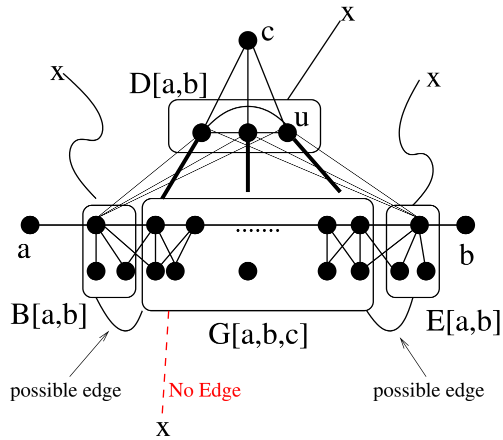

Figure 3: G[a,b,c] and outside

Lemma 3.11 *Let $S_{a,b,c}$ be a minimum AT. Then $D(a,b,c) \cup B[a,b] \cup E[a,b]$ separates $G[a,b,c]$ from the rest of the graph.*

Proof: We need to show that if $x \in G \setminus (\{v_1, v_p\} \cup D_0 \cup S_1 \cup D_1 \cup T_0 \cup S_p \cup D_{p-1} \cup T_{p-1} \cup D_p \cup D(a,b,c) \cup V(G[a,b,c]))$ then there is no edge from x to $y \in G[a,b,c]$. For contradiction suppose xy is an edge. We also note that $x \notin N(c)$ as otherwise by Lemma 3.5, x is a dominating vertex for $S_{a,b,c}$ and we get a contradiction. By definition of $G[a,b,c]$, y is adjacent to v_i, $3 \le i \le p-2$. First suppose $y \in \{v_2, v_{p-1}\}$. Now x is adjacent to v_2 or x is adjacent to v_{p-1}. Since x is adjacent to at most three consecutive vertices on the path $P_{a,b}$, x lies in $\{v_1, v_p\} \cup S_2 \cup T_0 \cup S_p \cup D_{p-1} \cup T_{p-1} \cup D_p$. This implies that $x \in B[a,b] \cup E[a,b]$. We continue by assuming that $y \in V(G[a,b,c]) \setminus \{v_2, v_{p-1}\}$. Now we apply Lemma 3.4(6) for y and we conclude that x is adjacent to one of the vertices $v_{i-2}, v_{i-1}, v_i, v_{i+1}, v_{i+2}$. Therefore x is in $N[v_r]$, $1 \le r \le p$. We observe that $r \in \{1, 2, p-1, p\}$ as otherwise by definition x is in $G[a,b,c]$. Therefore $x \in B[a,b] \cup E[a,b]$. $\diamond$

The following Lemma is obtained by applying similar argument in Lemma 3.5.

Lemma 3.12 *Let $S_{a,b,c}$ be a minimum AT with a path $P_{a,b} = a, v_1, v_2, \ldots, v_p, b$ and center vertex u (u, w). Then every chordless path from c to $d \in G[a,b,c]$ has length 2 and the intermediate vertex of this path is a dominating vertex for $S_{a,b,c}$.*

Lemma 3.13 *Let $S_{x,y,z}$ be a minimum AT in $G[a,b,c]$ with a path $P_{x,y} = x, w_1, w_2, \ldots, w_q, y$ ($x = w_0$, $y = w_{q+1}$) and center vertex u' (central vertices u', w' if of type 2). Then there exists $2 \le i \le p-1$, such that v_i is a dominating vertex for $S_{x,y,z}$.*

Proof: If $u' \in \{v_{i-1}, v_i, v_{i+1}\}$ then $u' = v_j \in \{v_{i-1}, v_i, v_{i+1}\}$ is a dominating vertex for $S_{x,y,z}$. Note that by definition of $G[a,b,c]$ we have $2 \le j \le p-1$. Therefore we may assume that u' is not on the path $P_{a,b}$.

We first show that $v_i \ne z$. For contradiction suppose $z = v_i$. Observe that the conditions of the Lemma 3.4(1) are applied for $S_{x,y,z}$ and hence v_{i-1} is adjacent to u' and v_{i+1} is adjacent to u'. First suppose v_{i+1} is not adjacent to any vertex w_j, $0 \le j \le q+1$. By replacing v_i with v_{i+1} we get a minimum AT, $S_{x,y,v_{i+1}}$ with the same number of vertices as S_{x,y,v_i}, and hence by

Lemma 3.4(1), v_{i+2} must be adjacent to u'. This implies that u' is adjacent to more than three vertices of the path $P_{a,b}$. Since $u' \notin N(c)$, we get a contradiction by Lemma 3.4(5). Therefore v_{i+1} must be adjacent to some w_j. Similarly we conclude that v_{i-1} must be adjacent to some $w_{j'}$, $0 \leq j' \leq q+1$. By applying the item (4) of Lemma 3.4 for S_{x,y,v_i}, v_{i-1} is a dominating vertex for S_{x,y,v_i} and similarly v_{i+1} is also a dominating vertex for $S_{x,y,v_{i+1}}$. But this is a contradiction because by Corollary 3.6 $v_{i-1}v_{i+1}$ is an edge. Therefore we have the following fact.

(f) For every minimum AT, $S_{x',y',z'} \subseteq G[a,b,c]$ we have $z' \neq v_i$, $2 \leq i \leq p-1$.

Now suppose $z \in S_i \cup D_i \cup T_i$ and $v_i z$ is an edge of G. Note that $v_i u'$ is also an edge by Lemma 3.4(1). Now if v_i is not adjacent to any vertex of the path $P_{x,y}$ then S_{x,y,v_i} is also a minimum AT with the same number of vertices as $S_{x,y,z}$ and we get a contradiction by (f). Thus we conclude that v_i is adjacent to some vertex w_j, $0 \leq j \leq q+1$ and hence by Lemma 3.4(4), v_i is a dominating vertex for $S_{x,y,z}$.

$\diamond$

Definition 3.14 *We say an AT, $S_{a,b,c}$ is* ripe *if there is no AT in $G[a,b,c]$, i.e., $G[a,b,c]$ is an interval graph.*

Remark : Note that a ripe AT may not be necessary a minimum AT. But for the purpose of the algorithm we often use a minimum ripe AT and by that we mean an AT which is ripe and it is minimum among all the ripe AT' in a subgraph of G. In what follows when we say ripe AT we mean minimum ripe AT.

Looking for a ripe AT, starting with a minimum AT S_{a_0,b_0,c_0}.

We start with minimum AT, S_{a_0,b_0,c_0}. If $G[a_0,b_0,c_0]$ is interval then S_{a_0,b_0,c_0} is the answer. Otherwise we find a minimum AT, S_{a_1,b_1,c_1} in $G[a_0,b_0,c_0]$. Note that according to Lemma 3.13 there is a vertex v_i (on the path P_{a_0,b_0}) that is a dominating vertex for S_{a_1,b_1,c_1}. We say S_{a_0,b_0,c_0} dominates S_{a_1,b_1,c_1} at v_i. Now we define the sets S_i^1, D_i^1, T_i^1 with respect to the vertices on the path $P_{a_1,b_1} = a_1, w_1, w_2, \ldots, w_q, b_q$ (the same way we define them for S_{a_0,b_0,c_0}). If necessary the search is continued for other AT dominated by S_{a_1,b_1,c_1}. See the Algorithm 1 for more details.

Algorithm 1 Looking for a ripe AT

1. Start with an arbitrary minimum AT, S_{a_0,b_0,c_0}, and set $i = 0$, $G_0 = G$.
2. Define $G_i[a_i,b_i,c_i]$ in $G_i \setminus N(c_i)$ (see definition 3.8) and set $G_{i+1} = G_i[a_i,b_i,c_i]$.
3. If there is no AT in G_{i+1}, report S_{a_i,b_i,c_i} as a ripe AT and exit.
4. If $i > k$ then report NO solution and exit.
5. Let $S_{a_{i+1},b_{i+1},c_{i+1}}$ be a minimum AT, in G_{i+1}
6. increase i by one and go to (2).

Lemma 3.15 *The Algorithm 1 reports a ripe AT and terminates after at most k steps.*

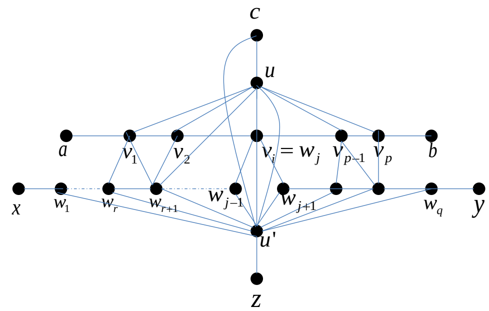

Figure 4: $u' \in D(a,b,c)$ and $P_{x,y} \cap B[a,b] \neq \emptyset$, $P_{x,y} \cap E[a,b] \neq \emptyset$

Proof: Suppose $S_{a,b,c}$ dominates S_{a^1,b^1,b^1} at some vertex v_i. Then by Lemma 3.13, v_i is a dominating vertex for S_{a^1,b^1,c^1}. If S_{a^1,b^1,c^1} also dominates S_{a^2,b^2,c^2} at some vertex w_j on the path P_{a^1,b^1} then by Lemma 3.4 item (2) or (3) the vertices of S_{a^2,b^2,c^2} are all adjacent to v_i (since v_i is a dominating vertex for $S_{a,b,c}$). Now it is easy to see that S_{a^r,b^r,c^r} does not dominate $S_{a,b,c}$, as otherwise the vertices on the path $S_{a,b,c}$ must be all adjacent to some vertex in the neighborhood of v_i which is not possible. Therefore there is no domination from an AT at step i in the Algorithm 1 to an AT at step $j < i$. Thus the algorithm reports a ripe AT after at most k steps. Note that the number of AT's found in the Algorithm 1 can not be more than k. $\diamond$

3.1 AT and AT interaction

Remark : In the following three Lemmas we consider the vertex intersection of a minimum AT, $S_{x,y,z}$ with a ripe AT, $S_{a,b,c}$. There are only four possible interaction configurations for these two ATs. In two of these configurations the central vertex (vertices) of $S_{x,y,z}$ lie in dominating set of $S_{a,b,c}$. In two of these configurations the path $P_{x,y}$ has no intersection with $G[a,b,c]$ and in one situation every vertex in $P_{a,b}$ has a neighbor in $P_{x,y}$. In two situations if $V(S_{x,y,z}) \cap N[v_i] = T$, for some $7 \leq i \leq p-6$ then $T = \{x\}$ or $T = \{y\}$. See the Figures 4,5,6.

Lemma 3.16 *Let $S_{a,b,c}$ be a ripe AT. Let $S_{x,y,z}$ be a minimum AT with a path $P_{x,y} = x, w_1, \ldots, w_q, y$, and center vertex (central vertices) u' (u', w') such that $u' \in D(a,b,c)$ $(u', w' \in D(a,b,c)$ if of type 2) and $V(S_{x,y,z}) \cap G[a,b,c] \neq \emptyset$. Then one of the following happens:*

1. *$P_{x,y} \cap B[a,b] \neq \emptyset$ and $P_{x,y} \cap E[a,b] \neq \emptyset$ and every v_i, $1 \leq i \leq p$ has a neighbor in $P_{x,y}$ (See the Figure 4).*
2. *$z \in G[a,b,c]$ and $P_{x,y} \cap G[a,b,c] = \emptyset$ and for every vertex $z' \in G[a,b,c]$, $S_{x,y,z'}$ is an AT with the same path $x, w_1, w_2, \ldots, w_q, y$ (See the Figure 5).*

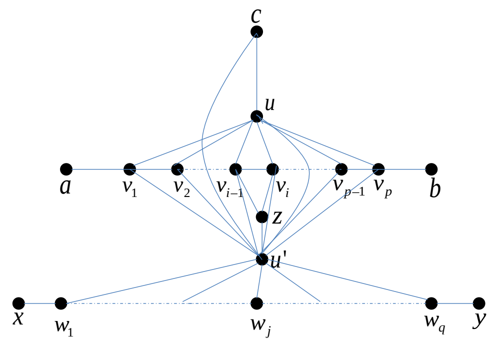

Figure 5: $z \in G[a, b, c]$ and $P_{x,y} \cap G[a, b, c] = \emptyset$

Proof: By Corollary 3.9 every dominating vertex is adjacent to every vertex in $G[a, b, c]$. Therefore none of the x, y is in $G[a, b, c]$ as otherwise xu' (xw' when $S_{a,b,c}$ is of type 2) or yu' is an edge. Moreover by Corollary 3.6 $x, y \notin D(a, b, c)$.

Now since $S_{x,y,z}$ has intersection with $G[a, b, c]$, we have two cases:

Case 1. $P_{x,y} \cap G[a, b, c] \neq \emptyset$. There exists some w_j, such that $w_j \in G[a, b, c]$. We show that $2 \leq j \leq q - 1$. Otherwise w.l.o.g assume that $w_1 \in G[a, b, c]$. Since xw_1 is an edge, by Lemma 3.10 x is adjacent to every vertex in $D(a, b, c)$ and in particular x is adjacent to u' (u', w') and hence we get a contradiction.

We continue by assuming that $w_j \in G[a, b, c]$ and $2 \leq j \leq p-2$. By definition of $G[a, b, c]$, w_j is adjacent to some vertex v_i, $3 \leq i \leq p - 2$.

We first show that $P_{x,y} \cap D(a, b, c) = \emptyset$. For contradiction suppose $w_t \in D(a, b, c)$, $1 \leq t \leq q$. Now by Corollary 3.9 w_t is adjacent to w_j and hence $t = j + 1$ or $t = j - 1$. W.l.o.g assume that $t = j + 1$. Since $w_{j-1}w_j$ is an edge of G and w_{j+1} is a dominating vertex for $S_{a,b,c}$, by Lemma 3.10, $w_{j-1}w_{j+1}$ is an edge of G, a contradiction. Therefore $P_{x,y} \cap D(a, b, c) = \emptyset$.

Since $x, y \notin G[a, b, c]$ and no vertex of $P_{x,y}$ is in $D(a, b, c)$, by Lemma 3.11 we conclude $B[a, b] \cap P_{x,y} \neq \emptyset$ or $E[a, b] \cap P_{x,y} \neq \emptyset$.

Observation 1. If for some v_i, $N[v_{i+1}] \cap P_{x,y} \neq \emptyset$ and $N[v_{i-1}] \cap P_{x,y} \neq \emptyset$ then $N[v_i] \cap P_{x,y} \neq \emptyset$. Otherwise we get cycle of length at least four with the vertices v_{i-1}, v_i, v_{i+1} and part of $P_{x,y}$ from $N[v_{i+1}]$ to $N[v_{i-1}]$.

Now by Observation 1 if $B[a, b] \cap P_{x,y} \neq \emptyset$ and $E[a, b] \cap P_{x,y} \neq \emptyset$ we conclude (1). Therefore w.o.l.g assume that $B[a, b] \cap P_{x,y} = \emptyset$ and $E[a, b] \cap P_{x,y} \neq \emptyset$.

By Observation 1 and because $B[a, b] \cap P_{x,y} = \emptyset$ we conclude that there exists a maximum number $3 \leq r \leq p - 2$ such that $N[v_r] \cap P_{x,y} \neq \emptyset$ and for every $1 \leq \ell \leq r - 1$, $N[v_\ell] \cap P_{x,y} = \emptyset$. Now let i' be the first index such that $w_{i'}$ is in $N[v_r]$ and j' is the last index such that $w_{j'}$

is in $N[v_r]$. Recall that $2 \le i', j' \le q-1$ (Note that j' could be the same as i'). However $v_{r-2}, v_{r-1}, v_r, w_{i'}, w_{i'-1}, w_{i'-2}, w_{j'}, w_{j'+1}, w_{j'+2}$ induce a small AT.

Case 2. $P_{x,y} \cap G[a,b,c] = \emptyset$. Since $G[a,b,c] \cap V(S_{x,y,z}) \neq \emptyset$, $z \in G[a,b,c]$. By definition of $G[a,b,c]$; z is adjacent to some vertex v_i, $3 \le i \le p-2$. We show that v_i is not adjacent to any vertex w_j, $0 \le j \le q+1$. Otherwise by applying Lemma 3.4(4) for $S_{x,y,z}$; v_i is a dominating vertex for $S_{x,y,z}$ and now $v_i w_1 \in E(G)$ implies that w_1 is in $G[a,b,c]$ which is a contradiction. (Note that w_j is not in $D(a,b,c)$ since zw_1 is not an edge). Therefore S_{x,y,v_i} is minimum AT and has the same number of vertices as $S_{x,y,z}$ and the same path $P_{x,y}$.

Now by repeating the same argument for S_{x,y,v_j} starting from $j = i$ and vertex v_{j+1} and vertex v_{j-1} (if they are in the range, v_3 and v_{p-2}) we conclude that :

(f) For every $3 \le j \le p-2$, S_{x,y,v_j} is a minimum AT and the same number of vertices as $S_{x,y,z}$ and the same path $P_{x,y}$.

Now by applying similar argument for $z' \in G[a,b,c] \cap N(v_i)$; $3 \le i \le p-2$. We conclude that z' is not adjacent to any vertex w_r, $0 \le r \le q+1$. Otherwise by applying Lemma 3.4(4) for S_{x,y,v_i}, z' is a dominating vertex for S_{x,y,v_i}. We note that since S_{x,y,v_j} is a minimum AT with the same path $P_{x,y}$, z' is also a dominating vertex for S_{x,y,v_j} and hence by Corollary 3.6 z' is adjacent to v_j. This implies that z' is adjacent to every vertex v_ℓ, $3 \le \ell \le p-2$, contradiction to $z' \in G[a,b,c]$. Therefore z' is not adjacent to any vertex on the path $P_{x,y}$ and hence $S_{x,y,z'}$ is a minimum AT with the same number of vertices as $S_{x,y,z}$. The proof of this case is complete.

$\diamond$

Lemma 3.17 *Let x_1, x_2, x_3 be three vertices in $G \setminus N(c)$ such that $v_i x_1, x_1 x_2, x_2 x_3$; $7 \le i \le p-6$ are edges of G. Then $x_3 \in N[v_j]$, $i-3 \le j \le i+3$*

Proof: By Lemma 3.4(6), x_2 is adjacent to one of the v_j, $i-2 \le j \le j+2$. If x_2 is adjacent to one of the v_{i-1}, v_i, v_{i+1} then by applying Lemma 3.4(6) for x_2, x_3 we conclude that x_3 is adjacent to some v_r, $i-3 \le r \le i+3$ and we are done. Thus w.l.o.g we may assume that x_2 is adjacent to v_{i-2} and not adjacent to any of v_{i-1}, v_i. Now x_1 is adjacent to v_{i-2}, v_{i-1}, v_i as otherwise we get a cycle of length 4 or 5 with the vertices $x_2, x_1, v_{i-1}, v_i, v_{i-2}$. Because $x_2 v_{i-2}, x_2 x_3$ are edges of G and $5 \le i-2 \le p-4$ by Lemma 3.4 (6) we conclude that x_3 is adjacent to one of the $v_{i-4}, v_{i-3}, v_{i-2}$. If x_3 is adjacent to v_{i-3} or v_{i-2} then we are done. Thus we may assume that x_3 is adjacent to v_{i-4} and not adjacent to any of v_{i-3}, v_{i-2}. Now in this case x_2 must be adjacent to v_{i-4} as otherwise we obtain a small cycle with the vertices $x_2, x_3, v_{i-4}, v_{i-3}, v_{i-2}$. However $a, v_1, \ldots, v_{i-4}, x_2, x_1, v_i, \ldots, v_p, b$ is shorter than $P_{a,b}$, a contradiction. $\diamond$

Lemma 3.18 *Let $S_{a,b,c}$ be a ripe AT. Let $S_{x,y,z}$ be a minimum AT, with a path $P_{x,y} = x, w_1, w_2, \ldots, w_q, y$ and a center vertex u' (central vertices u', w' if of type 2) such that $S_{x,y,z} \cap (N[v_i] \setminus N(c)) \neq \emptyset$ for some $7 \le i \le p-6$. Then one of the following happens :*

1. *$u' \in D(a,b,c)$ ($u', w' \in D(a,b,c)$) and $P_{x,y} \cap B[a,b] \neq \emptyset$ and $P_{x,y} \cap E[a,b] \neq \emptyset$ and every v_j, $2 \le j \le p-1$ has a neighbor in $P_{x,y}$ (See Figure 4).*
2. *$u' \in D(a,b,c)$, ($u', w' \in D(a,b,c)$) $z \in G[a,b,c]$ and $P_{x,y} \cap G[a,b,c] = \emptyset$ and for every vertex $z' \in G[a,b,c]$, $S_{x,y,z'}$ is an AT with the same path $x, w_1, w_2, \ldots, w_q, y$ (See Figure 5).*

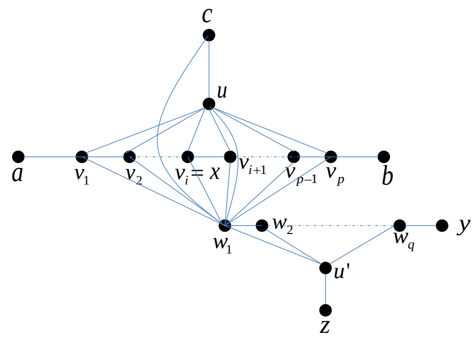

Figure 6: $x \in N[v_i]$, $w_1 \in D(a,b,c)$

3. *$x \in N[v_i]$, $w_1 \in D(a,b,c)$, ($u', w_1 \in D(a,b,c)$) and $V(S_{x,y,z}) \cap P_{x,y} \cap G[a,b,c] = \{x\}$ and for every $x' \in N[v_j] \setminus N(c)$; $5 \le j \le p-4$, $S_{x',y,z}$ is an AT with the path $P_{x',y} = x', w_1, \ldots, w_q, y$ (See Figure 6).*

4. *$y \in N[v_i]$, $w_q \in D(a,b,c)$, ($w', w_q \in D(a,b,c)$) and $V(S_{x,y,z}) \cap G[a,b,c] = \{y\}$ and for every $y' \in N[v_j] \setminus N(c)$; $5 \le j \le p-4$, $S_{x,y',z}$ is an AT with the path $P_{x,y'} = x, w_1, \ldots, w_q, y'$.*

Proof: First suppose u' (u', w' if $S_{x,y,z}$ is of type 2) the center vertex (central vertices) of $S_{x,y,z}$ is in $D(a,b,c)$. Since $V(S_{x,y,z}) \cap (N[v_i] \setminus N(c)) \neq \emptyset$, we have $G[a,b,c] \cap V(S_{x,y,z}) \neq \emptyset$. Thus the conditions of the Lemma 3.16 are satisfied and hence we have (1) or (2).

Therefore we may assume that $u' \notin D(a,b,c)$ when $S_{x,y,z}$ is of type 1 and $w' \notin D(a,b,c)$ when $S_{x,y,z}$ is of type 2. Recall that $x = w_0$ and $y = w_{q+1}$.

Case 1. Suppose w_j, $0 \le j \le q+1$ is in $N[v_i] \setminus D(a,b,c)$.

Claim 3.19 $w_j \in \{w_0, w_1, w_q, w_{q+1}\}$.

Proof: For contradiction suppose $2 \le j \le q-1$. Note that at least one of the w_1, w_q is not in $D(a,b,c)$, as otherwise w_1w_q is an edge by Corollary 3.9. W.l.o.g assume that $w_1 \notin D(a,b,c)$. By applying Lemma 3.4 (2,3) for $S_{x,y,z}$ we have that v_i is adjacent to u' (to w') as otherwise $S_{x,v_i,z}$ is a smaller AT and it has the condition of the lemma. Now we have the following implications.

(f_0) $u' \in N[v_i]$, ($w' \in N[v_i]$) and (f_1) $u' \notin N(c)$ ($w' \notin N(c)$).
Otherwise by Lemma 3.4(4) for the edges $cu', u'v_i$, ($cw', w'v_i$) $u' \in D(a,b,c)$ ($w' \in D(a,b,c)$).

(f_2) w_1 is not in $N(c)$.
Otherwise by applying Lemma 3.5 for c, w_1, u', v_i we conclude that $w_1 \in D(a,b,c)$. The same argument is applied using w' instead of u' when $S_{x,y,z}$ is of type 2.

(f_3) $x \notin D(a,b,c)$ and $x \notin N(c)$
Note that by (f_0), $u' \in G[a,b,c]$ ($w' \in G[a,b,c]$) and hence by Lemma 3.10 every vertex in $D(a,b,c)$ is adjacent to u' (w'). Since xu' (xw') is not an edge, $x \notin D(a,b,c)$. This implies that $x \notin N(c)$ as otherwise by considering path c, x, w_1, u', v_i (c, x, w_1, w', v_i) and applying Lemma 3.5 we conclude that $x \in D(a,b,c)$, a contradiction.

Now by Lemma 3.17 for u', w_1, x (w', w_1, x if of type (2)) we conclude that x is adjacent to some vertex v_r, $4 \le r \le p-3$ and hence $x \in G[a,b,c]$. This implies that w_q is not in $D(a,b,c)$ as otherwise $w_q x$ is an edge by Corollary 3.9. By similar argument for (f_2,f_3) we conclude that $w_q, y \notin N(c)$. Now by Lemma 3.17 for u', w_q, y (w', w_q, y if of type (2)) we conclude that y is adjacent to some vertex v_r, $3 \le r \le p-2$ and hence $y \in G[a,b,c]$.

It remains to observe that $w_{j-1} \notin D(a,b,c)$ as otherwise $w_{j-1}y$ would be an edge by Corollary 3.9. Similarly $w_{j+1} \notin D(a,b,c)$. Now none of the w_r, $2 \le r \le q-1$ is in $D(a,b,c)$ as otherwise by Corollary 3.9, $w_r x, w_r y$ are edges of G. By similar argument in (f_2) we conclude that $w_r \notin N(c)$. Since $u'w_r$ ($w'w_r$ if of type (2)) is an edge, Lemma 3.4 (6) implies that w_r is adjacent to some v_ℓ, $i-2 \le \ell \le i+2$ and hence $w_r \in G[a,b,c]$. Therefore when $S_{x,y,z}$ is of type 1 we have $V(S_{x,y,z}) \subset V(G[a,b,c])$, contradicting that $S_{a,b,c}$ is ripe.

Suppose $S_{x,y,z}$ is of type (2). We observe that since $y \in G[a,b,c]$ and $yu' \notin E(G)$, $u' \notin D(a,b,c)$ by Corollary 3.6. Because $u'w_j$ is an edge $u' \in G[a,b,c]$. These imply that $V(S_{x,y,z}) \subset V(G[a,b,c])$, contradicting that $S_{a,b,c}$ is ripe. $\diamond$

We assume $w_j \in \{w_0, w_1\}$, i.e., $x \in N[v_i] \setminus N(c)$ or $w_1 \in N[v_i] \setminus N(c)$. The other case is treated similarly.

To summarize we have the following :

(a) $x \in N[v_i] \setminus N(c)$ or $w_1 \in N[v_i] \setminus N(c)$.

(b) $u' \notin D(a,b,c)$ when $S_{x,y,z}$ is of type 1 and $w' \notin D(a,b,c)$ if $S_{x,y,z}$ is of type (2).

We proceed by proving that $x \notin D(a,b,c)$ and $w_1 \in D(a,b,c)$.

Claim 3.20 *$x \notin D(a,b,c)$.*

Proof: If $j = 0$, i.e., $w_j = x$ then clearly $x \notin D(a,b,c)$. If $j = 1$, i.e., $w_j = w_1$ then we show that x is not in $D(a,b,c)$. For contradiction suppose $x \in D(a,b,c)$. Now Lemma 3.4(6) for $v_i w_1, w_1 w_2$ implies that w_2 is in $N[v_r]$, $i-2 \le r \le i+2$ and hence $w_2 \in G[a,b,c]$. This would imply that xw_2 is an edge by Corollary 3.9, a contradiction. Therefore $x \notin D(a,b,c)$. $\diamond$

Claim 3.21 *$w_1 \in D(a,b,c)$ and $x \in N[v_i] \setminus N(c)$.*

Proof: In what follows we may assume that $S_{x,y,z}$ is of type 1. If $S_{x,y,z}$ is of type 2 we consider w' instead of u'. For contradiction suppose $w_1 \notin D(a,b,c)$. Recall items (a),(b) in summary of our assumption.

$u' \notin N(c)$. Otherwise when $w_1 \in N[v_i] \setminus N(c)$, Lemma 3.5 for path c, u', w_1, v_i implies that $u' \in D(a,b,c)$ and when $x \in N[v_i] \setminus N(c)$, Lemma 3.5 for path c, u', w_1, x, v_i implies that

$u' \in D(a,b,c)$.

$x \notin N[c]$. Otherwise Lemma 3.5 for path c, x, v_i when $x \in N[v_i] \setminus N(c)$ or for path c, x, w_1, v_i when $w_1 \in N[v_i] \setminus N(c)$ implies that $x \in D(a,b,c)$, a contradiction to Claim 3.20.

$w_1 \notin N[c]$. For contradiction suppose $w_1 \in N[c]$. If $w_1 \in\in N[v_i] \setminus N(c)$ then by applying Lemma 3.5 for path c, w_1, v_i we conclude that $w_1 \in D(a,b,c)$ a contradiction to our assumption. Therefore $w_1 \notin N[v_i] \setminus N(c)$ and hence $x \in N[v_i] \setminus N(c)$, according to (b). However Lemma 3.5 for path c, w_1, x, v_i implies that $w_1 \in D(a,b,c)$, again contradiction to our assumption.

We continue by having that $u', w_1, x \notin N(c)$.

We observe that $z \notin N(c)$ as otherwise by Lemma 3.5 for path c, z, u', w_1, x, v_i or path c, z, u', w_1, v_i we conclude that $z \in D(a,b,c)$ and hence one of the $zw_1, zx \in E(G)$.

By Lemma 3.17 for v_i, x, w_1, u' when $x \in N[v_i]$ or Lemma 3.4 (6) for v_i, w_1, u' when $w_1 \in N[v_i]$ we conclude that $u' \in N[v_r]$, $i-3 \leq r \leq i+3$. Since $u'z$ is an edge and $z \notin N(c)$, z is adjacent to some vertex $v_{r'}$, $i-5 \leq r' \leq i+5$. W.o.l.g assume that $r' \leq i$.

Now by considering the path $z, v_{r'}, v_{r'+1}, \ldots, v_i, w_j$ ($w_j \in \{w_0, w_1\}$) Lemma 3.5 implies that one of the v_ℓ, $r' \leq \ell \leq i$ is a dominating vertex for $S_{x,y,z}$ as otherwise we obtain a smaller AT that satisfies the condition of the lemma (in particular w_j is the same).

Now it is clear that $i-3 \leq r' \leq i$. Otherwise we get a shorter path $P'_{a,b} = a, v_1, \ldots, v_{r'}, w_j, v_i, \ldots, v_p, b$ when $j \neq 0$ and we get a shorter path $P''_{a,b} = a, v_1, \ldots, v_{r'}, w_1, w_0, v_i, \ldots, v_p, b$ when $j = 0$ (observe that we assumed that w_1 is not a dominating vertex).

Note that z is not adjacent to v_{i-5} as otherwise by Lemma 3.4(1) for $S_{x,y,z}$, v_{i-5} is adjacent to $v_{r'}$ ($i-3 \leq r'$). By applying Lemma 3.4 (7) for $v_{r'}, w_q, y$ we conclude that $y \in N[v_{\ell'}]$, $3 \leq \ell' \leq p-2$ or w_q is adjacent to v_{i-5}, v_{i-4}. However we obtain an AT, $S_{x,v_{i-5},z}$ with the path $P_{x,v_2} = x, w_1, w_2, \ldots, w_q, v_{i-5}$ and center vertex u'. Since $7 \leq i \leq p-6$, $S_{x,y,z} \subset G[a,b,c]$, a contradiction. The proof of the claim is complete. $\diamond$

We continue by having that $w_1 \in D(a,b,c)$ (a dominating vertex) and consequently by (b) $x \in N[v_i]$. Since w_1 is not adjacent to any of the vertices $z, w_3, \ldots, w_q, y$, by Lemma 3.10 none of these vertices is in $G[a,b,c]$. It is also easy to see that $u' \notin G[a,b,c]$ ($w' \notin G[a,b,c]$ when $S_{x,y,z}$ is of type 2) as otherwise because zu' is an edge Lemma 3.10 implies that w_1 is adjacent to z.

Remark : Observe that when $S_{x,y,z}$ is of type 2 then u' must be in $D(a,b,c)$. Otherwise because $v_i x, xu', u'w_3$ are edges of G by Corollary 3.17 we conclude that w_3 is adjacent to some v_r, $4 \leq r \leq p-3$ and hence $w_1 w_3$ is an edge by Lemma 3.4(7).

Finally it is easy to see that for $x' \in N(v_j) \setminus N(c)$; $5 \leq j \leq p-4$; $S_{x',y,z}$ is an AT with the path $P_{x',y} = x', w_1, \ldots, w_q, y$. This proves (3). Analogously if $w_j \in \{w_q, w_{q+1}\}$ then for every $y' \in N(v_j) \setminus N(c)$; $5 \leq j \leq p-4$, $S_{x,y',z}$ is an AT with the path $P_{x,y'} = x, w_1, \ldots, w_q, y'$. This shows (4).

Case 2. $z \in N[v_i] \setminus N(c)$. No vertex w_j, $0 \leq j \leq q+1$ is in $D(a,b,c)$ as otherwise $w_j z$ is an edge by Corollary 3.9. By our assumption $u' \notin D(a,b,c)$. We note that u' is adjacent to v_i by Lemma 3.4 (1). Now by applying Lemma 3.17 for v_i, u', w_1, x and for v_i, u', w_q, y we conclude that $w_1, w_q, x, y \in G[a,b,c]$. Moreover by applying Lemma 3.4 (6) for u', w_r where $2 \leq r \leq q-1$ we conclude that w_r is adjacent to some vertex v_ℓ, $i-2 \leq \ell \leq i+2$ and hence $w_r \in G[a,b,c]$. Therefore entire $S_{x,y,z}$ is in $G[a,b,c]$. This is a contradiction to $S_{a,b,c}$ is ripe. When $S_{x,y,z}$ is of type (2) $u' \notin D(a,b,c)$ as otherwise u' is adjacent to y, a contradiction. Moreover since $z \in N[v_i] \setminus N(c)$, $u' \in G[a,b,c]$ and hence $V(S_{x,y,z}) \subset V(G[a,b,c])$. $\diamond$

4 Vertex Deletion

4.1 From Chordal to Interval

In this subsection we assume that G is chordal and it does not contain small ATs. We design an FPT algorithm that takes G as an input and k as a parameter and turns G into interval graph by deleting at most k vertices. Recall that $P_{a,b} = a, v_1, v_2, \ldots, v_p, b$, and c is a shallow vertex for $S_{a,b,c}$ and u is one of the central vertices for $S_{a,b,c}$.

$Chordal - Interval(G,k)$ **Algorithm**

Input : Chordal graph G without small AT's and without cycle C, $4 \leq |C| \leq 9$.

Output : A minimum set F of G such that $|F| \leq k$ and $G \setminus F$ is interval graph OR report NOT exists (no such an F, more than k vertices need to be deleted).

1. If G is an interval graph then return $\emptyset$.
2. If $k \leq 0$ and G is not interval then report NOT exists.
3. Let $S_{a,b,c}$ be a ripe AT in G with the path $P_{a,b} = a, v_1, v_2, \ldots, v_p, b$ and center vertex u (u, w when it is of type 2).
4. Let X be a smallest set of vertices such that there is no path from v_6 to v_{p-5} in $G \setminus (X \cup N(c))$ and X contains a v_j, $7 \leq j \leq p-6$.
5. If $S_{a,b,c}$ is of type 1 then find a $w \in \{a, b, c, u, v_1, v_2, v_3, v_4, v_5, v_6, v_p$ $, v_{p-1}, v_{p-2}, v_{p-3}, v_{p-4}, v_{p-5}\}$ such that $F' = Chordal - Interval(G - w, k-1)$ exists and return $F' \cup \{w\}$.
6. If $S_{a,b,c}$ is of type 2 then find a $w \in \{a, b, c, u, w, v_1, v_2, v_3, v_4, v_5, v_6, v_p$ $, v_{p-1}, v_{p-2}, v_{p-3}, v_{p-4}, v_{p-5}\}$ such that $F' = Chordal - Interval(G - w, k-1)$ exists and return $F' \cup \{w\}$.
7. Let $S = \{w' \in N[v_j] \setminus N(c); |5 \leq j \leq p-4\}$. If $F' = Chordal - Interval(G \setminus S, k - |S|)$ exists then return $F' \cup S$.
8. If $F' = Chordal - Interval(G \setminus X, k - |X|)$ exists then return $F' \cup X$.

The following Lemma shows the correctness of the Algorithm Chordal-Interval(G,k).

Lemma 4.1 *Let G be a chordal graph without small ATs and let $S_{a,b,c}$ be a ripe AT in G with path $P_{a,b} = a, v_1, v_2, \ldots, v_p$, and a center vertex u. Let X be a minimum separator in $G \setminus N(c)$ that separates v_6 from v_{p-5} and it contains a v_i, $7 \leq i \leq p-6$. Then there is a minimum set of deleting vertices F such that $G \setminus F$ is an interval graph and at least one of the following holds:*

(i) If $S_{a,b,c}$ is of type 1 then F contains at least one vertex from

$$\{a, b, u, c, v_1, v_2, v_3, v_4, v_5, v_6, v_{p-5}, v_{p-4}, v_{p-3}, v_{p-2}, v_{p-1}, v_p\}$$

If $S_{a,b,c}$ is of type 2 then F contains at least one vertex from

$$\{a, b, u, w, c, v_1, v_2, v_3, v_4, v_5, v_6, v_{p-5}, v_{p-4}, v_{p-3}, v_{p-2}, v_{p-1}, v_p\}$$

(ii) F contains all vertices of $S = \{x' \in N(v_j) \setminus N(c) | 5 \leq j \leq p-4\}$;

(iii) F contains all the vertices in X.

Proof: Let $S_{a,b,c}$ be a ripe AT. Any optimal solution F must contains at least a vertex from $V(S_{a,b,c})$. Let H be a minimum set of deleting vertices such that $G \setminus H$ is an interval graph. We may assume that H does not contain all the vertices in S. Otherwise we set $F = H$. Moreover we may assume that H does not contain any vertex from $\{a, b, u, c, v_1, v_2, v_3, v_4, v_5, v_6, v_{p-5}, v_{p-4}, v_{p-3}, v_{p-2}, v_{p-1}, v_p\}$ and if $S_{a,b,c}$ is of type (2) we may assume that H does not contain w (the other center vertex of $S_{a,b,c}$). Otherwise we set $F = H$ and we are done. Recall that $G[a, b, c]$ is an interval graph since $S_{a,b,c}$ is a ripe AT in D.

Let $W = \{w | w \in H \cap G[a, b, c]\}$. Because $S_{a,b,c}$ is an AT in G there is no path from v_6 to v_{p-5} in $G \setminus H$. Hence, set W should contain a *minimal* v_6, v_{p-5}-separator X' that contains some vertex v_j, $7 \leq j \leq p-6$ in $G \setminus N(c)$. Since $G[a, b, c]$ is an interval graph, X' is in $N[v_j]$.

We define $F = (H \setminus X') \cup X$ and we observe that $|F| \leq |H|$.

In what follows, we prove that $I = G \setminus F$ is an interval graph. For a sake of contradiction, let us assume that I is not an interval graph. By Theorem 1.1, I contains either cycle of length more than three or an AT. It is clear that by deleting vertices from G no cycle would appear, since we have assumed that G is chordal. Therefore we consider the case that I may have an AT. Because we delete vertices and at the beginning G does not have small AT, we conclude that I does not have a small AT. Therefore we may assume that I contains a big AT, $S_{x,y,z}$ with the path $P_{x,y} = x, w_1, w_2, \ldots, w_q, y$ and a center vertex u'. We may assume that $S_{x,y,z}$ is of type 1. Similar argument is applied when $S_{x,y,z}$ is of type (2).

We conclude that $S_{x,y,z}$ has a vertex in X' and no vertex in X. Now according to the Lemma 3.18 one of the following happens:

1. $u' \in D(a, b, c)$ and $P_{x,y} \cap B[a, b] \neq \emptyset$ and $P_{x,y} \cap E[a, b] \neq \emptyset$ and every v_r, $2 \leq r \leq p-1$ has a neighbor in $P_{x,y}$.

2. $u' \in D(a, b, c)$, $z \in G[a, b, c]$ and $P_{x,y} \cap G[a, b, c] = \emptyset$ and for every vertex $z' \in G[a, b, c]$, $S_{x,y,z'}$ is an AT with the same path $x, w_1, w_2, \ldots, w_q, y$.

3. $x \in N[v_i]$, $w_1 \in D(a, b, c)$ and $V(S_{x,y,z}) \cap G[a, b, c] = \{x\}$ and for every $x' \in N[v_r] \setminus N(c)$; $5 \leq r \leq p-4$, $S_{x',y,z}$ is an AT with the path $P_{x',y} = x', w_1, \ldots, w_q, y$.

4. $y \in N[v_i]$, $w_q \in D(a, b, c)$ and $V(S_{x,y,z}) \cap G[a, b, c] = \{y\}$ and for every $y' \in N[v_r] \setminus N(c)$; $5 \leq r \leq p-4$, $S_{x,y',z}$ is an AT with the path $P_{x,y'} = x, w_1, \ldots, w_q, y'$.

If (1) happens then there exists a path in from v_6 to v_{p-5} in $G \setminus X$. This is a contradiction to X being a separator and hence there exists some delete vertex $w' \in X$, such that $w' \in \{x, w_1, w_2, \ldots, w_q, y\}$.

Suppose (2) happens. Then $V(P_{x,y}) \cap V(G[a,b,c]) = \emptyset$ and $u' \notin G[a,b,c]$. Therefore we may assume that $z \in X'$ and no other vertex of $S_{x,y,z}$ is in $H \setminus X'$. However by (2) for every vertex $z' \in G[a,b,c]$, $S_{x,y,z'}$ is an AT with the same path $x, w_1, w_2, \ldots, w_q, y$. Since $(P_{x,y} \cup \{u'\}) \cap X' = \emptyset$ and $S_{x,y,z'}$ is not an AT in $G \setminus H$, we conclude that H must contain all the vertices in $G[a,b,c]$. This is a contradiction because $S \subset V(G[a,b,c])$ and we assumed that H does not contain entire S.

Suppose (3) happens. Then $V(S_{x,y,z}) \cap V(G[a,b,c]) = \{x\}$ and $w_1 \in D(a,b,c)$. Since $X' \cap V(S_{x,y,z}) \neq \emptyset$, we have $x \in X'$. We may assume that no other vertex of $S_{x,y,z}$ is in $H \setminus X'$. However by (3) for every vertex $x' \in N[v_j] \setminus N(c)$, $5 \leq j \leq p-4$, we have that $S_{x',y,z}$ is an AT with the path $P_{x',y} = x', w_1, \ldots, w_q, y$. Since $S_{x',y,z}$ is not an AT in $G \setminus H$, we conclude that H must contain all the vertices in $S = \{x' \in N(v_j) \setminus N(c) | 5 \leq j \leq p-4\}$. This is a contradiction because we assumed that H does not contain entire S.

Analogously if (4) happens we get a contradiction. $\diamond$

4.2 When G is not Chordal, Structural Properties

In this subsection we assume that G does not contain small AT, as an induced subgraph and it does not contain cycle of length less than 9 and more than 3.
Let $C = v_0, v_1, .., v_{p-1}, v_0$ be a shortest cycle in G, $9 \leq p$. We say a vertex of G is a *dominating* vertex for C if it is adjacent to every vertex of the cycle C. Let $D(C)$ denotes the set of all dominating vertices of C.

Lemma 4.2 *Let x be a vertex in $V(G) \setminus V(C)$. Then one of the following happens :*

(1) x is adjacent to all vertices of C,

(2) x is adjacent to at most three consecutive vertices of C,

(3) Any path from $x \notin N[C]$ to C has intersection with $D(C)$.

Proof: If x is adjacent to all the vertices in $V(C)$ then (1) holds. Thus we may assume that x is not adjacent to every vertex in C.

(2) Suppose $x \in N(C)$. If x is adjacent to exactly one vertex in C then (2) holds. Therefore we may assume there are vertices $v_i \neq v_j$ of $V(C)$ such that $v_i x, v_j x$ are edges of G and none of the

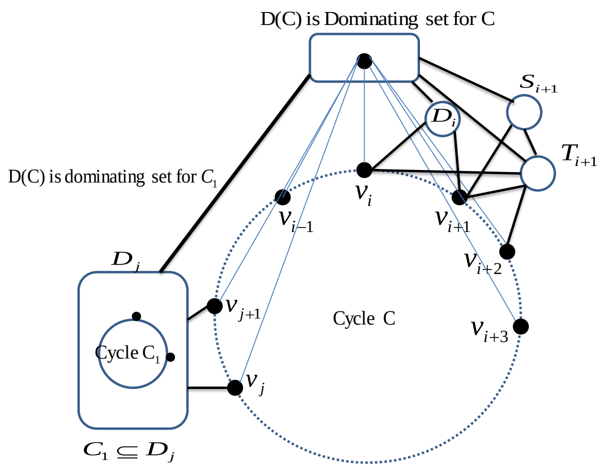

Figure 7: Cycles

vertices of C between v_i and v_j in the clockwise direction is adjacent to x. We get a shorter cycle, using the portion of C (in the clockwise direction) from v_i to v_j and x unless up to symmetry $v_j = v_{i+1}$ or $v_j = v_{i+2}$. If $v_j = v_{i+1}$ then (2) holds. If $v_j = v_{i+2}$ then x is also adjacent to v_{i+1} as otherwise we obtain an induced 4 cycle in G which is not the case. Thus (2) is proved.

(3) For contradiction let $x \notin N(C)$ be adjacent to a vertex $y \in N(v_i) \backslash D(C)$. Now $x, y, v_{i-1}, v_i, v_{i+1}$ induce a small AT unless yv_{i-1}, yv_{i-2} or yv_{i+1}, yv_{i+2} are edges of G. W.l.o.g assume that yv_{i-1}, yv_{i-2} are edges of G. Now by (2) y is not adjacent to any of v_{i+1}, v_{i-3} and hence the vertices $v_{i-3}, v_{i-2}, v_{i-1}, v_i, v_{i+1}, x, y$ induce a small AT v_{i-3}, v_{i+1}, x, a contradiction. $\diamond$

We introduce the following notations. For every $0 \leq i \leq p-1$, we define the following subsets of $N(C) \setminus D(C)$

- S_i vertices adjacent to v_i and not adjacent to any other v_j, $j \neq i$;
- D_i vertices adjacent to v_i, v_{i+1} and not adjacent to any other v_j, $j \neq i, i+1$, and
- T_i vertices adjacent to v_i, v_{i+1}, v_{i+2} only

See Figure 7 for illustration.

Lemma 4.3 *Consider the cycle C and the sets S_i, D_i, T_i, $0 \leq i \leq p-1$. Then the followings hold.*

1. *If there is an edge from a vertex in D_i to a vertex in D_j then v_i, v_j are consecutive on the cycle.*

2. *Every vertex in T_i is adjacent to every vertex in S_{i+1}.*

3. *There is no edge from S_i to $S_{i+1} \cup D_{i+1} \cup T_{i+1}$.*

Proof: (1) Let $x \in D_i$ and $y \in D_j$. Since cycle $v_{i-1}, v_i, x, y, v_{j+1}, v_{j+2}, \ldots, v_{i-2}, v_{i-1}$ is not shorter than C we have $v_j \in \{v_{i+1}, v_{i+2}, v_{i-1}, v_{i-2}\}$. We show that $v_j \neq v_{i+2}$. For contradiction suppose $v_j = v_{i+2}$. Now by definition none of the $v_{i+1}y$ and $v_{i+2}x$ is an edge of G and hence x, y, v_{i+1}, v_{i+2} induce a C_4 in G. This is a contradiction because G does not have induced C_4. Similarly $v_j \neq v_{i-2}$. Therefore $v_j = v_{i-1}$ or $v_j = v_{i+1}$.

(2) Suppose $s \in S_{i+1}$ is adjacent to $t \in T_i$. Then the vertices $v_{i-1}, v_i, v_{i+1}, v_{i+2}$, v_{i+3}, s, t induce a small AT v_{i-1}, v_{i+3}, s in G. This is a contradiction because G does not have small AT as an induced subgraph.

(3) Suppose $x \in S_i$ is adjacent to $y \in S_{i+1} \cup D_{i+1} \cup T_{i+1}$. Then the vertices x, v_i, v_{i+1}, y induce a C_4, a contradiction. $\diamond$

Lemma 4.4 *Every vertex $x \in D(C)$ is adjacent to every vertex in $N[C] - x$.*

Proof: Let x be a vertex in $D(C)$ and y be a vertex in $N[v_i]$, $0 \leq i \leq p-1$. Note that if $y = v_i$ then by definition of $D(C)$, xy is an edge.

If $y \in D(C) \cup T_i$ then xy is an edge as otherwise x, y, v_i, v_{i+2} induce a C_4. Thus by Lemma 4.2 (2) we may assume that $y \in S_i \cup D_i$. This implies that y is adjacent to v_i and not adjacent to v_{i-1} and not adjacent to v_{i+2}. However v_{i-1}, v_{i+2}, y is a small AT, with the vertices $v_{i-1}, v_i, v_{i+1}, v_{i+2}, x, y$. $\diamond$

4.2.1 Cycle-Cycle interaction

Lemma 4.5 *Let C_1 be cycle in $N[C]$. Then $V(C_1) \cap D(C) = \emptyset$ and one of the following happens:*

1. *For every $0 \leq i \leq p-1$, $N(v_i) \cap V(C_1) \neq \emptyset$.*

2. *$V(C_1) \subset S_i$ or $V(C_1) \subset V(D_i)$ (See Figure 7).*

Proof: Observe that according to our assumption $|V(C_1)| \geq 10$. By Lemma 4.4 every vertex x in $D(C)$ is adjacent to every vertex in $N[C] \setminus \{x\}$, we conclude that $D(C) \cap V(C_1) = \emptyset$.

Suppose (1) does not hold. Thus there exists some i such that $V(C_1) \cap N(v_i) = \emptyset$ but $V(C_1) \cap N(v_{i+1}) \neq \emptyset$.

Let $v \in V(C_1) \cap N(v_{i+1})$. Let x be the last vertex of C_1 after v in the clockwise direction such that $x \in N(v_{i+1})$ but x' the neighbor of x in C_1 (clockwise direction) is not in $N(v_{i+1})$. We show that if x exists then we obtain a small AT. Let y' be the first vertex after x in the clockwise direction such that $y' \in N(v_{i+1})$ but y the neighbor of y' in C_1 (clockwise direction) is in $N(v_{i+1})$.

Note that if x exists then y also exists. We note that none of the x', y' is adjacent to v_{i-1} as otherwise we obtain an induced small cycle with the vertices $v_{i-1}, v_i, v_{i+1}, x, x'$ or with the vertices $v_{i-1}, v_i, v_{i+1}, y, y'$.

Observe that $x' \neq y'$ as otherwise v_{i+1}, x, y, x' induce a C_4. Moreover $x'y'$ is not an edge of G as otherwise the vertices y', x', x, v_{i+1}, y induce a C_5. However we get a small AT v_{i-1}, x', y' with the vertices $x, y, v_{i+1}, v_i, v_{i-1}, x', y'$. Therefore x does not exist and hence $V(C_1) \subseteq D_{i+1} \cup S_{i+1} \cup T_{i+1}$.

First suppose $V(C_1) \cap S_{i+1} \neq \emptyset$ and $V(C_1) \cap (D_{i+1} \cup T_{i+1}) \neq \emptyset$. If $|V(C_1) \cap S_{i+1}| = 1$ then we get cycle x, x', y', v_{i+2} where $x \in V(C_1) \cap S_{i+1}$ and x', y' are the neighbors of x in $V(C_1) \cap D_{i+1} \cup T_{i+1}$.

Similarly if $|V(C_1) \cap D_{i+1} \cup T_{i+1}| = 1$ we get an induced C_4 in G. Thus we may assume that C_1 has at least two vertices in S_{i+1} and two vertices in $D_{i+1} \cup T_{i+1}$. Let $x \neq y$ be two vertices of C_1 in $D_{i+1} \cup T_{i+1}$. Now let xx', yy' be the edges of C_1 such that $x', y' \in S_{i+1} \setminus D_{i+1} \cup T_{i+1}$. If $x' = y'$ we obtain an induced 4 cycle with the vertices x', x, y, v_{i+2} (xy is not an edge as otherwise $|C_1| \leq 4$.) in G. If $x'y'$ is an edge then we obtain induced 5 cycle in G with the vertices x', y', x, y, v_{i+2} in G. Now we obtain a small AT v_{i+4}, x', y' with the vertices $x', y', x, y, v_{i+2}, v_{i+3}, v_{i+4}$. This is a contradiction and hence we have $V(C_1) \subseteq S_{i+1}$ or $V(C_1) \subseteq D_{i+1} \cup T_{i+1}$. If $V(C_1) \subseteq S_{i+1}$ then (2) is proved. Thus we may assume that C_1 has vertices in T_{i+1} and D_{i+1} only. Note that $|V(C_1) \cap T_{i+1}| \leq 2$. Again by similar argument and considering v_{i+3}, v_{i+4} and part of C_1 inside D_{i+1} we see a small AT. Therefore $V(C_1) \subseteq V(D_{i+1})$ and the proof is complete. $\diamond$

Definition 4.6 *We say a shortest cycle $C = v_0, v_1, \ldots, v_{p-1}, v_0$ in G is* clean *if for every cycle C_1 in $N[C]$, every vertex of C has a neighbor in C_1. We say a cycle C is* ripe *if it is clean and there is no AT in $N[C] \setminus D(C)$.*

Looking for a shortest clean cycle

We start with an arbitrary shortest cycle C and we construct $S_i, D_i, T_i, D(C)$ as defined and then we look for a shortest cycle C_1 in some S_i or D_i. If C_1 is clean we stop otherwise we consider S_i^1, D_i^1, T_i^1 of C_1 in D_i or S_i and we continue. After at most k steps we find a clean cycle C'.

4.2.2 Cycle and AT interaction

Lemma 4.7 *Let $S_{x,y,z}$ be a minimum AT with a path $P_{x,y} = x, w_1, w_2, \ldots, w_q, y$ such that $S_{x,y,z}$ contains a vertex outside $N[C]$ and a vertex from $N[C] \setminus D(C)$. Then up to symmetry one of the following happens.*

1. *The center vertex u (the central vertices u', w' when of type 2) of $S_{x,y,z}$ is a dominating vertex for C, $P_{x,y} \cap N[C] = \emptyset$, and $z \in N[C] \setminus D(C)$. Moreover for every vertex $z' \in N[C] \setminus D(C)$, $S_{x,y,z'}$ is an AT with the same number of vertices as $S_{x,y,z}$ and the same path $P_{x,y}$ (Figure 8 left)*
2. *$y \in N[C] \setminus D(C)$ and $w_q \in D(C)$ ($w_q, w \in D(C)$ when of type 2) and $N[C] \cap V(S_{x,y,z}) = \{y, w_q\}$ ($N[C] \cap V(S_{x,y,z}) = \{y, w_q, w\}$ when of type 2). Moreover for every vertex $y' \in N[C] \setminus D(C)$, $S_{x,y',z}$ is an AT with the same number of vertices as $S_{x,y,z}$ (Figure 8 right).*

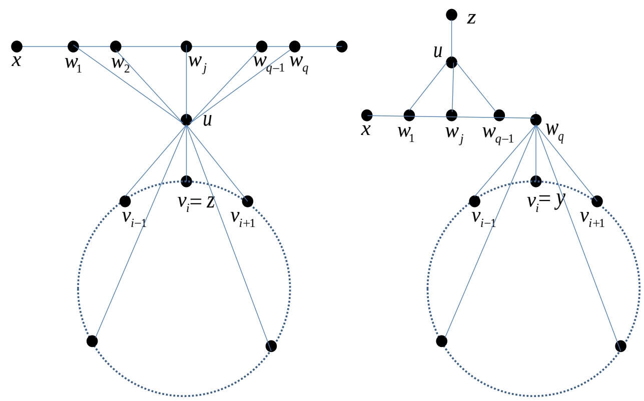

Figure 8: Cycle and AT outside

Proof: We note that since a vertex of $S_{x,y,z}$ is outside $N[C]$ and one vertex of $S_{x,y,z}$ is in $N[C] \setminus D(C)$, by Lemma 4.2 item (3) at least one vertex of $S_{x,y,z}$ is in $D(C)$.

First suppose $P_{x,y} \cap N[C] = \emptyset$. Now it is easy to see that since u (u, w) is (are) adjacent to $w_1, w_2, \dots, w_q, z$, we have $u \in D(C)$ $(u, w \in D(C))$ and $z \in N[C] \setminus D(C)$. Moreover for every $z' \in (N[C] \setminus D(C))$, $S_{x,y,z'}$ is a minimum AT with the same number of vertices as $S_{x,y,z}$.

Now suppose $P_{x,y} \cap N[C] \neq \emptyset$. We show that at least one of the x, y is in $N[C]$. For contradiction suppose $x, y \notin N[C]$. Let $1 \leq i \leq q$ be the first index such that $w_i \in N[C]$. By Lemma 4.2 (3) it is easy to see that w_i is a dominating vertex for C and hence by Lemma 4.4 none of the vertices $w_{i+2}, w_{i+3}, \dots, w_q, y$ is in $N[C]$. Moreover by assumption for i, none of the $x, w_1, \dots, w_{i-2}$ is in $N[C]$. Now $z \notin N[C]$ as otherwise $w_i z$ is an edge of G which is not possible. Now since $w_{i+2} \notin N[C]$ and $w_{i+1}w_{i+2}$ is an edge we conclude that $w_{i+1} \notin N[C] \setminus D(C)$ as otherwise we get a contradiction to item (3) of Lemma 4.2.

Therefore by assumption of the lemma we conclude that u. Now since $w_{i+2} \notin N[C]$ and $w_{i+1}w_{i+2}$ is an edge we get a contradiction to item (3) of Lemma 4.2. Since u is adjacent to $z, w_1, w_2, \dots, w_q$, we conclude that u (one of u', w' when type 2) must be in $D(C)$ (by Lemma 4.2) and hence we get a contradiction to the assumption of the lemma because no vertex of $S_{x,y,z}$ is in $N[C] \setminus D(C)$.

Therefore we conclude that at least one of the x, y is in $N[C]$. W.l.o.g. assume that $y \in N[C]$. We show that $y \in N[C] \setminus D(C)$. Otherwise none of the vertices $x, w_1, \dots, w_{q-1}, z, u \in N[C]$. Now by condition of the lemma we must have $w_1 \in N[C] \setminus D(C)$, and again because $w_1 w_2$ is an edge we get a contradiction to item (3) of Lemma 4.2. We continue by having $y \in N[C] \setminus D(C)$.

We show that at least one vertex of $P_{x,y}$ is outside $N[C] \setminus D(C)$. If this is not the case then either z or u (one of the u, w) is in $D(C)$ and by Lemma 4.4, we conclude that yz or yu $(yu, xw$ when $S_{x,y,z}$ is of type 2), a contradiction. Let $1 \leq i \leq q+1$ be the smallest index such that

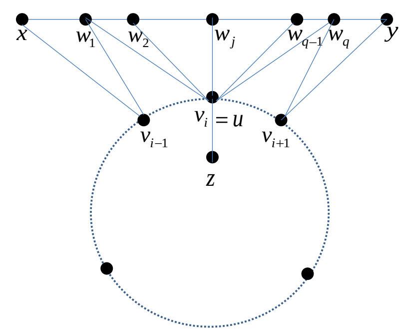

Figure 9: Cycle and AT inside N[C]

$w_i \in N[C] \setminus D(C)$ and $w_{i-1} \in D(C)$. By lemma 4.4, $w_{i-1}y$ is an edge of G and hence $i = q+1$. Now clearly the vertices in $P_{x,y} \setminus \{y, w_q\}$ are outside $N[C]$. Because $w_q z \notin E(G)$ lemma 4.4 implies that $z \notin N[C]$. Now since $uz \in E(G)$ $(uz, wz \in E(G))$, and $uy \notin E(G)$, we conclude that u is not in $N[C]$. Note that when $S_{x,y,z}$ is of type 2 then $w \in D(C)$. Now it is easy to see that for $y' \in N[C] \setminus D(C)$, $S_{x,y',z}$ is an AT with the same number of vertices as $S_{x,y,z}$. ◇

Lemma 4.8 *Let $S_{x,y,z}$ be a minimum AT with path $P_{x,y} = x, w_1, w_2, \ldots, w_q, y$ such that $V(S_{x,y,z}) \subset N[C]$. Then the followings happen:*

1. *$V(S_{x,y,z}) \subset N[C] \setminus D(C)$*
2. *There exists a $v_i \in V(C)$, $0 \le i \le p-1$ such that v_i is a dominating vertex for $S_{x,y,z}$*
3. *None of the vertices v_{i-2}, v_{i+2} is adjacent to any w_j, $2 \le j \le q-1$ (See Figure 9)*
4. *If $v_r \in V(C)$ (only $r = i-1, i, i+1$ is possible) is adjacent to some vertex w_j, $4 \le j \le q-3$ then v_r is a dominating vertex for $S_{x,y,z}$*
5. *Either $V(S_{x,y,z}) \subseteq N[\{v_{i-1}, v_i, v_{i+1}\}]$ or x, y can be replace by v_{i-2}, v_{i+2} and obtain an AT with the same number of vertices as $S_{x,y,z}$*

Proof: Observe that by Lemma 4.4 none of the u, z (u, w, z if $S_{x,y,z}$ is of type 2) is in $D(C)$ as otherwise one of the xu, zx (wx, zx if of type (2)) is an edge. Moreover none of the vertices in $P_{x,y}$ is in $D(C)$ as otherwise by Lemma 4.4 for some $w_j \in D(C)$, $w_j z$ is an edge. Therefore $V(S_{x,y,z}) \subset N[C] \setminus D(C)$.

First suppose $u = v_i$ for some $0 \le i \le p-1$. Now v_{i-2} is not adjacent to w_j, $2 \le j \le q-1$ as otherwise by Lemma 3.4(2,3) uv_{i-2} is an edge, a contradiction. Similarly v_{i+2} is not adjacent to w_j. In this case the proof is complete. Therefore we continue by assuming that $u \notin V(C)$.

We show that $z \notin V(C)$. For contradiction suppose $z = v_i$, $0 \le i \le p-1$. Since $u \notin V(C)$, we assume that $u \in S_i \cup D_i \cup T_i$. By Lemma 3.4 (1) both $v_{i-1}u$ and $v_{i+1}u$ are edges of G. Now

v_{i-1} is adjacent to some vertex w_j, $0 \leq j \leq q+1$. Otherwise $S_{a,b,v_{i-1}}$ is a minimum AT, with the same number of vertices as $S_{a,b,c}$ and hence by Lemma 3.4 (1) v_{i-2} must be adjacent to u; implying that u is adjacent to more than three consecutive vertices on the cycle, a contradiction to Lemma 4.2(2). Therefore v_{i-1} is adjacent to some w_j, $0 \leq j \leq q+1$ and hence by Lemma 3.4 (4), v_{i-1} is a dominating vertex for $S_{x,y,z}$. Similarly we conclude that v_{i+1} is a dominating vertex for $S_{x,y,z}$ and hence by Corollary 3.6 $v_{i-1}v_{i+1}$ must be an edge. This is a contradiction. Therefore we conclude the following :

(f_1) For every minimum AT, $S_{x',y',z'}$ such that $V(S_{x',y',z'}) \subset N[C] \setminus D(C)$ we have $z' \notin V(C)$.

We continue by assuming that $z \in N(v_i) \setminus V(C)$, $0 \leq i \leq p-1$. Now $v_i u$ is also an edge by Lemma 3.4(1). Note that v_i is adjacent to some vertex w_j, $0 \leq j \leq q+1$ on the path $P_{x,y}$ as otherwise S_{x,y,v_i} is a minimum AT with the same number of vertices as $S_{x,y,z}$ and we get a contradiction by (f_1). Since zv_i is an edge and v_i is adjacent to w_j, Lemma 3.4(4) implies that v_i is a dominating vertex for $S_{x,y,z}$. This proves 2.

Now v_{i-2} is not adjacent to any w_r, $2 \leq r \leq q-1$ as otherwise by Lemma 3.4(7) v_{i-2} must be adjacent to v_i which is not possible. Similarly v_{i+2} is not adjacent to any w_r, $2 \leq r \leq q-1$. This proves (3).

Now suppose some v_r is adjacent to a vertex w_j, $4 \leq j \leq q-3$. It is easy to see that $r = i-1, i, i+1$. We may assume that $v_{i-1}w_j$ is an edge. Now if $v_{i-1}z$ is an edge then by Lemma 3.4(4) v_{i-1} is a dominating vertex for $S_{x,y,z}$. Thus we may assume that $v_{i-1} \notin N(z)$. However by applying Lemma 3.4(6), v_{i-2} is adjacent to some vertex w_r, $2 \leq j-2 \leq r \leq j+2 \leq q-1$. Now v_{i-2} must be adjacent to v_i according to Lemma 3.4 (2) for $S_{x,y,z}$, a contradiction. Thus (4) is proved.

To see that (5), we note that since v_i is a dominating vertex for $S_{x,y,z}$, by Lemma 3.4(1) zv_i is an edge. We know that $xw_1, w_1v_i, yw_q, w_qv_i$ are edges of G. Now if $x \notin N(v_{i-1} \cup N(v_i) \cup N(v_{i+1})$ then xv_{i-2} or xv_{i+2} is an edge by Lemma 4.2. Suppose xv_{i-2} is an edge. Thus w_1v_{i-2} is an edge as otherwise we obtain a C_4 with x, w_1, v_{i-1}, v_{i-2} when w_1v_{i-1} is an edge or we obtain a C_5 with $x, w_1, v_i, v_{i-1}, v_{i-2}$ when w_1v_{i-1} is not an edge, a contradiction. Since v_iz is an edge, $v_{i-2}z$ is not an edge as otherwise by Lemma 3.4(1), $v_{i-2}v_i$ is an edge. Now by (3) we may replace v_{i-2} by x and obtain an AT.

◇

4.3 The Main Algorithm (Putting things together)

We branch on all the deleting vertices of each small AT. We also branch on by deleting vertices of each cycle C, $4 \leq |C| \leq 9$. After that if G is not interval we continue as follows.

Definition 4.9 *Let C be a ripe cycle. We say a set X of the vertices in $N[C] \setminus D(C)$ is a cycle-separator if there is no cycle in $N[C] \setminus (D(C) \cup X)$.*

In order to find set X, for every $0 \leq i \leq p-1$ we find a minimum set of vertices X_i that separates v_i from v_{i+3} in $W_i = N[\{v_{i+1}, v_{i+2}\}] \setminus D(C)$. X is the smallest set X_i. Note that W_i is an interval graph since C is ripe.

$Interval - Deletion(G, k)$ **Algorithm**

Input : Graph G without small AT's and without cycle C, $4 \leq |C| \leq 9$.
Output : A minimum set F of $V(G)$ such that $|F| \leq k$ and $G \setminus F$ is an interval graph OR report NOT exists (no such F, more than k vertices need to be deleted).

1. If G is an interval graph then return $\emptyset$.
2. If $k \leq 0$ and G is not an interval graph then report NOT exists.
3. If G is chordal then set $F = Chordal - Interval(G, k)$. If F exists then return F otherwise report NOT exists.
4. Let C be a shortest ripe cycle in G. Let X be a minimum cycle-separator in $N[C] \setminus D(C)$.
5. If $F = Interval - Deletion(G \setminus C, k - |C|)$ exists then return $F \cup C$. Else report NOT exists.
6. If $F = Interval - Deletion(G \setminus X, k - |X|)$ exists then return $F \cup X$. Else report NOT exists.
7. Let $C = v_0, v_1, \ldots, v_{p-1}$ be a shortest clean cycle in G.
8. Let $S = \{x | x \in (N[v_{i-1}] \cup N[v_i] \cup N[v_{i+1}]) \setminus D(C)\}$ for some $0 \leq i \leq p-1$ such that $G[S]$, contains an AT.
9. Set $F = Chordal - Interval(G[S], k)$. Set $F' = Interval - Deletion(G \setminus F, k - |F|)$. If $|F' \cup F| \leq k$ then return $F \cup F'$. Else report NOT exists.

If there is no cycle in G then we apply the Chordal-to-Interval Algorithm and as we argued in Lemma 4.1 there is an optimal solution that contains the solution of the Chordal-to-Interval Algorithm. Otherwise let C be a clean cycle in G. If C is ripe then we argue in Lemma 4.10 that there exists a minimum set X of the vertices in $N[C] \setminus D(C)$ and there is a minimum set of deleting vertices F such that $G \setminus F$ is an interval graphs and $X \subseteq F$. Therefore the steps 4,5 are justified.

Lemma 4.10 *Let $C = v_0, v_1, \ldots, v_{p-1}, v_0$ be a ripe cycle and let X be a minimum cycle-separator in $N[C] \setminus D(C)$. Then there is a minimum set of deleting vertices F such that $G \setminus F$ is an interval graph and at least one of the following holds:*

(i) F contains all the vertices of the cycle C.

(ii) F contains all vertices in X.

Proof: Let H be a minimum set of deleting vertices such that $G \setminus H$ is an interval graph. If H contains all the vertices in C then we set $F = H$ and we are done. Thus we suppose H does not contain all the vertices of C. Let $W = \{w | w \in H \cap (N[C] \setminus D(C))\}$.

Since H does not contain all the vertices of C, set W should contain a *minimal* cycle-separator X' in $N[C] \setminus D(C)$. Now define $F = (H \setminus X') \cup X$. We observe that $|F| \leq |H|$. We prove that $I = G \setminus F$ is an interval graph.

First suppose I contains cycle C_1. We note that $V(C_1) \cap X' \neq \emptyset$ and $V(C_1) \cap X = \emptyset$. Since C is ripe, by Lemma 4.5 (1) for every $0 \leq i \leq p-1$, $N[v_i] \cap V(C_1) \neq \emptyset$. But this a contradiction to X being a cycle-separator.

Therefore we may assume that I contains an AT. Since $G \setminus F$ has less vertices than G and G does not have small ATs, I does not have small ATs. Thus we may assume that I contains a big AT. Consider a minimum big AT, $S_{x,y,z}$ with path $P_{x,y} = x, z_1, z_2 \ldots, z_m, y$ in I. We note that $S_{x,y,z}$ must have some vertices in $X' \setminus X$ and none of the vertices of $S_{x,y,z}$ is in $F \setminus X'$. Since $X' \subset N[C] \setminus D(C)$ and cycle C is ripe, $V(S_{x,y,z})$ does not lie entirely in $N[C]$ and hence the conditions of the Lemma 4.7 in G are satisfied. According to Lemma 4.7 (up to symmetry) one of the following happens.

1 The center vertex u (the central vertices u, w when of type 2) of $S_{x,y,z}$ is a dominating vertex for C, $P_{x,y} \cap N[C] = \emptyset$, and $z \in N[C] \setminus D(C)$. Moreover for every vertex $z' \in N[C] \setminus D(C)$, $S_{x,y,z'}$ is an AT with the same number of vertices as $S_{x,y,z}$ and the same path $P_{x,y}$.

2 $y \in N[C] \setminus D(C)$ and $w_q \in D(C)$ ($w_q, w \in D(C)$ when of type 2) and $N[C] \cap V(S_{x,y,z}) = \{y, w_q\}$ ($N[C] \cap V(S_{x,y,z}) = \{y, w_q, w\}$ when of type 2). Moreover for every vertex $y' \in N[C] \setminus D(C)$, $S_{x,y',z}$ is an AT with the same number of vertices as $S_{x,y,z}$.

If 1 happens then $z \in X'$ and for every vertex v_i, $0 \leq i \leq p-1$ of C, S_{x,y,v_i} is a minimum AT with the same number of vertices as $S_{x,y,z}$. Since S_{x,y,v_i} is no longer an AT in $G \setminus H$ and $u \notin H$, $V(S_{x,y,v_i}) \setminus \{v_i\}) \cap (N[C] \setminus D(C)) = \emptyset$, we conclude that H must contain v_i. Therefore H must contain all the vertices in $V(C)$, a contradiction.

If 2 happens then $y \in X'$ and for every vertex v_i , $0 \leq i \leq p-1$ of C, $S_{x,v_i,z}$ is an AT with the same number of vertices as $S_{x,y,z}$. Since $S_{x,v_i,z}$ is no longer an AT in $G \setminus H$ and $u \notin H$, $V(S_{x,v_i,z}) \setminus \{v_i\}) \cap (N[C] \setminus D(C)) = \emptyset$, we conclude that H must contain v_i. Therefore H must contain all the vertices in $V(C)$, a contradiction.

$\diamond$

When C is not ripe then there is a minimum AT, $S_{x,y,z}$ such that $V(S_{x,y,z}) \subseteq (N[C] \setminus D(C)) \neq \emptyset$. According to item (5) of Lemma 4.8 we may assume that $V(S_{x,y,z}) \subseteq W = N[\{v_{i-1}, v_i, v_{i+1}\}]$ for three consecutive vertices v_{i-1}, v_i, v_{i+1} in C. We apply the Algorithm Chordal-to-Interval on the subgraph induced by W and hence we ripen the cycle C. We need to argue that we can apply the Lemma 4.1 when G is not chordal. Let $P_{x,y} = x, w_1, w_2, \ldots, w_q, y$. Let X' be a minimum separator in $G \setminus N(z)$ that separates w_6 from w_{q-5} and it contains a vertex w_j, $7 \leq j \leq q-6$. According to Lemma 4.8 (2) v_i is a dominating vertex for $S_{x,y,z}$ and according to items (2,3,4) of Lemma 4.8 no vertex v_r of cycle C belongs to X'. In other words none of the vertices in X' are used to break cycle C. These allow us to apply the Lemma 4.1 for subgraph $G[W]$.

Overall the running time of the algorithm is $O(c^k n(m+n)$ where $c = min\{18, k\}$. In order to find an AT we apply the algorithm in [14]. Now we focus on the correctness of the Interval-Deletion(G,k) algorithm.

5 Interval Completion

5.1 AT and AT Edge Interaction

In this subsection we may assume that G is chordal and G does not contain any small AT.

Lemma 5.1 *Let $S_{a,b,c}$ be a minimum AT with path $P_{a,b} = a, v_1, v_2, \ldots, v_p, b$ and center vertex u (central vertices u, w). Let $Q = u, x_1, x_2, \ldots, x_r, a$ be a chordless path from u to a (not including v_1) (from w to a if $S_{a,b,c}$ is of type 2). Then for every vertex x_i, $2 \leq i \leq r$, $S_{x_i,b,c}$ is a minimum AT with the same number of vertices as $S_{a,b,c}$, and path $P_{x_i,b} = x_i, v_1, v_2, \ldots, v_p, b$*

Proof: Since there is no cycle of length more than 3 in G, v_1 must be adjacent to x_i, $1 \leq i \leq r$. Now c is not adjacent to any x_i, $2 \leq i \leq r$. Otherwise by item (4) of Lemma 3.4 (cx_i, x_iv_1 are edges) x_i is a dominating vertex for $S_{a,b,c}$ and hence by Corollary 3.6 x_i would be adjacent to u (w) contradiction to Q being chordless. We note that x_i is not adjacent to v_j, $2 \leq j \leq p+1$ as otherwise we get a smaller AT $S_{x_i,b,c}$ with the path $x_i, v_j, v_{j+1}, \ldots, v_p, b$. If $S_{x,y,z}$ is of type (2) we note that u is adjacent to x_i as otherwise we get an AT with the vertices $c, u, x_i, v_1, v_2, \ldots, v_p, b$ and has fewer vertices than $S_{a,b,c}$. Thus cu is an edge when $S_{a,b,c}$ is of type (2). Now regardless of type of $S_{a,b,c}$ we conclude that $S_{x_i,b,c}$ is also a minimum AT with path $P_{x_i,b} = x_i, v_1, \ldots, v_p, b$. $\diamond$

Analogous to the Lemma 5.1 we have the following Lemma.

Lemma 5.2 *Let $S_{a,b,c}$ be a minimum AT with path $P_{a,b} = a, v_1, v_2, \ldots, v_p, b$ and center vertex u (central vertices u, w). Let $Q = u, x_1, x_2, \ldots, x_r, b$ be a chordless path from u to b (not including v_p) (from u to b if $S_{a,b,c}$ is of type 2). Then for every vertex x_i, $2 \leq i \leq r$, $S_{a,x_i,c}$ is a minimum AT with the same number of vertices as $S_{a,b,c}$, and path $P_{a,x_i} = a, v_1, v_2, \ldots, v_p, x_i$.*

Definition 5.3 *Let $S_{a,b,c}$ be a minimum AT in graph G. We refer to a fill-in edge cv_i, $0 \leq i \leq p+1$, as a* long fill-in edge *and we refer to a fill-in edge v_iv_j, $0 \leq i, j \leq p+1$, as a* bottom fill-in edge *of $S_{a,b,c}$. Note that ac, bc are long fill-in edges when $S_{a,b,c}$ is of type 2 and that ab is a bottom fill-in edge.*

By cross fill-in edges *of $S_{a,b,c}$, we call fill-in edges au, bu when $S_{a,b,c}$ is of type 1 and aw, bu, when $S_{a,b,c}$ is of type 2.*

Let us remark that in a graph G' obtained from G by adding either long or cross fill-in edge, subgraph $S_{a,b,c}$ does not induce a cycle of length more than 3 and it does not induce an AT.

Lemma 5.4 *Let $S_{a,b,c}$ be a ripe AT. Let $S_{x,y,z}$ be a minimum AT, with path $P_{x,y} = x, w_1, \ldots, w_q, y$ such that a long fill-in edge cd, $d \in G[a,b,c]$ is a fill-in edge of $S_{x,y,z}$. Then cd is a long fill-in edge of $S_{x,y,z}$ and one of the following happens :*

1. *$z = c$, $P_{x,y} \cap B[a,b] \neq \emptyset$, $P_{x,y} \cap E[a,b] \neq \emptyset$, and every v_i, $2 \leq i \leq p-1$ has a neighbor on $P_{x,y}$ (See Figure 10).*

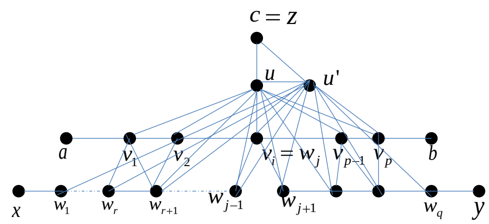

Figure 10: $u' \in D(a,b,c)$ and $P_{x,y} \cap B[a,b] \neq \emptyset$, $P_{x,y} \cap E[a,b] \neq \emptyset$

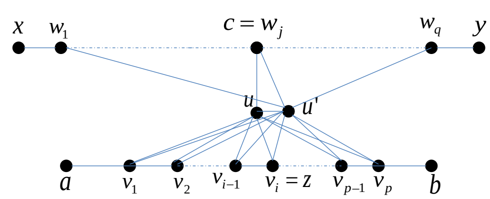

Figure 11: $z \in G[a,b,c]$ and $P_{x,y} \cap G[a,b,c] = \emptyset$

2. $z = d$*, and for every vertex* $z' \in G[a,b,c]$*,* $S_{x,y,z'}$ *is an AT with the same path* $P_{x,y} = x, w_1, w_2, \ldots, w_q, y$ *(See Figure 11).*

Proof: By definition of $G[a,b,c]$, d is adjacent to some v_i, $3 \le i \le p-2$. We first show that cd is not a bottom fill-in edge for $S_{x,y,z}$. Otherwise up to symmetry we may assume that $x = c$ and $w_q = d$ when $S_{x,y,z}$ is of type 1 and $y = d$ when $S_{x,y,z}$ is of type 2. Now by Lemma 3.12 for the path $c, w_1, w_2, \ldots, w_q$ ($cw_1, w_2, \ldots, w_q, y$ when of type 2) ,we conclude that w_1 is a dominating vertex for $S_{a,b,c}$ and hence by Lemma 3.4 (7) w_1w_q (w_1y when $S_{x,y,z}$ is of type 2) is an edge of G, yielding a contradiction.

We show that cd is not a cross fill-in edge for $S_{x,y,z}$. For contradiction suppose cd is a cross fill-in edge for $S_{x,y,z}$. Let u' be one of the center vertices of $S_{x,y,z}$. Now consider the path c, x', d that is corresponding to one the paths x, w_1, u' and y, w_q, u' and u', w_1, x and u', w_1, y in $S_{x,y,z}$. By Lemma 3.12 for the path c, x', d we conclude that x' is a dominating vertex for $S_{a,b,c}$. W.l.o.g assume that c, x', d is corresponding to path u', w_1, x or path x, w_1, u'. Thus w_1 is a dominating vertex for $S_{a,b,c}$. We observe that $u' \neq c$ as otherwise because $u'z = cz$ is an edge, Corollary 3.6 implies that zw_1 is an edge, a contradiction. Therefore we have $c = x$ and $u' = d$ ($w' = d$ when $S_{x,y,z}$ is of type 2). Because $u' \in G[a,b,c]$ and $u'w_3$ is an edge by Lemma 3.10 w_3 is adjacent to every vertex in $D(a,b,c)$ and in particular w_3w_1 is an edge; a contradiction. Therefore cd is not a cross fill-in edge for $S_{x,y,z}$.

We conclude that cd is a long fill-in edge. By considering the path c, u', d and applying the Lemma 3.12 we conclude that u' is a dominating vertex for $S_{a,b,c}$. Now the conditions of the Lemma 3.16 are satisfied and hence one of the (1),(2) holds.

◇

Lemma 5.5 *Let* $S_{a,b,c}$ *be a ripe AT with path* $P_{a,b} = a, v_1, v_2, \ldots, v_p, b$*. Let* $S_{x,y,z}$ *be a minimum AT, with center vertex* u' *(central vertices* u', w' *is of type 2). If bottom fill-in edge* ab *(when* $S_{a,b,c}$ *is of type 2) (*av_p *when* $S_{a,b,c}$ *is of type 1) is a fill-in edge of* $S_{x,y,z}$ *with path* $P_{x,y} = x, w_1, \ldots, w_q, y$ *then one of the following happens :*

1. ab *(when* $S_{a,b,c}$ *is of type 2) (*av_p *when* $S_{a,b,c}$ *is of type 1) is a cross fill-in edge,* $a = x$*,* $b = u'$ *(*$a = x$ *and* $v_p = u'$ *when type 1) and for every* v_i*,* $1 \le i \le p$*, (* $1 \le i \le p-1$ *when type 1)* $S_{v_i,y,z}$ *is an AT with the same number of vertices as* $S_{x,y,z}$ *and the same center vertex* u'*.*

2. ab *(when* $S_{a,b,c}$ *is of type 2) (*av_p *when* $S_{a,b,c}$ *is of type 1),* $b = y$*,* $a = u'$ *(*$b = y$ *and* $v_1 = u'$ *when type 1) and for every* v_i*,* $1 \le i \le p$*, (* $2 \le i \le p$ *when type 1)* $S_{x,v_i,z}$ *is an AT with the same number of vertices as* $S_{x,y,z}$ *and the same center vertex* u'*.*

3 up to symmetry for av_p*,* bv_1 *we have the followings :*

3.1 $S_{x,y,z}$ *is of type (2) and* $a = x$*,* $b = z$ *(* $a = x$*,* $v_p = z$ *when* $S_{a,b,c}$ *is type 1) and for every* v_i*,* $1 \le i \le p$*, (*$1 \le i \le p-1$ *when type 1)* $S_{v_i,y,z}$ *is an AT with the same number of vertices as* $S_{x,y,z}$ *and the same central vertices* u', w'*.*

3.2 $S_{x,y,z}$ *is of type (2) and* $a = z$*,* $b = y$ *(* $v_p = x$*,* $a = z$ *when* $S_{a,b,c}$ *is type 1) and for every* v_i*,* $1 \le i \le p$*, (*$1 \le i \le p-1$ *when type 1)* $S_{x,v_i,z}$ *is an AT with the same number of vertices as* $S_{x,y,z}$ *and the same central vertices* u', w'*.*

Proof: We prove the theorem when $S_{a,b,c}$ is of type 2. The proof when $S_{a,b,c}$ is of type 1 is similar.

First suppose ab is a cross fill-in edge of $S_{x,y,z}$. Therefore up to symmetry we are left with the case $a = x$ and $b = u'$. By Lemma 5.1 for $S_{x,y,z}$ we conclude that for every v_i, $S_{v_i,y,z}$ is an AT with the same number of vertices as $S_{x,y,z}$.

Now suppose that ab is a long fill-in edge for $S_{x,y,z}$. If $a = w_j$ for some $1 \leq j \leq q$ and $b = z$ then by Lemma 3.5 for $S_{x,y,z}$ and path $z, v_p, v_{p-1}, \ldots, v_1, a$ we conclude that v_p is a dominating vertex for $S_{x,y,z}$ and hence $v_p v_1$ is an edge. This is a contradiction. Thus up to symmetry we may assume that $a = x$ and $b = z$. We note that in this case $S_{x,y,z}$ is of type (2). Now again by Lemma 3.5, v_p is a dominating vertex for $S_{x,y,z}$. We note that v_p is not adjacent to a and hence v_p must be adjacent to y otherwise we obtain a smaller AT with the vertices $z, v_p, x, w_1, \ldots, w_q, y$, contradicting the minimality of $S_{x,y,z}$. Observe that by replacing w' with v_p we obtain an AT $(S_{x,y,z})'$ with the same number of vertices as $S_{x,y,z}$. However by applying Lemma 5.2 for $(S_{x,y,z})'$ with the path $v_p, v_{p-1}, ...v_1, x$ we conclude that for every v_i, $S_{v_i,y,z}$ is an AT with the same number of vertices as $S_{x,y,z}$.

$\diamond$

5.2 Algorithm For Interval Completion

$Interval - Completion(G, K)$ **Algorithm**

Input : Graph G, and parameter k.

Output : A minimum set F of edges such that $|F| \leq k$ and $G \cup F$ is an interval graph OR report NOT exists.

1. If G is an interval graph then return $\emptyset$.

2. If $k \leq 0$ and G is not interval graph then report NOT exists.

3. Let C be cycle with $|C| \geq 4$. For every minimal triangulation F of C set $F' = Interval - Completion(G \cup F, k - |C| + 3)$. If F' exists then return $F \cup F'$.

4. Let S be a small AT in G. For every edge e (at most 9 ways) such that $S \cup \{e\}$ is not an AT in G set $F = Interval - Completion(G \cup \{e\}, k - 1)$. If F exists then return $F \cup \{e\}$.

5. Let S_{a_0,b_0,c_0} be a minimum AT in G. Apply the Algorithm 1 to obtain a ripe AT, $S_{a,b,c}$ with the path $P_{a,b} = a, v_1, v_2, \ldots, v_p, b$ and center vertex u (u, w when is of type 2).

6. Let X be a smallest set of vertices in $G \setminus N(c)$ such that X contains a vertex $v_i \in X$, $7 \leq i \leq p - 6$, and there is no path from v_6 to v_{p-5} in $G \setminus (X \cup N(c))$.

7. Let $S = \{au, bu\}$ when $S_{a,b,c}$ is of type 1 otherwise let $S = \{aw, bu\}$. Set $F = Interval - Completion(G \cup \{e\}, k - 1)$. If F exists then return $F \cup \{e\}$.

8. If $S_{a,b,c}$ is of type 1 then for **each** of the long fill edge $e = cv_i$, $i \in \{1, 2, 3, 4, 5, 6, p - 5, p - 4, p - 3, p - 2, p - 1, p\}$ set $F = Interval - Completion(G \cup \{e\}, k - 1)$. If F exists then return $F \cup \{e\}$.

9. If $S_{a,b,c}$ is of type 2 then for **each** of the long fill edge $e \in ca, cv_i, cb$, $i \in \{1,2,3,4,5,6,p-5,p-4,p-3,p-2,p-1,p\}$ set $F = Interval - Completion(G \cup \{e\}, k-1)$. If F exists then return $F \cup \{e\}$.
10. Let $S = \{av_i | 2 \leq i \leq p\} \cup \{ab\}$. Set $S_1 = S$ when $S_{a,b,c}$ is of type 2 otherwise $S_1 = S \setminus \{ab\}$. Set $F = Interval - Completion(G \cup S_1, k - |S_1|)$. If F exists then return $F \cup S_1$.
11. Let $S = \{bv_i | 1 \leq i \leq p-1\} \cup \{ab\}$. Set $S_1 = S$ when $S_{a,b,c}$ is of type 2 otherwise $S_1 = S \setminus \{ab\}$. Set $F = Interval - Completion(G \cup S_1, k - |S_1|)$. If F exists then return $F \cup S_1$.
12. Let U be the set of vertices adjacent to u and not adjacent to any vertex on the path $P_{a,b} = a, v_1, v_2, \ldots, v_p, b$. Let C be a connected component of $G[U]$, containing c
13. Let S_3 be the set of all edge $c'x$ for some $c' \in C$ and $x \in X$. Set $F = Interval - Completion(G \cup S_3, k - |S_3|)$. If F exists then return $F \cup S_3$.

We now focus on the correctness of the Interval-Completion algorithm. In what follows we consider the ripe AT, $S_{a,b,c}$ with the path $P_{a,b} = a, v_1, v_2, \ldots, v_p, b$ and center (central vertices) u (u, w if of type 2).

Definition 5.6 *For center vertex u in $S_{a,b,c}$ let U be the set of vertices adjacent to u and not adjacent to any vertex on the path $P_{a,b} = a, v_1, v_2, \ldots, v_p, b$. Let C be a connected component of $G[U]$, containing c (by Lemma 3.4 (4) for every $c' \in C$, $S_{a,b,c'}$ is a minimum AT, with the same number of vertices as $S_{a,b,c}$).*

Now we are ready to prove the following Lemma.

Lemma 5.7 *Let X' be a minimal separator in $G[a,b,c]$ such that $v_i \in X'$, $7 \leq i \leq p-6$. Set $E'_X = \{c'x' | c' \in C, x' \in X'\}$. Then $G \cup E'_X$ does not contain a minimum AT, S (small or big) containing some edges of E'_X.*

Proof:

For contradiction we assume there exists a minimum AT, that uses one edge from E'_X and it is not contained in some other AT. We show that there exists an AT in G such that it has a vertex from $N[v_i]$, $7 \leq i \leq p-6$ but none of the items (1,2,3,4) of the Lemma 3.18 holds for this AT and consequently $G[a,b,c]$ is not an interval graph, i.e., $S_{a,b,c}$ is not ripe.

Observation 1. Since $G[a,b,c]$ is an interval graph X' is a clique and $G[a,b,c] \cup \{cx' | x' \in X'\}$ induce interval graph in G.

Let S be a minimum AT with the vertices x', y', z' such that cd, $c \in C$ and $d \in X'$ is an edge of S. We assume that $d \in N[v_i]$, $7 \leq i \leq p-6$. W.l.o.g assume that cd is an edge of the shortest path P_1 from x' to y' outside the neighborhood of z'. We observe that $z' \notin C$ as otherwise $z'd$ is an edge. Since d is adjacent to every vertex $c' \in C$, we conclude that P_1 has only one vertex c in C.

Let d_1 be the neighbor of d in P_1 and d_2 be the neighbor of c in P_1. First suppose $c \neq y'$. Now d_2 is not in $D(a,b,c)$ as otherwise by Corollary 3.6 dd_2 is an edge , a contradiction to the minimality of the length of P_1. $d_2 \notin C$ as otherwise dd_2 is an edge in $E_{X'}$ and we get a shorter path. Thus we conclude that $d_2 \in X'$ and since X' is a clique, dd_2 is an edge and hence we replace dc, cd_2 by dd_2 in P_1 and we get a shorter path. This contradicts the minimality of the AT. Therefore we assume that $c = y'$.

We note that $P_1 \cap D(a,b,c) = \emptyset$ as otherwise we get a shorter path from x' to y'. Let P_2 be the shortest path from y' to z' that avoids the neighborhood of x' and P_3 be the shortest path from z' to x' that avoids the neighborhood of y'. Let cd_2 be the first edge of P_2. By Corollary 3.6 $P_3 \cap D(a,b,c) = \emptyset$ as otherwise P_3 does not avoid $y' = c$. We also note that P_3 does not uses any edge z_1z_2, $z_2 \in X'$ as otherwise $y'z_2$ is an edge in $E_{X'}$.

We show that P_2 goes through some vertex in $D(a,b,c)$. For contradiction suppose $P_2 \cap D(a,b,c) = \emptyset$. If P_2 does not use any of the edges in E'_X then $P_2P_3P_1$ contains an induce shortest path Q from $c = y'$ to vertex $d \in G[a,b,c]$. Thus lemma 3.12 implies that d_2 is a dominating vertex for $S_{a,b,c}$, a contradiction.

Therefore we assume that P_2 uses some edge $y_1y_2 \in E'_X$, $y_2 \in X'$. Note that there is only one edge of P_2 in E'_X as otherwise we get a shorter path because $c = y'$ is adjacent to all the vertices in X'. Now consider part of path P_2 from y_2 to z' and part of P_1 from x' to d and the path P_3 and edge dy_2 (both $d, y_2 \in X'$) we get cycle of length more than three or an AT S' in G. None of the vertices of this AT is in $D(a,b,c)$ and none of the edges of S' in E'_X. This means we get an AT in G such that it contains a vertex from $N[v_i]$, $7 \leq i \leq p-6$. Now we get a contradiction by Lemma 3.18 because at least one vertex of the AT, S' must be in $D(a,b,c)$ or S' is in $G[a,b,c]$, implying that $G[a,b,c]$ is not ripe.

We conclude that P_2 contains a vertex from $D(a,b,c)$.

Before we proceed we summarize the followings.

(1) $d_2 \in D(a,b,c)$. Since every vertex in $D(a,b,c)$ is adjacent to $c = y'$ and P_2 is the shortest path.

(2) x' is not adjacent to any vertex in $G[a,b,c]$. Otherwise by Lemma 3.10 $x'd_2$ is an edge.

(3) $P_1 \cap D(a,b,c) = \emptyset$.

(4) We may assume that $B[a,b] \cap P_1 \neq \emptyset$. Note that P_1 is a path from $x' \notin G[a,b,c]$ to vertex $d \in G[a,b,c]$. Therefore by Lemma 3 we may assume that $B[a,b] \cap P_1 \neq \emptyset$.

Let P'_2 be a path from d to d_2 and then following P_2 to z'.

Case 1. z' is adjacent to some v_j, $0 \leq j \leq p+1$.

We show that $j \leq i$. For contradiction suppose $j > i$. Now P_3 from z' to x' must contain a vertex from X'. Otherwise we would have a path v_jP_3Q outside the neighborhood of $y' = c$ where Q is part of P_1 from x' to v_1. But this would be a contradiction to X' is a separator.

We continue by having $j \leq i$. If $j \leq 3$ then we get an AT S with the vertices d, z', x' as follows: x' is joined with z' via part of P_3 from x' to the first time P_3 reaches to v_j and then to z' (note that since $d \in N[v_i]$, $7 \leq i \leq p-6$ then dv_j is not an edge). d is joined with z' via P'_2 and finally d is joined with x' via path P_1. We note that none of the edges S belongs to

$E_{X'}$. However since $d_2 \in D(a,b,c)$ and $d \in N[v_i]$, $7 \leq i \leq p-6$ the conditions of the Lemma 3.18 are met while none of the (1,2,3,4) consequences of Lemma 3.18 holds and hence we get a contradiction to $S_{a,b,c}$ is ripe. If $j > 3$ then we get an AT, v_2, c, z' as follows: v_2 is joined with c via part of P_1 from the neighborhood of v_2 to c. There is a path from v_2 to z' using the vertices of P_1 $v_2, v_3, \ldots, v_j, z'$ and then z' to c via the vertices $v_j, v_{j+1}, \ldots, v_i, d, c$ yielding an AT, inside $G[a,b,c] \cup \{cx' | x' \in X'\}$. This is a contradiction according to Observation 1.

Case 2. z' has no neighbor in the path $P_{a,b}$.
If the path $P_3 \cap G[a,b,c] = \emptyset$ then we obtain a smaller AT S with the vertices d, x', z' as follows. x' is joined with d via part of P_1 from x to d and P_3 joins x' to z' and P'_2 joins d_2 and z'. We note that none of the edges of S belongs to $E_{X'}$ and S contains a vertex d from $N[v_i]$, $7 \leq i \leq p-6$. Now the conditions of the Lemma 3.18 are met while none of the (1,2,3,4) consequences of Lemma 3.18 holds and hence we get a contradiction to $G[a,b,c]$ is an interval, i.e., $S_{a,b,c}$ is ripe. Therefore $P_3 \cap G[a,b,c] \neq \emptyset$.

Since $P_3 \cap D(a,b,c) = \emptyset$ and $P_3 \cap G[a,b,c] \neq \emptyset$, by Lemma 3 $P_3 \cap (B[a,b] \cup E[a,b]) \neq \emptyset$. Consider the first time that P_3 visits a vertex in $P_{a,b}$. Either we have $P_3 \cap E[a,b] \neq \emptyset$ or $P_3 \cap B[a,b] \neq \emptyset$. Recall that by our assumption $B[a,b] \cap P_1 \neq \emptyset$. If $B[a,b] \cap P_3 = \emptyset$ and $E[a,b] \cap P_3 \neq \emptyset$ then P_3 must contain a vertex from X'. Otherwise we would have a path $v_p P'_3 P'_1 v_1$ outside the neighborhood of $y' = c$ where P'_3 is part of P_3 from a vertex in the neighborhood of v_1 to x' and P'_1 is part of P_1 from x' to a vertex in the neighborhood of v_p. But this would be a contradiction to X' is a separator.

We continue by having $P_3 \cap B[a,b] \neq \emptyset$. Now consider the first time P_3 has a vertex from $N[v_1]$, and the first time P_1 contains a vertex from $N[v_1]$. We obtain a path from x' to z' that avoids the neighborhood of d. Now we get an AT d, x', z', and similarly we get a contradiction.

◇

Lemma 5.8 *Let G be a chordal graph without small ATs and let $S_{a,b,c}$ be a ripe AT with the path $P_{a,b} = a, v_1, v_2, \ldots, v_p, b$. Let X be a minimum separator in $G \setminus N(c)$ that separates v_6 from v_{p-5} and it contains a vertex v_i, $7 \leq i \leq p-6$. Then there is a minimum set of fill-in edges F such that $G \cup F$ is an interval graph and at least one of the following holds:*

(i) If $S_{a,b,c}$ is of type 1 then F contains at least one fill-in edge from

$$\{bu, au, cv_1, cv_2, cv_3, cv_4, cv_5, cv_6, cv_{p-5}, cv_{p-4}, cv_{p-3}, cv_{p-2}, cv_{p-1}, cv_p\}$$

If $S_{a,b,c}$ is of type (2) then F contains at least one fill-in edge from

$$\{bu, aw, ca, cb, cv_1, cv_2, cv_3, cv_4, cv_5, cv_6, cv_{p-5}, cv_{p-4}, cv_{p-3}, cv_{p-2}, cv_{p-1}, cv_p\}$$

(ii) F contains all the edges av_i, $2 \leq i \leq p$ and ab when $S_{a,b,c}$ is of type 2.

(iii) F contains all the edges bv_i, $1 \leq i \leq p-1$ and ab when $S_{a,b,c}$ is of type 2.

(iv) F contains all the edges cf, $f \in G[a,b,c]$

(v) F contains all edges $E_X = \{c'x | c' \in C, x \in X\}$.

Proof: Let H be a minimum set of fill-in edges such that $G \cup H$ is an interval graph. If $S_{a,b,c}$ is of type 1 and H contains an edge from

$$\{bu, au, cv_1, cv_2, cv_3, cv_4, cv_5, cv_6, cv_{p-5}, cv_{p-4}, cv_{p-3}, cv_{p-2}, cv_{p-1}, cv_p\}$$

or when $S_{a,b,c}$ is of type (2) and H contains an edge from

$$\{bu, aw, ca, cb, cv_1, cv_2, cv_3, cv_4, cv_5, cv_6, cv_{p-5}, cv_{p-4}, cv_{p-3}, cv_{p-2}, cv_{p-1}, cv_p\}$$

then we set $F = H$.

Suppose H contains edge ab when $S_{a,b,c}$ is of type 2 and without loss of generality H contains edge av_p if $S_{a,b,c}$ is of type 1. We argue that H also contains all the edges av_i, $2 \leq i \leq p$ ($i \leq p-2$ when $S_{a,b,c}$ is of type 1) or all the edges bv_i, $1 \leq i \leq p-1$ when $S_{a,b,c}$ is of type 2. Mow we may assume that there exists a minimum AT, $S_{x,y,z}$ such that ab (av_p when $S_{a,b,c}$ is of type 1) is a fill-in edge for $S_{x,y,z}$. By Lemma 5.5 (1,2) ab (av_p) is a cross fill-in edge for $S_{x,y,z}$ and up to symmetry we have $a = x$ and $b = u'$ ($a_p = u'$) and for every $x' \in G[a,b,c]$, $S_{x',y,z}$ is an AT with the same number of vertices as $S_{x,y,z}$. This implies that every optimal solution must also add the edges $u'v_1, u'v_2, \ldots, u'v_{p-1}$ and in particular H contains all the edges bv_i, $1 \leq i \leq p-1$ and ab (when $S_{a,b,c}$ is of type 2) this case we also set $F = H$. If one if the items (3) and (4) of the Lemma 5.5 holds again we conclude that H must contains all the edges bv_i, $1 \leq i \leq p-1$ and ab (when of type 2) or H must contain all the edges av_i, $2 \leq i \leq p$ and ab (when of type 2).

We will proceed by assuming that H does not contain any of the edges au, bu (aw, bu if of type 2), ab, cv_r, $0 \leq r \leq 6$, and cv_r, $p-5 \leq r \leq p+1$. Moreover we assume that H does not contain all the edges cf, $f \in G[a,b,c]$ as otherwise we set $F = H$.

Let $W = \{w | cw \in H\}$ be the set of vertices adjacent to c via fill-in edges. Because $S_{a,b,c}$ is an AT there is no path from v_6 to v_{p-5} in $(G \cup H) \setminus N(c)$. Hence, set W should contain a *minimal* v_6, v_{p-5}-separator X' in $G \setminus N(c)$, containing a vertex v_i, $7 \leq i \leq p-6$.

Claim 5.9 *Let c' is a vertex adjacent to c and not adjacent to any vertex on the path $P_{a,b}$ in G. Then H contains all the edges $c'x'$, $x' \in X'$.*

Proof: Indeed, by Lemma 3.4, c' is adjacent to u, and thus $S_{a,b,c'}$ is also a minimum AT with the same number of vertices as $S_{a,b,c}$. Since $S_{a,b,c'}$ is no longer an AT in $G \cup H$ and none of the au, bu is an edge in H and ab (av_p, bv_1 when $S_{a,b,c}$ is of type 2) is not an edge in H we have that H contains at least one edge $c'v_j$. Let us assume that $v_j \neq v_i$ is the closest vertex to v_i such that cv_j is not in H. (Observe that we assumed that the H is a minimal set of fill edges such that $G \cup H$ is an interval graph and cv_i is in H since $S_{a,b,c}$ is an AT in G). Let P be part of $P_{a,b}$ from v_i to v_j. By our assumption for H, P has no chords. No vertex of P except v_j and v_i is adjacent to c or c'. Thus the cycle c, P, c', c' is a chordless cycle in $G \cup H$, which is a contradiction. Therefore we conclude that $c'v_i$ is an edge. No let $W' = \{w' | c'w' \in H\}$. Since $S_{a,b,c'}$ is an AT and none of the au, bu, cv_r, $v_r \in \{v_1, \ldots, v_6, v_{p-5}, \ldots, v_p\}$ is in H (note that $7 \leq i \leq p-6$) and there is no path from v_6 to v_{p-5} in $(G \cup H) \setminus N(c)$, set W' should contain a *minimal* v_6, v_{p-5}-separator X'' containing $v_i = v_j$ in $G \setminus N(c)$. Because $G[a,b,c]$ is an interval graphs and both X' and X'' contain v_i we have $X' = X''$. ◇

Now by applying the Claim 5.9 for every $c'' \in C$ we conclude that $c''x'$, $x' \in X'$ is in H. Let $E'_X = \{c'x' | c' \in C, x' \in X'\}$. We observe that X' is a clique because $G[a,b,c]$ is an interval graph.

We define $F = (H \setminus E'_X) \cup E_X$. Let us note that because none of the sets E_X and E'_X contains edges of G and because X is a minimum separator, we have that $|F| \leq |H|$. In what follows, we prove that $I = G \cup F$ is an interval graph. For a sake of contradiction, let us assume that I is not an interval graph. We note that by Lemma 5.7 adding the edges $c'x'$, $c' \in C$ and $x' \in X'$ would not add new AT in G. Therefore by Theorem 1.1 we may assume that I contains cycle of length more than three or a big AT with the edges in G.

Case 1. I contains big AT.

Let $S_{x,y,z}$ be an AT in I. We assume that vertices x and y are connected in $S_{x,y,z}$ by an induced path $x, w_1, w_2, \ldots, w_q, y$, where $q \geq 6$. Because $S_{x,y,z}$ is not an AT in $G \cup H$, set $E'_X \setminus E_X$ must contain some fill-in edge cd, for $S_{x,y,z}$. By definition of $G[a, b, c]$, d is adjacent to some v_i, $3 \leq i \leq p-2$. Every fill-in edge of $S_{x,y,z}$ is either long, cross, or bottom, see Definition 5.3.

Claim A. cd is not a cross fill-in edges of $S_{x,y,z}$.
By Lemma 5.4 the fill-in edge cd is a long fill-in edge of $S_{x,y,z}$ and not a cross fill-in edge.

Claim B. cd is not a bottom fill-in edge of $S_{x,y,z}$.
For contradiction suppose cd is a bottom fill-in edge for $S_{x,y,z}$. Thus we have $c = x$ and $y = d$ or $c = y$ and $x = d$. W.l.o.g assume that $c = x$ and $y = d$. Now there is a path $Q = c, w_1, w_2, \ldots, w_q, d$ from c to d. Since Q is chordless, by Lemma 3.12 w_1 is a dominating vertex for $S_{a,b,c}$ and by Corollary 3.9 $w_1 d$ is an edge. This is a contradiction because $q > 2$.

We conclude that cd is a long fill-in edge for $S_{x,y,z}$. Now we have $z = c$ or $z = d$. If $z = d$ then by Lemma 5.4 (2) for every vertex $f \in G[a, b, c]$, $S_{x,y,f}$ is an AT with the same path $x, w_1, w_2, \ldots, w_q, y$. Since H does not contain any of the edge au, bu, cv_i, $i \in \{1, \ldots, 6, p-5, \ldots, p\}$, and $S_{a,b,f}$ is an AT, H must contain the edge cf. But this is a contradiction as we assumed that H does not contain all the edges cf, $f \in G[a, b, c]$.

Therefore we suppose $z = c$. We argue that there exists a fill-in edge $cd' \in E_X$, such that $d' \in \{x, w_1, w_2, \ldots, w_q, y\}$. Since $z = c$, Lemma 5.4 (1) implies that $P_{x,y} \cap B[a, b] \neq \emptyset$ and $P_{x,y} \cap E[a, b] \neq \emptyset$ and every v_i, $2 \leq i \leq p-1$ has a neighbor in $P_{x,y}$. Therefore there would be a path from v_2 to v_{p-1}. This is a contradiction to X being a v_6, v_{p-5} separator and hence there exists some fill-in edge $cd' \in E_X$, such that $d' \in \{x, w_1, w_2, \ldots, w_q, y\}$.

Case 2. I contains a cordless cycle of length more than three. This implies that there exists a path Q from c to $d \in G[a, b, c]$. However by Lemma 3.12 the second vertex of Q say d' is a dominating vertex for $S_{a,b,c}$ and since $d \in G[a, b, c]$ by Corollary 3.9 $d'd$ is an edge. This implies that the length of Q is 2, a contradiction.

$\diamond$

In the following Lemma we address the correctness and complexity of the Algorithm.

Theorem 5.10 *The Branching Algorithm is optimal and its running time is $O(c^k n(n+m))$, $c \in \min\{17, k\}$ for parameter k.*

Proof: The correctness of the Branching Algorithm is justified by Lemma 5.8. By Lemma 5.8(1,2) there are five ways of adding one fill in edge for AT $S_{a,b,c}$ of type 1 (type 2). Either we add one of the edges au, bu, cv_i, $1 \leq i \leq 6$ or $p-5 \leq i \leq p$ or we add at least one edge from c

to set X or we add edge ab and hence the Algorithm needs to add $p-2$ other fill in edges. Note that once we add ab then the parameter k decrease by at least 5. In order to get the maximum number of branching we may assume that no bottom fill-in edge ab is added. By looking at the small AT, together with the AT's of type 1 and type 2 there are at most $\max\{17, k\}$ possible ways to add a fill in edge to $S_{a,b,c}$ and at each step the parameter k is decreased by at least one. We may deploy the algorithms developed in [14] with the running time $O(n(n+m))$ to find ATs. Therefore the running time of the algorithm is $O(c^k n(n+m))$, $c \in \min\{17, k\}$. ◇

6 Conclusion and future work

We have shown that there exist single exponential FPT algorithms for k-interval deletion problem. The obstruction for the class of interval graphs is not finite but the obstructions can be partitioned into a constant number of families.

Let Π be a class of graphs. We say Π has *family bounded* property if the forbidden subgraphs for this class can be partitioned into a constant number of families. Let $\Pi + kv$ denotes the problem of deleting k vertices (edges) from (into) input graph G such that the resulting graphs becomes a member of Π. It would be interesting to study the following problem.

Problem 6.1 *For which classes Π of graphs with family bounded property, the problem $\Pi + kv$ is FPT ?*

Remark : We have heard that Cao and Marx have solved the k-interval deletion problem. They start by branching on small interval graph obstructions and then start breaking the cycles first and then deleting the big AT's. They have made some comments about an earlier version of this paper and they had some concerns for the cycle breaking procedure. I have said that the structure of the subgraph induced by $N[C] \setminus D(C)$ is simple and it is a circular arc graph. This statement led them to a confusion and I make it clear in this version. I also would like to thank Yixin Cao for a useful comment in the k-interval completion algorithm.

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