Interval H-graphs: Recognition and forbidden obstructions

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Abstract

We introduce the class of *interval H-graphs*, which is the generalization of interval graphs, particularly interval bigraphs. For a fixed graph H with vertices $a_1, a_2, \ldots, a_k$, we say that an input graph G with given partition $V_1, \ldots, V_k$ of its vertices is an interval H-graph if each vertex $v \in G$ can be represented by an interval I_v from a real line so that $u \in V_i$ and $v \in V_j$ are adjacent if and only if $a_i a_j$ is an edge of H and intervals I_u and I_v intersect. G is called interval k-graph if H is a complete graph on k vertices. and interval bigraph when $k = 2$. We study the ordering characterization and forbidden obstructions of interval k-graphs and present a polynomial-time recognition algorithm for them. Additionally, we discuss how this algorithm can be extended to recognize general interval H-graphs. Special cases of interval k-graphs, particularly comparability interval k-graphs, were previously studied in [2], where the complexity interval k-graph recognition was posed as an open problem.

1 Introduction and Problem Definition

The vertex set of a graph G is denoted by $V(G)$ and the edge set of G is denoted by $E(G)$. A graph G is called an *interval graph*, if there exists a family I_v, $v \in V(G)$, of intervals (from the real line) such that, for all different $x, y \in V(G)$ the vertices x and y are adjacent in G if and only if I_x and I_y intersect. A bigraph G is a bipartite graph with a fixed bipartition into *black* and *white* vertices. We sometimes denote these sets as B and W, and view the vertex set of G as partitioned into (B, W). A bigraph G is called an *interval bigraph* if there exists a family I_v, $v \in B \cup W$, of intervals (from the real line) such that, for all $x \in B$ and $y \in W$, the vertices x and y are adjacent in G if and only if I_x and I_y intersect. Then, this family of intervals is called an *interval representation* of the bigraph G.

Interval bigraphs were introduced in [14] and have been studied in [5, 15, 21]. They are closely related to interval digraphs introduced by Sen *et al.* [23]. Interval bigraphs and interval digraphs have become of interest in such new areas as graph homomorphisms, *e.g.* [11].

A *co-circular arc bigraph* is a bipartite graph whose complement is a circular arc graph (see [4] for the definitions of graph classes not introduced here). The class of interval bigraphs is a subclass of co-circular arc bigraphs. Indeed, the former class consists exactly of those bigraphs whose complement is the intersection of a family of circular arcs no two of which cover the circle [15]. There is a linear-time recognition algorithm for co-circular arc bigraphs [20]. On

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the other hand, the class of interval bigraphs is a super-class of proper interval bigraphs (also known as bipartite permutation graphs [24] or monotone graphs [10]), for which there is also a linear-time a linear time recognition algorithm [15, 24].

Interval bigraphs can be recognized in polynomial time using the algorithm developed by Müller [21]. Müller's algorithm runs in time $\mathcal{O}(nm^6(n+m)\log n)$ where m is the number of edges and n is the number of vertices on input bigraph G. A faster algorithm was developed in [22]; with running time $\mathcal{O}(mn)$. But there are several linear time algorithms for recognition of *interval graphs*, are known, *e.g.*, [3, 6, 7, 13, 19].

We use the ordering characterization of interval bigraphs in [15]. A bigraph G is an interval bigraph if and only if its vertices admit a linear ordering $<$ without any of the forbidden patterns in Figure 1. Hence, we will rely on the existence of a linear ordering $<$ such that if $v_a < v_b < v_c$ (not necessarily consecutively) and v_a, v_b have the same color and opposite to the color of v_c then $v_a v_c \in E(G)$ implies that $v_b v_c \in E(G)$.

Figure 1: Forbidden patterns for interval bigraphs

We call such an ordering $<$ a *bi-interval ordering* for G. There are several graph classes that can be characterized by the existence of an ordering without a number of forbidden patterns. One such class is the class of interval graphs. A graph G is an interval graph if and only if there exists an ordering $<$ of $V(G)$ such that none of the following patterns appears [8, 9].

- $v_a < v_b < v_c$, $v_a v_c, v_b v_c \in E(G)$ and $v_a v_b \notin E(G)$
- $v_a < v_b < v_c$, $v_a v_c \in E(G)$ and $v_b v_c, v_a v_b \notin E(G)$

Some of the other classes of graphs that have ordering characterizations without forbidden patterns are proper interval graphs, comparability graphs, co-comparability graphs, chordal graphs, convex bipartite graphs, co-circular arc bigraphs, permutation bigraphs, and proper interval bigraphs [17]. We, in particular, mention the ordering characterization of permutation bigraphs and co-circular arc bigraphs without forbidden patterns.

A bigraph $G = (A, B, E)$ is a co-circular arc bigraph if there is a linear ordering $a_1 < \cdots < a_p < b_1 < b_2 < \cdots < b_q$ (with $A = \{a_1, \ldots, a_p\}, B = \{b_1, \ldots, b_q\}$) so that if $a_i b_j, a_{i'} b_{j'} \in E$ with $i < i'$ and $j' < j$ then $a_i b_{j'} \in E$. Such an ordering $<$ is called a *min ordering* of G [16]. The forbidden pattern corresponding to a min ordering are given in Figure 2.

Figure 2: Forbidden Patterns for co-circular arc bigraphs

Likewise, a bigraph $G = (A, B, E)$ is a permutation bigraph if there is a linear ordering $a_1 < \cdots < a_p < b_1 < b_2 < \cdots < b_q$ (with $A = \{a_1, \ldots, a_p\}, B = \{b_1, \ldots, b_q\}$) so that if $a_i b_j, a_{i'} b_{j'} \in E$ with $i < i'$ and $j' < j$ then $a_i b'_j, a'_i b_j \in E$. Such an ordering $<$ is called a *min-max ordering* of G [12, 18]. The forbidden pattern corresponding to a min-max ordering are given in Figure 3.

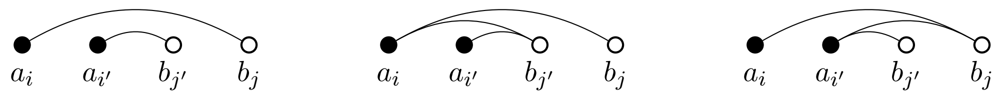

Figure 3: Forbidden Patterns for bipartite permutation graphs (proper interval bigraphs)

Let $G = (A, B, E)$ be an interval bigraph. Let $x_1 \prec \cdots \prec x_n$ be a linear ordering of vertices of G without the forbidden patterns in Figure 1. Let $<$ be the reverse of $\prec$; that is, let $x_n < \cdots < x_1$. Let $a_1 < a_2 < \cdots < a_p$ be the vertices in A under ordering $<$, and $b_1 < b_2 < \cdots < b_q$ be the vertices in B under ordering $<$. Then, it is easy to see that $a_1 < \cdots < a_p < b_1 < b_2 < \cdots < b_q$ provides a min ordering for G. Thus, interval bigraphs are a subclass of co-circular arc bigraphs.

Similarly, let $G = (A, B, E)$ be a permutation bigraph, and let $a_1 < \cdots < a_p < b_1 < \cdots < b_q$ be a min-max ordering of G (with $A = \{a_1, \ldots, a_p\}$, $B = \{b_1, b_2, \ldots, b_q\}$). Then, $<$ is a bi-interval ordering for G. Thus, permutation bigraphs are a subclass of interval bigraphs.

Following this line of research, we introduce a class of interval k-graphs where we still have a total ordering of the vertices of the input graph G, but there are partitions of the vertices, and the forbidden patterns are defined with colored vertices. We consider the graph G, which admits a k-coloring with the given coloring (given partitions), and we say G is an interval k-graph according to the following definition.

Definition 1 (interval k-graphs)**.** *Let G be a k-partite graph ($k \geq 2$) with the given partite sets $V_1, V_2, \ldots, V_k$. We say that G is an* interval k-graph *if there is a family of real line intervals I_v, $v \in V(G)$, so that for all $u, v \in V(G)$ from different partite sets, $uv \in E(G)$ if and only if I_u, I_v intersect.*

Herein, we identify $V_1, \ldots, V_k$ with a k-coloring of G and simply say that two vertices have different colors whenever they belong to different partite sets.

Notice that an interval graph G is an interval k-graph where $k = |G|$. Thus, interval k-graphs generalize interval graphs. Let G be a k-partite graph with the given partite sets $V_1, V_2, \ldots, V_k$. We will show that G is an interval k-graph if and only if G admits an ordering $u_1 < u_2 < \cdots < u_n$ of its vertices without the forbidden patterns depicted in Figure 4.

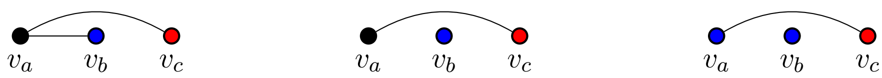

Figure 4: Forbidden patterns for interval k-graphs.

Obstruction for interval k-graphs: We have seen some of the forbidden obstructions of interval bigraphs in [15]. They are called *exobicliques*. The bigraph $G = (B, W)$ is an exobiclique if the following hold.

- B contains a nonempty part B_1 and W contains a nonempty part W_1 such that $B_1 \cup W_1$ induces a biclique in G;

- $B \setminus B_1$ contains three vertices with incomparable neighborhood in W_1 and $W \setminus W_1$ contains three vertices with incomparable neighborhoods in B_1 (an examples given in Figure 5).

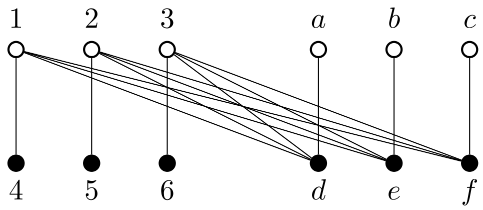

Figure 5: Exobicliques: Here, $B = \{4, 5, 6, d, e, f\}$, $W = \{1, 2, 3, a, b, c\}$ and $B_1 = \{d, e, f\}$, $W_1 = \{1, 2, 3\}$ and $B \setminus B_1 = \{4, 5, 6\}$, $W \setminus W_1 = \{a, b, c\}$.

However, the obstruction for interval bigraphs (interval 2-graphs) are not limited to exobiclique, there is a family of these obstruction considered in [22]. Figure 6 is an obstruction of the interval bigraph that is not an exobiclique.

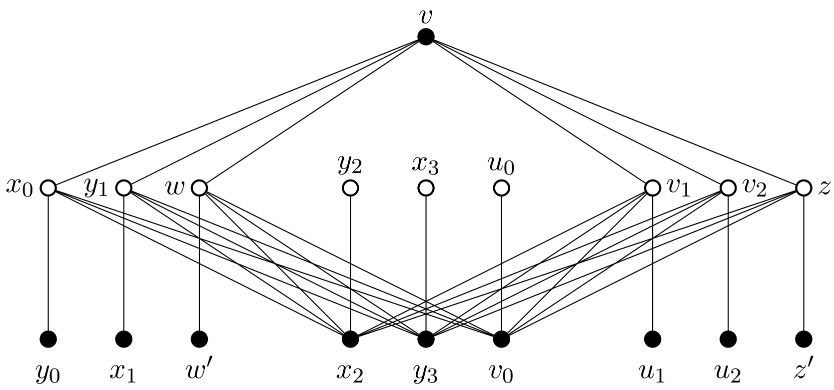

Figure 6: Forbidden Patterns

There are some new obstructions for interval k-graphs $k > 2$, depicted in Figure 7.

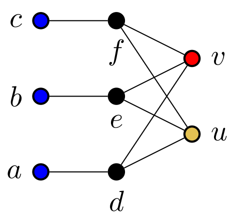

Figure 7: Obstruction for interval 4-graphs

1.1 Our Results and Future Work

Our primary contribution is the development of a recognition algorithm for interval k-graphs.

Theorem 2. *Let G be a graph with a given partition of its vertices into k partite sets $V_0, V_1, \ldots, V_{k-1}$. Then, it can be determined in $O(|V(G)||E(G)|)$ time whether G is an interval k-graph.*

An interesting challenge arises when the k-coloring of G is not provided, yet we still seek to determine whether G is an interval k-graph. However, we do not have a conclusive indication that this problem becomes NP-complete when the k-coloring (a k-partition of its vertices) is not given. The authors of [2] investigated this problem for specific cases where $k = 2, 3$. This leads us to propose the following open problem:

Open Problem 3. *Can one determine in polynomial time whether a given graph G admits a k-partition $V_1, V_2, \ldots, V_k$ such that G, along with this partitioning, forms an interval k-graph?*

Identifying forbidden obstructions for interval k-graphs remains a significant challenge, even for $k = 2$. As discussed in the previous subsection, the forbidden obstructions for interval bigraphs cannot be categorized into a finite number of families. This raises an important question for interval k-graphs: However, it may be possible to enumerate them systematically.

Open Problem 4. *What are the forbidden obstructions for interval k-graphs?*

2 Basic definitions and some preliminary results

Note that for $k > 1$, a k-partite graph G is an interval k-graph if and only if each connected component of it is an interval k-graph. In the remainder of this paper, we assume that $G = (V, E)$ is a connected k-partite graph with a fixed partition $V_1, V_2, \ldots, V_k$. By set of edges in the complete k-partite graph with partite sets $V_1, \ldots, V_k$ that are not present in G we denote by

$$\bar{E} = \{uv \mid u \in V_i, v \in V_j, 1 \leq i < j \leq k\} \setminus E\,.$$

We define *pair-digraph* G^+ of G corresponding to the forbidden patterns in Figure 4, as follows. The set of vertexes of G^+ consists of all pairs (u, v) such that $u, v \in V(G)$ and $u \neq v$. For clarity, we will often refer to vertices of G^+ as *pairs* (in G^+). The arcs in G^+ are of one of the following types:

- $(u, v)(u', v)$ is an arc of G^+ when u and v belong to the same V_i and $uu' \in E(G)$, and $vu' \notin E(G)$.
- $(u, v)(u', v)$ is an arc of G^+ if $uu' \in E(G)$ and $u'v \in \bar{E}(G)$ and u, v, v' all belong to different V_i.
- $(u, v)(u, v')$ is an arc of G^+ when u and v' belong to the same V_i with $vv' \in E(G)$, and $uv \notin E(G)$.
- $(u, v)(u, v')$ is an arc of G^+ if $vv' \in E(G)$ and $uv \in \bar{E}(G)$ and u, v, v' all belong to different V_i.

Observe that if there is an arc from (u, v) to (u', v'), then both uv and $u'v'$ are non-edges of G. For two pairs $(x, y), (x', y') \in V(G^+)$ we say (x, y) *dominates* (x', y') (or (x', y') is *dominated by* (x, y)) and write $(x, y) \to (x', y')$ if there exists an arc (directed edge) from (x, y) to (x', y') in G^+. One should note that if $(x, y) \to (x', y')$ in G^+ then $(y', x') \to (y, x)$, to which property we will refer to as *skew-symmetry*.

Remark. This idea of a pair-digraph can also be applied to the forbidden patterns in Figure 2. With $M = (A, B, E)$ being a connected bigraph, we define the *pair-digraph* M^* of M corresponding to the forbidden pattern in Figure 2 as follows. We set $V(M^*) = \{(u, v) \mid u, v \in A \text{ or } u, v \in B\}$ and $A(M^*) = \{(u, v)(u', v') \mid uu', vv' \in E,\ uv' \notin E\}$. Notice that if $(u, v)(u', v') \in A(M^*)$ then $(u', v')(u, v'), (u, v')(u, v) \in A(M^+)$. Then, it is easy to see that all vertices of each strong component of M^* belong to the same strong component of M^+.

Lemma 5. *Let $<$ be an ordering of G without the forbidden patterns in Figure 4, and let $(u, v) \to (u', v')$ with $u < v$. Then, $u' < v'$.*

Proof. According to the definition of G^+, we either have

Case (1) u, v have the same color, $v = v'$, $uu' \in E(G)$, and $vu' \notin E(G)$; or

Case (2) u, v have different colors, $u = u'$, $vv' \in E(G)$, and $uv \notin E(G)$

In Case (1) (resp. Case (2)), if $v' < u'$, then vertices v', u, v (resp. u, v, u')— in that order— would induce a forbidden pattern in G, a contradiction. Hence, in both cases we will have $u' < v'$, as desired. □

We shall generally refer to a strong component of G^+ simply as a *component* of G^+. We shall also identify a component by its vertex (pair) set. A component in G^+ is called *non-trivial* if it contains more than one pair. For any component S of G^+, we define its *couple component*, denoted S', to be $S' = \{(u, v) : (v, u) \in S\}$.

The skew-symmetry property of G^+ implies the following fact.

Lemma 6. *If S is a component of G^+ then so is S'.*

In light of Lemma 6, for each component S of G^+, S and S' are couple components of each other and we shall collectively refer to them as *coupled components*. It can be easily shown that coupled components S and S' are either disjoint or equal. In the latter case, we say component S is *self-coupled*.

Definition 7 (circuit)**.** *For $n \geq 1$, a sequence $(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n),\ (x_n, x_0)$ of pairs in a set $D \subseteq V(G^+)$ is called a* circuit *in D.*

Lemma 8. *If a component of G^+ contains a circuit then G is not an interval k-graph.*

Should say with respect to the fixed k partition.

Proof. Let $(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, x_0)$ be a circuit in a component S of G^+. Since S is strongly connected, for all non-negative integers i and j there exists a directed walk $W_{i,j}$ in G^+ from (x_i, x_{i+1}) to (x_j, x_{j+1}), where indices are mod $n+1$. Now, for all $i, j \geq 0$, following the sequence of pairs on $W_{i,j}$ and using Lemma 5, we conclude that $x_j < x_{j+1}$ whenever $x_i < x_{i+1}$. Hence, we must either have $x_i < x_{i+1}$ for all i, or $x_i > x_{i+1}$ for all i. However, since $x_{n+1} = x_0$, either case implies $x_0 \neq x_0$; a contradiction. □

If G^+ contains a self-coupled component then G is not an interval k-graph. This is because a self-coupled component of G^+ contains two such pairs as (u, v) and (v, u), which comprise a circuit of length 2 (corresponding to $n = 1$ in the definition of a circuit). However, we will show that these are not the only obstructions to interval k-graphs (an in particular interval bigraphs), that is, there are bigraphs G which are not interval bigraphs, despite G^+ not having any self-coupled component. In contrast, the obstructions to co-circular arc bigraphs are precisely the components of G^* containing both pairs $(x, y), (y, x)$.

Theorem 9. *[16] The bigraph G is a co-circular arc bigraph if and only if it admits a min ordering and if and only if G^* does not contain pairs (x, y) and (y, x) belonging to the same strong component of G^*.*

A *tournament* is a directed graph that can be obtained from a complete undirected graph by orienting each edge in one of the two possible directions. A tournament is called *transitive* if it is acyclic; i.e., if it does not contain a directed cycle.

Lemma 10. *Suppose that G^+ contains no self-coupled components, and let D be any subset of $V(G^+)$ containing exactly one component from each pair of coupled components. Then, D is the set of arcs of a tournament on $V(G)$. Moreover, such a D can be chosen to be a transitive tournament if and only if G is an interval k-graph for any $k \geq 2$.*

In what follows, by a *component* we mean a non-trivial (strong) component unless we specify otherwise. For simplicity, we shall use a set S of pairs in G^+ to also denote the sub-digraph of G^+ induced by S, when no confusion arises.

We shall say two edges ab and cd of G are *independent* if the subgraph of G induced by the vertices a, b, c, and d has just the two edges ab and cd. We shall say two disjoint induced subgraph G_1 and G_2 of G are *independent* if there is no edge of G with one endpoint in G_1 and another endpoint in G_2.

First, to describe the algorithm, we introduce some technical definitions.

Definition 11 (reachability closure)**.** *Let R be a subset of the pairs of G^+. Let $N^+[R]$ denote the set of all pairs in G^+ that are reachable (via a directed path in G^+) from a pair in R. (Notice that $N^+[R]$ contains R.) We call $N^+[R]$ the* reachability closure *of R. We say that a pair (u, v) is* implied *by R if $(u, v) \in N^+[R] \setminus R$. If $R = N^+[R]$, we say that R is* closed under reachability*.*

Definition 12 (envelope)**.** *Let R be a set of pairs of G^+. The* envelope *of R, denoted $N^*[R]$, is the smallest set of pairs that contains R and is closed under both reachability and transitivity (if $(u, v), (v, w) \in N^*[R]$ then $(u, w) \in N^*[R]$).*

Lemma 13. *Let S, S' be the coupled components in G^+, so that both $N^*[S]$ and $N^*[S']$ contain a circuit. Then G is not an interval k-graph.*

Proof. According to Lemma 10 the final set D must be a total ordering with transitivity property. Therefore, one of S and S' must be in D. To find a total ordering that avoids the patterns in Figure 1, one of the $N^*[S]$, $N^*[S']$ must be in D, which is impossible. □

We show the an ordering charaterization of interval k-graphs.

Theorem 14. *For a fixed $k \geq 2$, let G be a k-partite graph with the given partite sets $V_1, V_2, \ldots, V_k$. G is an interval k-graph if and only if G admits an ordering $u_1 < u_2 < \cdots < u_n$ of its vertices without the forbidden patterns depicted in Figure 4.*

Proof. We denote the right and left end-points of an interval I by $r(I)$ and $\ell(I)$, respectively.

First, suppose there is an interval representation $I_v,\ v \in G$ of G. Now, consider the total ordering $v_1 < v_2 < \cdots < v_n$ of G where for all i, j we have $v_i < v_j$ when either $r(I_{v_i}) < r(I_{v_j})$, or $r(I_{v_i}) = r(I_{v_j})$ and $\ell(I_{v_i}) \le \ell(I_{v_j})$. (In other words, we have $v_i < v_j$ whenever $(r(I_{v_i}), \ell(I_{v_i})) < (r(I_{v_j}), \ell(I_{v_j}))$ in the lexicographic ordering of pairs of real numbers.)

Now consider three indexes $a < b < c$. Assume $v_a v_c$ is an edge of G, and vertices v_b and v_c have different colors. Since $v_a v_c \in E(G)$, I_{v_a} and I_{v_c} intersect and, hence, $\ell(I_{v_c}) < r(I_{v_a})$. Moreover, since $v_a < v_b$, we have $r(I_{v_a}) \le r(I_{v_b})$. Therefore, $\ell(I_{v_c}) < r(I_{v_b})$; i.e., I_{v_b} and I_{v_c} intersect. Thus, $v_b v_c$ is an edge of G, implying that none of the forbidden patterns in Figure 4 occurs.

In contrast, let $v_1 < v_2 < \cdots < v_n$ be an ordering of the vertices of G without the forbidden patterns in Figure 4. For each i set $r(J_{v_i}) = i$ and $\ell(J_{v_i}) = \min(\{i\} \cup \{j :\ v_j < v_i,\ v_i v_j \in E(G)\})$. One can easily see that $J_v,\ v \in G$, is an interval representation for G. □

One can observe that if u and v are two vertices of G so that they have in-comparable neighborhoods then (u, v) is in a component of G^+. Indeed, with $uu', vv' \in E(G)$ and $uv', u'v \notin E(G)$ we get $(u, v) \to (u', v) \to (u', v') \to (u, v') \to (u, v)$. Using the same reasoning, if a pair (x, y) outside component S is dominated by a vertex in S, then $N(x) \subseteq N(y)$.

3 Structural Properties of Strong components in G^+

The following Lemma follows from the definition of G^+.

Lemma 15. *If uu' and vv' are independent edges in G then the pairs (u, v), (u', v), (u', v'), and (u, v') form a directed four-cycle of G^+ in the given order (resp. in reverse order). In particular, (u, v), (u', v), (u', v'), and (u, v') belong to the same component of G^+.*

Lemma 16. *Let S be a component of G^+ containing a pair (u, v) then one of the following occurs.*

1. *$uv \in E(G)$ or u and v have the same colors and there exist u', v' where uu', vv' are edges of G, $uv', vu' \notin E(G)$. Furthermore, the four pairs $(u, v), (u, v')$, (u', v), and (u', v') are contained in S.*

2. *u and v have different colors and $uv \notin E(G)$ and there exists u' such that $uu' \in E(G)$, $vu' \notin E(G)$ and u, u', v all have different colors.*

Proof. Since S is a component, (u, v) dominates some pair of S and is dominated by some pair of S. Firstly, suppose that u and v have the same color in G. Then (u, v) dominates some $(u', v) \in S$ and is dominated by some $(u, v') \in S$. Now uu' and vv' must be edges of G, and $uv, uv', u'v$, and $u'v'$ must be non-edges of G. Thus, uu' and vv' are independent edges in G. In this case, according to Lemma 15, S contains the directed cycle $(u, v) \to (u', v) \to (u', v') \to (u, v') \to (u, v)$.

Secondly, suppose $uv \in E(G)$. Now there must be a pair $(u', v) \in S$ dominated by (u, v) where in this case we have $uu' \in E(G)$, $uv' \notin E(G)$, and u, v' have different colors. Analogously, there must be some pair $(u, v') \in S$ that dominates (u, v) where in this case we have $vv' \in E(G)$, $u'v \notin E(G)$, and u', v have different colors. Now $(u, v) \to (u', v) \to (u', v') \to (u, v') \to (u, v)$.

Finally, suppose that u and v have different colors. We note that (u, v) dominates some $(u, v') \in S$, and hence, $uv \notin E(G)$ and vv' is an edge of G. Now, if u, v, v' have different colors, then (u, v') also dominates (u, v), implying that $(u, v), (u, v')$ are in the same component S. □

The structure of the components of G^+ is quite special, and the trivial components interact with them in simple ways. A trivial component will be called a *source* if its unique pair has in-degree zero, and a *sink* if its unique pair has out-degree zero. Herein, we further explore these properties by establishing several lemmas. To do this, we need the following definition of the reachability of pairs in G^+.

Definition 17 (reachability closure). *Let R be a subset of the pairs of G^+. Let $N^+[R]$ denote the set of all pairs in G^+ that are reachable (via a directed path in G^+) from a pair in R. (Notice that $N^+[R]$ contains R.) We call $N^+[R]$ the* reachability closure *of R. We say that a pair (u,v) is* implied *by R if $(u,v) \in N^+[R] \setminus R$. If $R = N^+[R]$, we say that R is* closed under reachability.

Lemma 18. *A pair (a,c) is in $N^+[S] \setminus S$ (implied by S) for some component S of G^+ if one of the following occurs.*

1. *a and c have the same color. $N(a) \subseteq N(c)$, and there exist $bd, dc \in E(G)$ so that a, b, d all have different colors and $ab, ad \notin E(G)$, and $(a,b), (a,d)$ are in S.*
2. *a and c have the same color. $N(a) \subseteq N(c)$ and G contains path a, b, c, d, e, such that and $ad, be \notin E(G)$, and $(a,d), (a,e), (b,d), (b,e)$ lie in S.*
3. *$ac \in E(G)$ and G contains a path b, a, c, d such that $N(a) \setminus \{c\} \subseteq N(c)$ and $ad, bd \notin E(G)$, $(a,d), (b,d)$ lie in S.*
4. *$ac \in E(G)$ and G contains path b, d, c such that $N(a) \setminus \{c\} \subseteq N(c)$, and $ab, ad \notin E(G)$ and a, d have different colors and a, b have different colors. Furthermore, $(a,d), (a,b) \in S$.*

Proof. Suppose that (a,c) is implied by a component S.
First suppose $ac \notin E(G)$. We show that a and c must have the same color. Suppose that this is not the case. Let $(a,d) \in S$ such that $(a,d) \to (a,c)$ or let $(b,c) \in S$ so that $(b,c) \to (a,c)$. In the former case $ad \notin E(G)$, $cd \in E(G)$, and hence $(a,d) \to (a,c) \to (a,d)$, which implies that (a,c) is in S, a contradiction. On the other hand if $(b,c) \in S$ such that $(b,c) \to (a,c)$, we have $bc \notin E(G)$, and $ab \in E(G)$. Thus, we have $(a,c) \to (b,c) \to (a,c)$, and hence $(a,c) \in S$, a contradiction. Therefore, a and c have the same color. In this case, there is $(a,d) \in S$ so that $(a,d) \to (a,c)$, where $ad \notin E(G)$, and $cd \in E(G)$.

Now we must have $N(a) \subseteq N(c)$ as otherwise, if a has a neighbor a' where $ca' \notin E(G)$, then $(a,c) \to (a',c) \to (a',c) \to (a',d) \to (a,d) \to (a,c)$, a contradiction.

Since $(a,d) \in S$, there is some $(a,b) \to (a,d)$ in S or $(b,d) \to (a,d)$. In the former case we have $cd, bd \in E(G)$, and $ad, ab \notin E(H)$, and hence, $(a,d) \to (a,b) \to (a,d)$ which proves that (1) occurs.

If $(b,d) \in S$ so that $(b,d) \to (a,d)$ then $ad \in E(G)$. Let $(b,e) \in S$ such that $(b,e) \to (b,d)$. Observe that $be \notin E(G)$ and $de \in E(G)$. Now it is easy to see that $(b,d) \to (a,d) \to (a,e) \to (b,e) \to (b,d)$. This shows that (2) occurs.
Second suppose $ac \in E(G)$. In this case we should have some $(a,d) \in S$ so that $(a,d) \to (a,c)$ with $dc \in E(G)$, and $ad \notin E(G)$. Since $(a,d) \in S$, there is either $(a,d) \to (b,d) \in S$ or $(a,d) \to (a,b) \in S$. Suppose $(a,d) \to (b,d) \in S$. We have $bd \notin E(G)$, $ab \in E(G)$, and hence $(b,d) \to (a,d) \in S$. Furthermore, we should have $bc \in E(G)$, otherwise, $(b,c) \to (b,d) \to (b,c)$, and $(a,c) \to (b,c)$, $(a,d) \to (a,c)$ implying that $(a,c) \in S$. a contradiction. Therefore, we have a path b, a, c, d with $N(a)\{c\} \subseteq N(c)$, and hence (3) is established.

Now assume that $(a, d) \to (a, b) \in S$. We have $bd \in E(G)$, and $ad \notin E(G)$ and hence (4) occurs. □

We emphasize that ab and de from Lemma 18 are independent edges. Inclusion $N(a) \subseteq N(c)$ implies the following corollary.

Corollary 19. *If there is an arc from a component S of G^+ to a pair $(x, y) \notin S$ then (x, y) forms a trivial component of S that is a sink component. If there is an arc to a component S of G^+ from a pair $(x, y) \notin S$ then (x, y) forms a trivial component of G^+ that is a source. In particular, if there is a directed path in G^+ from component S_1 to component S_2, then $S_1 = S_2$.*

4 Interval k-graph recognition

In this section, we present our algorithm for the recognition of interval bigraphs. Firstly, to describe the algorithm, we introduce some technical definitions.

Definition 20. *(envelope, $N^*[D]$) Let R be a set of pairs of G^+. The* envelope *of R, denoted $N^*[R]$, is the smallest set of pairs that contains R and is closed under both reachability and transitivity (if $(u, v), (v, w) \in N^*[R]$ then $(u, w) \in N^*[R]$).*

Remark and $N_l^*[D]$ definition For the purposes of the proofs, we visualize taking the envelope of R as divided into consecutive *levels*, where in the zero-th level we just replace R by its reachability closure, and in each subsequent level we replace R by the rechability closure of its transitive closure. The pairs in the envelope of R can be thought of as forming the arc of a digraph on $V(G)$, and each pair can be thought of as having a label corresponding to its level. The pairs (arcs of the digraph) in R, and those implied by R have label 0, arcs obtained by transitivity from the arcs labeled 0, as well as all arcs implied by them have label 1, and so on. More precisely, $N^*[R]$ is the disjoint union of $R^0, R^1, \ldots, R^k$, where $R^0 = N^+[R]$ (level zero), and each R^i (level $i \geq 1$) consists of every pair (u, v) such that either (u, v) is obtainable by transitivity in R^{i-1} (meaning that there is some sequence $(u, u_1), (u_1, u_2), \ldots, (u_{r-1}, u_r), (u_r, v)$ in R^{i-1}), or (u, v) is dominated by a pair (u', v') obtainable by transitivity in R^{i-1}. Let $N_l^*[D] = \bigcup_{i=0}^{i=l} R^i$. Note that $R \subseteq N^+[R] \subseteq N^*[R]$.

Definition 21 (dictator component)**.** *Let $\mathcal{R} = \{R_1, R_2, \ldots, R_k, S\}$ be a set of components of G^+ such that $N^*[\bigcup_{A \in \mathcal{R}} A]$ contains a circuit. We say S is a* dictator *if for every subset W of $\mathcal{R} \setminus \{S\}$, there exist a circuit in the envelope of $(\bigcup_{A \in W'} A) \cup (\bigcup_{B \in \mathcal{R} \setminus W} B)$, where $W' = \{R'_i \mid R_i \in W\}$. In other words, S is a dictator if by replacing some of the R_is with R'_is in $\mathcal{R}$ and taking the envelope of the union of elements we still get a circuit.*

Definition 22 (complete set)**.** *A set $D_1 \subseteq V(G^+)$ is called* complete *if for every pair of coupled components R, R' of G^+, exactly one of $R \subseteq D_1$ and $R' \subseteq D_1$ holds.*

A component S is a dictator if and only if the envelope of every complete set D_1 containing S has a circuit.

Definition 23 (simple pair, complex pair)**.** *A pair $(x, y) \in G^+$ is* simple *if it belongs to $N^+[S]$ for some component S, otherwise we call it* complex.

Algorithm 1 Algorithm for recognition of interval k-graphs

function INTERVAL-K-GRAPH(G)
  Input: A connected k-partite graph G
  Output: An ordering of $V(G)$ without patterns in Figure 4 or return false
  Construct the pair-digraph G^+ of G, and compute its components; if any component is self-coupled report false
  Stage1 : Adding (non-trivial strong) components
  Initialize D to be the empty set
  let $v_1, v_2, \ldots, v_n$ be an ordering of the vertices of G such that $i < j$ implies $c(v_i) < c(v_j)$
  while $\exists S_{v_i,v_j}$ and S_{v_j,v_i}, $i < j$ components in $G^+ \setminus D$ **do**
    if $D \cup N^+[S_{v_i,v_j}]$ does not have a circuit **then**
      add $N^+[S_{v_i,v_j}]$ to D, remove $N^+[S_{v_j,v_i}]$ from further consideration in this step ▷ add X to D means add all the pairs of X into D
      for all $(x,y) \in N^+[S_{v_i,v_j}]$ **do** $Dic(x,y) = S_{v_i,v_j}$
    else
      if $D \cup N^+[S_{v_j,v_i}]$ does not have a circuit **then**
        add $N^+[S_{v_j,v_i}]$ into D, delete $N^+[S_{v_i,v_j}]$ from further consideration here
        for all $(x,y) \in N^+[S_{v_j,v_i}]$ **do** set $Dic(x,y) = S_{v_j,v_i}$
      else report that G is not an interval k-graph
    increase i by one
  Stage2 : Computing $N^*[D]$ and detecting dictator components
  Set $En = N^*[D]$, and $\mathcal{DT} = \varnothing$ ▷ $\mathcal{DT}$ is a set of components
  while $\exists (x,y) \in En \setminus D$ **do** ▷ we consider the pairs in En level by level
    Move (x,y) into D and set $Dic(x,y) =$ DICTATOR(x,y,D)
    if $D \cup \{(x,y)\}$ contains a circuit **then** add $Dic(x,y)$ into $\mathcal{DT}$ ▷ (x,y) is a complex pair
  Stage3 : Adding dual of dictator components
  Let $D_1 = \varnothing$
  for all components $S \in \mathcal{DT}$ **do** add $N^+[S']$ into D_1
  for all components $R \in D \setminus \mathcal{DT}$ **do** add $N^+[R]$ into D_1
  Set $D = N^*[D_1]$
  if there is a circuit in D **then** report G is not an interval k-graph
  Stage4 : Adding remaining trivial components, returning an ordering
  while $\exists$ trivial component S outside D, and S is a sink component **do**
    Add S into D and remove S' from further consideration
  Output the final ordering
  for all $(u,v) \in D$ **do** set $u \prec v$ ▷ yielding an ordering of $V(G)$ without the patterns from Figure 1
  Return the ordering $v_1 \prec v_2 < \cdots \prec v_n$ of $V(G)$

function DICTATOR(x, y, D)

  if $(x, y) \in N^+[S]$ for some component S in D **then** return S

  if x, y have different colors and $(u, y) \in D$ dominates (x, y) **then** return DICTATOR(u, y, D) ▷ we mean the earliest pair (u, y)

  if x, y have the same color and $(x, w) \in D$ dominates (x, y) **then** return DICTATOR(x, w, D)

  if x, y have the same color and (x, y) is by transitivity on $(x, w), (w, y) \in D$ **then** return DICTATOR(w, y, D)

  if x, y have different colors and (x, y) is by transitivity on $(x, w), (w, y) \in D$ **then** return DICTATOR(x, w, D)

Definition 24. *Let D be a complete set and let C be a circuit in $N^*[D]$. We say C is a* minimal circuit *if first, the latest pair in C is created as early as possible (the smallest possible level) during the execution of $N^*[D]$; second, C has the minimum length; third, no pair in C is by transitivity.*

Definition of color $c(u)$. For every vertex u of G we denote by $c(u)$ index of the partite class containing u. This is the color of u; $c(u) = k$ if $u \in V_k$.

High level overview of the algorithm The algorithm begins by constructing G^+. If any component of G^+ is self-coupled, then G is not an interval k-graph. Next, we initialize an empty set D (which will store selected pairs). If we add a pair (x, y) into D, it means that in the final ordering, x must appear before y.

The core of the algorithm involves selecting components of G^+ based on the following principles:

- If $(x, y) \in D$, then (y, x) must be discarded,
- If $(x, y) \in D$ and $(x, y)(x', y') \in A(G^+)$, then (x', y') must also be in D, Thus, once a component (x, y) is added to D, all pairs within its corresponding component $S_{x,y}$ are also included in D.
- If $(x, y), (y, z) \in D$, then (x, z) must also be in D.

Stage 1: Selecting Components: We first compute an ordering $v_1, v_2, \ldots, v_n$ of the vertices of G such that if $v_i < v_j$, then the color of v_i is the same as or smaller than the color of v_j. The algorithm proceeds in steps $i = 1, 2, \ldots$, where at each step, we consider components S_{v_i,v_j} (with $i < j$) for inclusion in D. The selection follows these rules: If adding S_{v_i,v_j} to D (along with its outgoing neighbors $N^+[S_{v_i,v_j}]$) does not create a *circuit*, then we include $N^+[S_{v_i,v_j}]$ in D and discard S_{v_j,v_i}, Otherwise, we try to add $N^+[S_{v_j,v_i}]$ to D. If this results in a *circuit*, then G is not an interval k-graph. If neither S_{v_i,v_j} nor S_{v_j,v_i} can be added to D, then G contain an *exobiclique* or the forbidden structures in Figure 7.

Stages 2 and 3: Closure and Dictator Components: Next, we compute the *closure* of D, ensuring that for each pair of components S and S', exactly one of them is in D. If a circuit C is found in $N^*[D]$, we identify a dictator component S, which cannot be included in any complete set. Consequently, we remove all these dictator components $S_1, S_2, \ldots, S_t$ from D and define a new set D_1, which includes their dual components $S'_1, S'_2, \ldots, S'_t$ along with all other elements of $D \setminus (S_1 \cup S_2 \cup \cdots \cup S_t)$. If $N^*[D_1]$ contains a *circuit*, then G is not an interval k-graph; otherwise, we update D as D_1 and set $D = N^*[D]$.

Stage 4: Handling Trivial Components Finally, we add any remaining trivial components from $G^+ \setminus (D \cup D')$ (where D' is the *dual* of D), starting with the *sink components*. This step does not introduce *circuits*, so we do not need to check for conflicts. At the end of the algorithm, we derive a final order $\prec$ setting $u \prec v$ whenever $(u, v) \in D$.

4.1 Proof of the correctness of the Algorithm

Proof of Theorem 2 The correctness of Algorithm 1 follows from Lemma 27. We denote the degree of a vertex z of H by d_z. In order to construct the digraph G^+, we need to list all neighbors of each pair in G^+. If the vertices x and y in G have different colors, then the pair (x, y) of G^+ has d_y out-neighbors; and if x and y have the same color, then the pair (x, y) has d_x out-neighbors in G^+. For a fixed vertex x with $c(x) = 0$, the number of all pairs that are neighbors of all pairs (x, z), $z \in V(H)$, is $nd_x + d_{y_1} + d_{y_2} + \cdots + d_{y_n}$, where $y_1, y_2, ..., y_n$ are all of different colors than $c(x)$. We can use a linked list structure to represent G^+, therefore, overall, it takes time $\mathcal{O}(mn)$ to construct G^+. Notice that in order to check whether a component S is self-coupled, it suffices to choose any pair (a, b) in S and check if (b, a) is also in S. The latter task can be done in time $\mathcal{O}(mn)$, using Tarjan's strongly-connected component algorithm. Since we maintain a partial order on D, once we add a new pair to D, we can decide whether that pair closes a circuit or not. Computing $N^*[D]$ also takes time $\mathcal{O}(n(n+m)) = \mathcal{O}(mn)$ since there are $\mathcal{O}(mn)$ edges in G^+ and $\mathcal{O}(n^2)$ vertices in G^+. Note that the envelope of D is computed at most twice.

Once a pair (x, y) is added to D, we put an arc from x to y in the partial order and give the arc xy a time label (also called level). Once a circuit is formed at Stage 2, we can find a dictator component S using the DICTATOR function, and store S in the set $\mathcal{DT}$. Therefore, we spend at most $\mathcal{O}(nm)$ time to find all the dictator components. Stage 4, in which we add the remaining pairs, takes time at most $\mathcal{O}(n^2)$. Therefore, the overall running time of the algorithm is $\mathcal{O}(nm)$. □

We start by giving some technical definition that are used in the correctness proof.

Definition 25. *Let (x, y) be a pair in D by transitivity at the (earliest) level $l \geq 0$. By a minimal chain between x and y we mean a sequence $(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n)$ of minimum length (n) of pairs in D with $x_0 = x$ and $x_n = y$, such that each $(x_i, x_{i+1}) \in D$ for $0 \leq i \leq n-1$ at some level before l by reachability (and not by transitivity). We also say (x_0, x_n) is by transitivity on the minimal chain $(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n)$.*

Definition 26. *Let $CH = (x_0, x_1), \ldots, (x_{n-1}, x_n)$ be a minimal chain between x_0 and x_n in $N^*_l[D]$, $0 \leq l$. We say that pair (x_i, x_{i+1}) is a **tail** pair if there exists $(x_i, a_{i+1}) \in N^*_l[D]$ such that $(x_i, a_{i+1}) \to (x_i, x_{i+1})$ with $x_{i+1}a_{i+1} \in E(G)$, and $x_i a_{i+1} \notin E(G)$ (x_i, a_{i+1} have different colors). On the other hand, we say (x_i, x_{i+1}) is a **head** pair if there exists $(a_i, x_{i+1}) \in N^*_l[D]$ so that $(a_i, x_{i+1}) \to (x_i, x_{i+1})$ with $a_i x_i \in E(G)$ and $x_i x_{i+1} \notin E(G)$, and $a_i x_i \in E(G)$, and x_i, x_{i+1} have different colors.*

Now we are ready to prove the following correctness lemma for our algorithm.

Lemma 27. *Algorithm* 1*, produces an ordering of the vertices of G without the forbidden patterns depicted in Figure 1 if and only if G is an interval k-graph.*

Proof. Suppose we encounter a circuit while creating $N^*[D]$. The main ingredient of the proof is to assume this circuit is minimal. That is, C contains a pair that added to $N^*[D]$ was the earliest pair, and no pair within the circuit arises purely by transitivity. Among possible choices, C is assumed to have the shortest length. Since transitive closure is applied, we first analyze minimal chains of the form $CH : (x, y_1), (y_1, y_2), \ldots, (y_{n-1}, y_n), (y_n, y)$, where if each pair belongs to $N^*[D]$, then (x, y) is also included in $N^*[D]$.

Note that each pair in the circuit C, say (x_i, x_{i+1}), is derived from the reachability of some (x, y), where (x, y) originates by transitivity, and thus, it can be assumed that (x, y) is formed via a minimal chain. Lemma 28 in the appendix show some essential properties about a minimal CH, while a sequence of Lemmas (29,30,31,32) examines consecutive pairs (y_i, y_{i+1}) and (y_{i+1}, y_{i+2}) within a chain CH and demonstrates that their type (head of tail) must alternate when the chain length is at least 4. Furthermore, additional structural properties of minimal chains, proven in Lemmas 33, 34, 35, and 36, lead to the conclusion that the length of a minimal chain is at most 3 pairs. Consequently, the length of a minimal circuit is at most 4. In Theorem 38, analyzing the presence of a circuit in Stage 1 of the algorithm reveals that such a circuit either detects an exobiclique (if its length is 4) or identifies a new obstruction (as shown in Figure 7) if its length is 3. This establishes the correctness of Algorithm 1 in Stage 1.

Additional structural insights emerge if a circuit appears in Stage 2 of the algorithm. Lemma 41 establishes the existence of a component X (so-called dictator component) such that, regardless of how other components are chosen in Stage 1, a circuit inevitably forms in Stage 2 as long as we keep X in D. Thus, X must not be included in D. If adding X' and removing X from D also results in a circuit in $N^*[D]$, then G cannot be an interval k-graph. □

We present a series of lemmas discussing the structural properties of a minimal chain and minimal circuit during the computation of $N^*[D]$.

Lemma 28. *Suppose a pair $(x, y) \in N^*_{l+1}[D]$ is obtained by a minimal chain $CH = (x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, x_{n+1})$ ($x_0 = x$ and $x_{n+1} = y$) in $N^*_l[D]$, and there is no circuit formed in $N^*_{l+1}[D]$ by adding (x, y). Suppose x_i and x_{i+1} have different colors and they are not adjacent. Then $x_{i+1}x_j \notin E(G)$.*

Proof. Assume for contradiction that $x_{i+1}x_j \in E(G)$. Since $j \neq i, i+1$, the arc $(x_i, x_{i+1})(x_i, x_j)$ exists in G^+. As $(x_i, x_{i+1}) \in N^*[D]$ and $(x_i, x_j) \in N^*[D]$, the chain $(x_{i-1}, x_i), (x_i, x_j), (x_j, x_{j+1}), \ldots, (x_{i-2}, x_{i-1})$ when $j > i$ is a shorter chain, contradicting the minimality of CH. Similarly, for $i < j$, circuit $(x_j, x_{j+1}), \ldots, (x_{i-1}, x_i), (x_i, x_j)$ is in $N^*[D]$, a contradiction to our assumption that the current $N^*[D]$ does not have a circuit. □ □

Lemma 29. *Suppose a pair $(x, y) \in N^*_{l+1}[D]$ is obtained by a minimal chain $CH = (x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, x_{n+1})$ ($x_0 = x$ and $x_{n+1} = y$) in $N^*_l[D]$, and there is no circuit formed in $N^*_{l+1}[D]$ by adding (x, y). If (x_i, x_{i+1}) is a head pair and (x_{i+1}, x_{i+2}) is a tail pair, then x_{i+1}, x_{i+2} have the same color and different from the color of x_i. Furthermore, $x_i x_{i+2} \in E(G)$.*

Proof. By definition, there exist $(a_i, x_{i+1}) \to (x_i, x_{i+1})$ in $N_l^*[D]$ with $a_i x_i \in E(G)$ and $x_i x_{i+1} \notin E(G)$. Similarly, there exist $(x_{i+1}, a_{i+2}) \to (x_{i+1}, x_{i+2})$ in $N_l^*[D]$ with $a_{i+2}x_{i+2} \in E(G)$ and $x_{i+1}x_{i+2} \notin E(G)$. Now $a_i a_{i+2} \notin E(G)$, otherwise $(x_{i+1}, a_{i+2}) \to (x_{i+1}, a_i)$, a circuit in $N_l^*[D]$.

For $l > 1$, $x_i x_{i+2} \in E(G)$ must hold. Otherwise, independent edges $a_i x_i$ and $a_{i+2}x_{i+2}$ would imply that $S_{x_i,x_{i+2}}$ is a component, and should have been selected earlier according to the algorithm's rules, a contradiction to the minimality of CH.

Now consider the case where $l = 1$. Observe that $x_{i+1}x_{i+2} \notin E(G)$ by Lemma 28. Consider the case where $a_i x_{i+1} \notin E(G)$ and $c(a_i) \neq c(x_{i+1})$. Since $(a_i, x_{i+1}), (x_i, x_{i+1})$ belong to the same component and $(x_i, x_{i+1}) \in N_l^*[D]$, we conclude that $c(x_i) < c(x_{i+1})$ or $c(a_i) < c(x_{i+1})$. Additionally, since $(a_i, x_{i+2}), (x_i, x_{i+2}), (a_i, a_{i+2}), (x_i, a_{i+2})$ belong to the same component, $S_{x_i,x_{i+2}}$ should have been selected before component S where $(x_{i+1}, x_{i+2}) \in N^+[S]$, leading to a shorter chain.

In the case where $c(a_i) = c(x_{i+1})$, there is some b_{i+1} so that $x_i a_i$ and $b_i x_{i+1}$ are independent edges of G, meaning that $a_i b_{i+1}, x_i x_{i+1} \notin E(G)$. Observe that $(x_i, b_{i+1}), (a_i, b_{i+1}), (x_i, x_{i+1}), (a_i, x_{i+1})$ are in a same component. Furthermore, $b_{i+1}x_{i+2} \notin E(G)$, as otherwise, $(x_i, b_{i+1}) \to (a_i, b_{i+1}) \to (a_i, x_{i+2}) \to (x_i, x_{i+2})$, and hence, $(x_i, x_{i+2}) \in N^+[S_{a_i,x_{i+1}}]$, a shorter chain. Now (x_{i+1}, x_{i+2}) and (x_i, x_{i+2}) are in components. As argued earlier, $S_{x_i,x_{i+2}}$ should have been selected before $S_{x_{i+1},x_{i+2}}$, contradicting the minimality of CH. □ □

Lemma 30. *Suppose a pair $(x, y) \in N_{l+1}^*[D]$ is obtained by a minimal chain $CH = (x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, x_{n+1})$ ($x_0 = x$ and $x_{n+1} = y$) in $N_l^*[D]$, and there is no circuit formed in $N_{l+1}^*[D]$ by adding (x, y). If (x_i, x_{i+1}) is a tail pair and (x_{i+1}, x_{i+2}) is a tail pair, then there exist a_{i+1}, a_{i+2} so that $a_{i+1}x_{i+1}, a_{i+2}x_{i+2} \in E(G)$ and $x_{i+2}a_{i+1}, a_{i+2}x_i, a_{i+2}x_{i+1} \notin E(G)$, and $S_{x_i,x_{i+1}}$ and $S_{x_i,x_{i+2}}$ are components.*

Proof. By definition, there exist $a_{i+1}x_{i+1}, a_{i+2}x_{i+2}$ edges of G such that $x_i a_{i+1}, x_{i+1}a_{i+2} \notin E(G)$. Now $a_{i+1}x_{i+2} \notin E(G)$, otherwise $(x_i, a_{i+1}) \to (x_i, x_{i+2})$, a shorter circuit. Notice that by minimality of CH, (x_i, x_{i+1}) is not by transitivity. Now, let $(a_i, x_{i+1}) \in N_{l+1}^*[D]$ such that $(a_i, x_{i+1}) \to (x_i, x_{i+1})$. Since $a_i x_i \in E(G)$, we observe that $a_i x_{i+2} \notin E(G)$, as otherwise, the sequence $(a_i, a_{i+1}) \to (x_{i+2}, a_{i+1}) \to (x_{i+2}, x_{i+1})$ would form a shorter circuit. Since $a_i x_i$ and $a_{i+2}x_{i+2}$ are independent edges, (x_i, x_{i+2}) should have been chosen earlier, contradicting the minimality of CH.

Now, let $(a_i, x_{i+1}) \in N_{l+1}^*[D]$ with $(a_i, x_{i+1}) \to (x_i, x_{i+1})$. We have $a_i x_i \in E(G)$. Now it is easy to see that $a_i x_{i+2} \notin E(G)$ otherwise, $(a_i, a_{i+1}) \to (x_{i+2}, a_{i+1}) \to (x_{i+2}, x_{i+1})$, is a shorter circuit. Now $a_i x_i, a_{i+2}x_{i+2}$ are independent edges and hence (x_i, x_{i+2}) should have been chosen. Moreover, $a_i x_{i+1} \notin E(G)$ otherwise, $(x_{i+1}, x_{i+2}) \to (a_i, x_{i+2}) \to (a_i, a_{i+2}) \to (x_i, a_{i+2}) \to (x_i, x_{i+2})$, a shorter circuit. These means we should have chosen (x_i, x_{i+1}) before (x_{i+1}, x_{i+2}).

If there is some b_{i+1} so that $(x_i, b_{i+1}) \to (x_i, a_{i+1})$, then $S_{x_i,a_{i+1}}$ is in a component, and we can replace x_{i+1} by a_{i+1} in the chain CH, meaning that the condition of the lemma holds for the replaced chain. □ □

Lemma 31. *Suppose a pair $(x, y) \in N_{l+1}^*[D]$ is obtained by a minimal chain $CH = (x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, x_{n+1})$ ($x_0 = x$ and $x_{n+1} = y$) in $N_l^*[D]$, and there is no circuit formed in $N_{l+1}^*[D]$ by adding (x, y). If (x_i, x_{i+1}) is a head pair and (x_{i+1}, x_{i+2}) is a head pair, then x_i, x_{i+2} have the same color, $x_i x_{i+1}, x_{i+1}x_{i+2} \notin E(G)$. Furthermore, there exists $x_{i+1}a_{i+1} \in E(G)$ so that $a_{i+1}x_{i+2} \notin E(G)$, that is, (x_{i+1}, x_{i+2}) is in a component.*

Proof. By definition, there exist $(a_i, x_{i+1}) \to (x_i, x_{i+1})$ in $N_l^*[D]$ with $a_i x_i \in E(G)$, and $x_i x_{i+1} \notin E(G)$. In addition, there exist $(a_{i+1}, x_{i+2}) \to (x_{i+1}, x_{i+2})$ with $a_{i+1}x_{i+1} \in E(G)$, and $x_{i+1}x_{i+2} \notin E(G)$. We observe that $a_i x_{i+2} \notin E(G)$ otherwise $(x_{i+1}, x_{i+2}) \to (x_{i+1}, a_i)$, a shorter circuit. Notice that $(x_i, x_{i+1}) \to (x_i, a_{i+1})$. So we may replace the pairs (x_i, x_{i+1}), (x_{i+1}, x_{i+2}) by (x_i, a_{i+1}), (a_{i+1}, x_{i+2}), obtained chain CH'. Notice that (a_{i+1}, x_{i+2}) is not by transitivity; else CH' contradicts the minimality of CH. Therefore, (a_{i+1}, x_{i+2}) is in a component and by Lemma 18, (x_{i+1}, x_{i+2}) is in a component.

The first case is when a_{i+1} and x_{i+2} have different colors and $a_{i+1}x_{i+2} \notin E(G)$. If $c(x_i) \neq c(x_{i+2})$, then by Lemma 28, $x_i x_{i+2} \notin E(G)$, and hence $(x_i, x_{i+2}), (a_i, x_{i+2})$ are in a component. According to the rules of the algorithm, (x_i, x_{i+1}) must be in a component and therefore, according to the rules of the algorithm (x_i, x_{i+2}) should have been chosen before the component $S_{x_{i+1},x_{i+2}}$, hence a shorter circuit. Therefore, $c(x_i) = c(x_{i+2})$, and we are done here.

So we may continue by assuming that there exists $x_{i+2}a_{i+2} \in E(G)$ such that $a_{i+1}a_{i+2} \notin E(G)$. Now again, $x_i a_{i+2} \notin E(G)$, otherwise we have $(x_i, x_{i+1}) \to (x_i, a_{i+1})$, and $(a_{i+1}, a_{i+2}) \to (a_{i+1}, x_i)$, a shorter circuit. Now $S_{x_i,x_{i+2}}$ is in a component. As we argued in the previous case, we must have $c(x_i) = c(x_{i+2})$. □ □

Lemma 32. *Suppose $(x, y) \in N_{l+1}^*[D]$ is obtained by a minimal chain $CH = (x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, x_{n+1})$ ($x_0 = x$ and $x_{n+1} = y$) in $N_l^*[D]$, and there is no circuit formed in $N_{l+1}^*[D]$ by adding (x, y). For $i < n - 2$, if (x_i, x_{i+1}) is a head pair, then (x_{i+1}, x_{i+2}) is a tail pair, and if (x_i, x_{i+1}) is a tail pair then (x_{i+1}, x_{i+2}) is a head pair.*

Proof. According to Lemma 30 we can assume that there is a chain CH' by replacing a_{i+1} by x_{i+1}. Now (x_i, a_i) and (a_i, x_{i+2}) are tail and head, respectively. Thus, we consider the case where (x_i, x_{i+1}) is a head pair and (x_{i+1}, x_{i+2}) is also a head pair. First, suppose (x_{i+2}, x_{i+3}) is a tail pair. Now, by Lemma 29, we have $x_{i+1}x_{i+3} \in E(G)$. However, $(x_i, x_{i+1}) \to (x_i, x_{i+3})$, a shorter chain. Thus, we may assume (x_{i+2}, x_{i+3}) is also a head pair. According to Lemma 31 there exists $x_{i+1}a_{i+1} \in E(G)$ such that $x_{i+1}x_{i+2}, a_{i+1}x_{i+2} \notin E(G)$ and x_{i+2} have different colors than x_{i+1}, a_{i+1}. According to Lemma 31 there exists $x_{i+2}a_{i+2} \in E(G)$ such that $x_{i+2}x_{i+3}, a_{i+2}x_{i+3} \notin E(G)$ and x_{i+3} have different colors than x_{i+2}, a_{i+2}. Notice that $x_i x_{i+1} \notin E(G)$. Now $x_{i+1}a_{i+2} \notin E(G)$, otherwise $(x_{i+1}, x_{i+2}) \to (x_i, a_{i+2})$, and hence we consider the chain $CH' = (x_0, x_1), \ldots, (x_i, a_{i+2}), (a_{i+2}, x_{i+3}), \ldots, (x_{n-1}, x_0)$, which is shorter than CH, a contradiction. Now $(x_{i+1}, a_{i+2}), (x_{i+1}, x_{i+2}), (a_{i+1}, x_{i+2})$ are in a component.

Note that there exists $(a_i, x_{i+1}) \to (x_i, x_{i+1})$ in $N_l^*[D]$. Observe that $a_{i+2}a_i \notin E(G)$, otherwise, $(x_{i+1}, a_{i+2}) \to (x_{i+1}, a_i)$ in $N_l^*[D]$, a circuit in $N_l^*[D]$. With the same line of reasoning $x_{i+2}a_i, x_i x_{i+2} \notin E(G)$. Therefore, now (x_i, x_{i+2}) is in a component and according to the rules of the algorithm, we should have selected (x_i, x_{i+2}), no later than (x_{i+1}, x_{i+2}), a shorter chain. □ □

Lemma 33. *Suppose $(x, y) \in N_{l+1}^*[D]$ is obtained by a minimal chain $CH = (x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, x_{n+1})$ ($x_0 = x$ and $x_{n+1} = y$) in $N_l^*[D]$, and there is no circuit formed in $N_{l+1}^*[D]$ by adding (x, y). Suppose $x_i x_{i+1} \notin E(G)$ and x_i, x_{i+1} have different colors. Then one of the following occurs:*

- *$x_i x_{i+2} \in E(G)$ and x_{i+1}, x_{i+2} have the same color.*
- *$n = 2$, x_i, x_{i+2} have the same color, and both $(x_{i+1}, x_{i+2}), (x_i, x_{i+1})$ are in components.*

Proof. Case 1. First, assume that (x_i, x_{i+1}) is a head pair. Thus, there exists $(y_{i+1}, x_{i+1}) \to (x_i, x_{i+1})$ in $N^*[D]$ with $x_i y_{i+1} \in E(G)$. From Lemma 28, $x_{i+1}x_{i+2} \notin E(G)$. Notice that $y_{i+1}x_{i+2} \notin E(G)$, otherwise, $(x_{i+1}, x_{i+2}) \to (y_{i+1}, x_{i+1})$, and hence, $(y_{i+1}, x_{i+1}) \in N^*[D]$ while (y_{i+1}, x_{i+1}) is also in $N^*[D]$, contradiction to our assumption.

Subcase 1.1 (x_{i+1}, x_{i+2}) is a tail pair. Thus, there exists $(x_{i+1}, y_{i+2}) \to (x_{i+1}, x_{i+2})$ in $N_l^*[D]$ with $y_{i+2}x_{i+2} \in E(G)$ and $x_{i+1}y_{i+2} \notin E(G)$. Notice that $y_{i+2}y_{i+1} \notin E(G)$, otherwise, $(x_{i+1}, y_{i+2}) \to (x_{i+1}, y_{i+1})$, implying $(x_{i+1}, y_{i+1}) \in N_l^*[D]$, while $(y_{i+1}, x_{i+1}) \in N_l^*[D]$. Now if $x_{i+2}x_i \notin E(G)$ or x_{i+1}, x_{i+2} have different colors, we have $x_i y_{i+1}, y_{i+2}x_{i+2}$ as pair of independent edges, and hence, (x_i, x_{i+2}) is in a (strong non-trivial) component and is in D, yielding a shorter chain.

Subcase 1.2. (x_{i+1}, x_{i+2}) is a head pair. There exists $(y_{i+2}, x_{i+2}) \to (x_{i+1}, x_{i+2})$ in $N_l^*[D]$ with $y_{i+2}x_{i+1} \in E(G)$ and $y_{i+2}x_{i+2} \notin E(G)$. Now, by Lemma 32 we must have $i \geq n-2$, and hence, x_{i+3} does not exists.

Now suppose x_{i-1} exists. Thus by Lemma 32 (x_{i-1}, x_i) is a tail pair. Thus we have $(x_{i-1}, y_i) \to (x_{i-1}, x_i)$ in $N_l^*[D]$. Now, $y_i x_{i+2} \notin E(G)$ else $(x_{i-1}, y_i) \to (x_{i-1}, x_{i+2})$ and hence, a shorter chain. Moreover, $y_i y_{i+2} \notin E(G)$, else $(y_{i+2}, x_{i+2}) \to (y_i, x_{i+2})$ in $N_l^*[D]$, and we have a shorter chain, by using $(x_{i-1}, y_i), (y_i, x_{i+2})$ instead of $(x_{i-1}, x_i), (x_i, x_{i+1}), (x_{i+1}, x_{i+2})$. This implies that $x_i y_i, y_{i+2}x_{i+1}$ are independent edges, and we obtain chain CH' by replacing $(x_{i-1}, x_i), (x_i, x_{i+1}), (x_{i+1}, x_{i+2})$ with $(x_{i-1}, y_i), (y_i, x_{i+1}), (x_{i+1}, x_{i+2})$. Now (x_{i-1}, y_i) is not by transitivity as otherwise, CH' contradicts the minimality of CH. Thus, there exists some $(y_{i-1}, y_i) \to (x_{i-1}, y_i)$ in $N_l^*[D]$ with $y_{i-1}x_{i-1} \in E(G)$. It turns out $y_{i-1}y_{i+2} \notin E(G)$, else $(y_{i-1}, y_i) \to (y_{i+2}, y_i) \to (y_{i+2}, x_i)$, implying a circuit in $N_l^*[D]$. Now $x_{i-1}y_{i-1}, x_{i+1}y_{i+2}$ are independent edges, implying $(x_{i-1}, x_{i+1}) \in N_l^*[D]$, and a shorter chain. Therefore, $n = 2$, and x_i, x_{i+2} have the same color.

Case 2. (x_i, x_{i+1}) is a tail pair. This means, $(x_i, y_{i+1}) \to (x_i, x_{i+1})$ in $N_l^*[D]$ with $x_i y_{i+1} \notin E(G)$, $x_{i+1}y_{i+1} \in E(G)$, and x_i, x_{i+1}, y_{i+1} all have different colors. Now, by Lemma 32 we must have $i \geq n-2$, and hence, x_{i+3} does not exists.

Subcase 2.1. (x_{i+2}, x_{i+3}) is a tail pair. Now by Lemma 32 we must have $(x_{i+2}, y_{i+3}) \to (x_{i+2}, x_{i+3})$ in $N^*[D]$ with $x_{i+3}y_{i+3} \in E(G)$, and $x_{i+2}y_{i+3} \notin E(G)$, and x_{i+2}, y_{i+3} have different colors. We notice that $x_{i+1}x_{i+3} \notin E(G)$ because of Lemma 28. Now, $y_{i+1}y_{i+3} \in E(G)$, otherwise, $x_{i+1}y_{i+1}, x_{i+3}y_{i+3}$ are independent edges, and hence, (x_{i+1}, x_{i+3}) is in a component, and it must be in D (else we have a circuit with $N_l^*[D]$), a shorter chain. Now x_i, y_{i+1}, y_{i+3} have different colors and $(x_i, y_{i+1}), (x_i, y_{i+3})$ are in the same component. Notice that $(x_i, y_{i+3}) \to (x_i, x_{i+3})$, and hence $(x_i, x_{i+3}) \in N_l^*[D]$, a shorter circuit.

We now consider the case where x_{i-1} exists. Suppose $(y_{i-1}, x_i) \to (x_{i-1}, x_i)$ in $N_l^*[D]$ with $y_{i-1}x_{i-1} \in E(G)$, $(x_{i-1}x_i \notin E(G)$. Notice that $y_i x_{i-1} \notin E(G)$ nor $y_{i-1}x_{i+1}$, else $(x_i, x_{i+1}) \to (y_{i-1}, x_i)$ or $(x_i, y_i) \to (x_i, x_{i-1})$, a circuit in $N_l^*[D]$. Thus, $y_{i-1}x_{i-1}, y_i x_{i+1}$ are independent edges of G, which implies $(x_{i-1}, x_{i+1}) \in N_l^*[D]$. We continue by considering the case where $(x_{i-1}, y_{i-1}) \to (x_{i-1}, x_i)$ in $N_l^*[D]$. The contradiction follows analogously to the argument in Subcase 1.2. for the case where x_{i-1} exists. □ □

Lemma 34. *Suppose $(x, y) \in N^*[D]$ is obtained by a minimal chain $CH = (x_0, x_1), (x_1, x_2),$*

$\ldots,(x_{n-1},x_n),(x_n,x_{n+1})$ ($x_0=x$ and $x_{n+1}=y$) in $N^*[D]$ at level l, and there is no circuit formed in $N^*[D]$ by adding (x,y) (level 0 to level l pairs). Suppose $x_ix_{i+1}\notin E(G)$ and x_i,x_{i+1} have different colors. Then $x_ix_{i+2}\in E(G)$ and if x_{i+3} exists then we have $x_{i+2}x_{i+3}\notin E(G)$.

Proof. By Lemma 33 when x_{i+3} exists we have $x_ix_{i+2}\in E(G)$. By Lemma 28 (1), $x_{i+1}x_{i+3}\notin E(G)$. For contradiction suppose $x_{i+2}x_{i+3}\in E(G)$. First assume x_i,x_{i+3} have different colors. Now $x_ix_{i+3}\notin E(G)$, otherwise, $(x_i,x_{i+1})\to(x_{i+3},x_{i+1})$ in $N_l^*[D]$, and hence, $(x_{i+3},x_{i+1})\in N_l^*[D]$, implying a circuit $(x_i,x_{i+1}),(x_{i+1},x_{i+2}),(x_{i+2},x_{i+3}),(x_{i+3},x_i)$ in $N_l^*[D]$. Now we have $(x_{i+2},x_{i+3})\to(x_i,x_{i+3})$ in $N_l^*[D]$ (because $x_{i+2}x_i\in E(G)$, $x_ix_{i+3}\notin E(G)$), and we get a shorter chain by passing x_{i+1},x_{i+2}. This shows that x_i,x_{i+3} must have the same color. Let $(x_{i+2},y_{i+2})\to(x_{i+2},x_{i+3})$ in $N^*[D]$ so that $x_{i+2}y_{i+2}\notin E(G)$ and $y_{i+2}x_{i+3}\in E(G)$ (notice that the other option is not possible since $x_{i+2}x_{i+3}\in E(G)$. Now, $x_iy_{i+2}\notin E(G)$, as otherwise, $(x_{i+2},y_{i+2})(x_{i+2},x_i)$ is an arc of G^+, and hence $(x_i,x_{i+1}),(x_{i+1},x_{i+2}),(x_{i+2},x_i)$ is a circuit. On the other hand, $(x_{i+2},y_{i+2})(x_i,y_{i+2})$ and $(x_i,y_{i+2})(x_i,x_{i+3})$ are arcs of G^+, and hence, we get a shorter chain bypassing x_{i+1},x_{i+2}, a contradiction. Therefore, $x_{i+2}x_{i+3}\notin E(G)$. □ □

Lemma 35. *Suppose $(x,y)\in N^*[D]$ is obtained by a minimal chain $CH=(x_0,x_1),(x_1,x_2),\ldots,(x_{n-1},x_n),(x_n,x_{n+1})$ ($x_0=x$ and $x_{n+1}=y$) in $N_l^*[D]$ and there is no circuit formed in $N_{l+1}^*[D]$ by adding (x,y). Then for every x_i, ($i<n-1$) of CH, $x_ix_{i+1}\notin E(G)$.*

Proof. For contradiction suppose $x_ix_{i+1}\in E(G)$ where x_i and x_{i+1} have different color. We consider two cases.
Case 1. x_{i+1},x_{i+2} have the same color. Now $x_ix_{i+2}\in E(G)$, as otherwise, $(x_{i+1},x_{i+2})(x_i,x_{i+2})$ is an arc of G^+, and hence $(x_i,x_{i+2})\in N^*[D]$, a shorter chain bypassing x_{i+1}. Let $(x_i,y_i)\in N_l^*[D]$ such that $(x_i,y_i)(x_i,x_{i+1})$ is an arc of G^+, $x_iy_i\notin E(G)$, and $y_ix_{i+1}\in E(G)$. Let $(y_{i+1},x_{i+1})\in N^*[D]$ such that $(x_{i+1},y_{i+1})(x_{i+1},x_{i+2})$ is an arc of G^+ with $y_{i+1}x_{i+2}\in E(G)$, and $x_{i+1}y_{i+1}\notin E(G)$. Now, $y_ix_{i+2}\notin E(G)$, as otherwise, $(x_i,y_i)(x_i,x_{i+2})$, and hence, $(x_i,x_{i+2})\in N^*[D]$, a shorter circuit. Now $y_ix_{i+1},y_{i+1}x_{i+2}$ are independent edges. Thus, we have $(x_i,y_i)(x_{i+2},y_i)$ and $(x_{i+2},y_i)(x_{i+2},x_{i+1})$ are arcs of G^+, implying that $(x_{i+2},x_{i+1})\in N_l^*[D]$, a contradiction.

Case 2. x_{i+1},x_{i+2} have different colors. By Lemma 33 $x_{i+1}x_{i+2}\in E(G)$.

By Case 1 and Lemma 33, $x_{i+2}x_{i+3}\in E(G)$. Let $(x_{i+2},y_{i+2})\in N^*[D]$ such that there is an arc $(x_{i+2},y_{i+2})(x_{i+2},x_{i+3})$ in G^+ with $y_{i+2}x_{i+3}\in E(G)$ and $x_{i+2}y_{2+1}\notin E(G)$ (x_{i+2},y_{i+2} have different colors). Notice that $y_{i+2}x_{i+1}\notin E(G)$, as otherwise, $(x_{i+2},y_{i+2})(x_{i+2},x_{i+1})$ is an arc of G^+. and $(x_{i+2},x_{i+1})\in N^*[D]$, a contradiction. Moreover, $y_ix_{i+3}\notin E(G)$, otherwise, $(x_i,y_i)(x_i,x_{i+3})$ is an arcs of G^+, and hence, $(x_i,x_{i+3})\in N^*[D]$, a shorter circuit. Now $(x_{i+2},y_{i+2})(x_{i+1},y_{i+2})$, $(x_{i+1},y_{i+2})(x_{i+1},x_{i+3})$ are arcs of G^+, and hence $(x_i,x_{i+3})\in N^*[D]$, a shorter circuit. □ □

Lemma 36. *Suppose $(x,y)\in N^*[D]$ is obtained by a minimal chain $CH=(x_0,x_1),(x_1,x_2),\ldots,(x_{n-1},x_n),(x_n,x_{n+1})$ ($x_0=x$ and $x_{n+1}=y$) in $N_l^*[D]$, and there is no circuit formed in $N_{l+1}^*[D]$ by adding (x,y) (level 0 to level l pairs). Suppose $x_ix_{i+1}\notin E(G)$ and x_i,x_{i+1} have different colors. Then the following hold.*

1. *$n=2$ and $(x_i,x_{i+1}),(x_{i+1},x_{i+2})$ are in components.*
2. *If $n>2$, x_{i+1},x_{i+2} have the same color and $x_ix_{i+2}\in E(G)$.*

3. *If* x_{i+3} *exists then* $x_{i+2}x_{i+3} \notin E(G)$ *and* x_i, x_{i+3} *have the same color and* x_{i+1}, x_{i+2} *have the same color different from the color of* x_i.

4. $n = 3$.

Proof. If x_{i+3} and x_{i-1} do not exist, then by Lemma 33 $n = 2$ and (x_i, x_{i+1}), (x_{i+1}, x_{i+2}) are in components. By Lemma 33 and Lemma 34 we have $x_i x_{i+2} \in E(G)$. Suppose x_{i+3} exists. We show that x_{i+2}, x_{i+3} must have different colors. Suppose that this is not the case. Now we have $x_i x_{i+3} \in E(G)$, as otherwise, $(x_{i+2}, x_{i+3}) \to (x_i, x_{i+3})$ in $N_l^*[D]$, and hence, we get a shorter chain bypassing x_{i+1}, x_{i+2}. Let $(x_{i+1}, y_{i+1}) \to (x_{i+1}, x_{i+2})$ in $N^*[D]$ with $y_{i+1}x_{i+2} \in E(G)$ and $x_{i+1}y_{i+1} \notin E(G)$. Now $y_{i+1}x_{i+3} \notin E(G)$, as otherwise, $(x_{i+1}, y_{i+1}) \to (x_{i+1}, x_{i+3})$ in $N_l^*[D]$, and hence, by passing x_{i+2} we get a shorter chain. Now $(x_{i+2}, x_{i+3}) \to (y_{i+1}, x_{i+3})$ and $(y_{i+1}, x_{i+3}) \to (y_{i+1}, x_i)$ in $N_l^*[D]$, implying that $(y_{i+1}, x_i) \in N^*[D]$, and getting a circuit $(x_i, x_{i+1}), (x_{i+1}, y_{i+1}), (y_{i+1}, x_i)$ in $N_l^*[D]$, a contradiction.

Therefore, x_{i+2}, x_{i+3} must have different colors and x_i, x_{i+3} have the same color. This proves 3.

Suppose $x_{i+4} \neq x_i$ exists. By Lemma 28, $x_{i+3}x_{i+4} \notin E(G)$ and $x_i x_{i+4} \notin E(G)$.

First, assume that x_{i+3}, x_{i+4} have different colors. Let $(y_{i+4}, x_{i+4}) \in N^*[D]$, so that the arc $(y_{i+4}, x_{i+4})(x_{i+3}, x_{i+4})$ is in G^+ (here $y_{i+4}x_{i+3} \in E(G)$). Notice that $x_i y_{i+4} \notin E(G)$, as otherwise, $(y_{i+4}, x_{i+4})(x_i, x_{i+4})$, a shorter chain bypassing $x_{i+1}, x_{i+2}, x_{i+3}$. Now, $x_i x_{i+2}, y_{i+4}x_{i+3}$ are independent edges, and hence (x_i, x_{i+3}) is in a strong component. Therefore, it has been selected to be in D, yielding a shorter circuit.

Second, let us assume x_{i+3} and x_{i+4} have the same color. Thus, there is $(x_{i+3}, y_{i+4}) \in N^*[D]$, so that $(x_{i+3}, y_{i+4})(x_{i+3}, x_{i+4})$ is an arc of G^+. Observe that $y_{i+4}x_i \notin E(G)$, otherwise, $(x_{i+3}, y_{i+4})(x_{i+3}, x_i)$ is an arc of G^+, yielding a circuit $(x_i, x_{i+1}), (x_{i+1}, x_{i+2}), (x_{i+2}, x_{i+3})$, (x_{i+3}, x_i). Now $x_{i+2}x_{i+4}$ must be an edge of G, as otherwise, $x_i x_{i+2}, x_{i+4}y_{i+4}$ are independent edges, and hence, (x_i, x_{i+4}) is in a strong component, and hence, (x_i, x_{i+4}) must be in D, according to the rules of the algorithm. Therefore, $n = 3$. □ □

Lemma 37. *Let* $C = (x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, x_0)$ *be a minimal circuit in* $N_l^*[D]$. *Then the number of tail pairs in* C *is two and* $n \leq 3$.

Proof. First suppose there are more than two tail pairs in C. Let $(x_i, x_{i+1}), (x_j, x_{j+1}), (x_k, x_{k+1})$, $0 \leq i \leq j \leq k \leq n$ (here $x_{n+1} = x_0$) be tail pairs in C. Notice that by Lemma 32 $j - i > 1$ and $k - j > 1$. Since $(x_i, x_{i+1}), (x_j, x_{j+1})$, (x_k, x_{k+1}) are tail pairs, there are $(x_i, a_{i+1}) \to (x_i, x_{i+1})$, $(x_j, a_{j+1}) \to (x_j, x_{j+1})$ and $(x_k, a_{k+1}) \to (x_k, x_{k+1})$ in $N_l^*[D]$ with $a_{i+1}x_{i+1}, a_{j+1}x_{j+1}, a_{k+1}x_{k+1} \in E(G)$, and $x_i a_{i+1}, x_j a_{j+1}, x_k a_{k+1} \notin E(G)$. Let $c(x)$ denote the partite set of x (color of the vertex $x \in G$). Suppose $c(x_{i+1}) \neq c(a_{j+1})$ and $c(a_{i+1}) \neq c(x_{j+1})$. Now $a_{i+1}x_{j+1} \notin E(G)$ otherwise $(x_i, a_{i+1}) \to (x_i, x_{j+1})$ in $N_l^*[D]$ a shorter circuit. Similarly $a_{j+1}x_{i+1} \notin E(G)$. Now (x_{i+1}, x_{j+1}) is in a component and, according to the rules of the algorithm, it should have been selected no later than (x_{i+2}, x_{i+3}), and hence a shorter circuit in $N_l^*[D]$. Therefore, by similar argument the following should occur.

- $c(a_{i+1}) = c(x_{j+1})$ (or $c(x_{i+1}) = c(a_{j+1})$)
- $c(a_{j+1}) = c(x_{k+1})$ (or $c(x_{k+1}) = c(a_{j+1})$)

- $c(a_{i+1}) = c(x_{k+1})$ (or $c(x_{i+1}) = c(a_{k+1})$)

So we may assume $c(a_{i+1}) = c(x_{j+1})$ (or $c(x_{i+1}) = c(a_{j+1})$). First suppose $c(a_{j+1}) = c(x_{k+1})$. Since $x_{j+1}a_{j+1}$ is an edge, $c(x_{j+1}) = c(a_{i+1}) \neq c(x_{k+1})$. Thus we must have $c(a_{k+1} = c(x_{i+1})$. Note that $a_{k+1}a_{j+1} \notin E(G)$, otherwise $(x_k, a_{k+1}) \to (x_k, x_{j+1})$, a shorter circuit in $N_l^*[D]$. Now $x_{j+1}x_{k+1}$ is an edge of G, otherwise $(x_{j+1}, a_{k+1}), (x_{j+1}, x_{k+1})$ are in a component, and hence according to the rules of the algorithm, $(x_{j+1}, x_{k+1}) \in N_l^*[D]$, a shorter circuit. By Lemma 32, (x_{j+1}, x_{j+2}) is a head pair, and hence $x_{j+1}x_{j+2} \notin E(G)$. By Lemma 28, $x_{j+2}x_{k+1} \notin E(G)$. Now, $(x_{j+1}, x_{j+2}), (x_{k+1}, x_{j+2})$ are in a same component, a shorter circuit (note that $j + 2 \neq k + 1$). By Lemma 32 we conclude that $n \leq 3$. □ □

Theorem 38. *If in stage 1 of the algorithm we encounter a component S such that we cannot add $N^+[S]$ and $N^+[S']$ to the current D, then G has an exobiclique as an induced subgraph, and it is not an interval k-graph.*

Proof. The inability to add $N^+[S]$ and $N^+[S']$ arises because their inclusion creates circuits in $D \cup N^+[S]$ and $D \cup N^+[S']$, respectively. Suppose that adding $N^+[S]$ with $S = S_{v_i,v_j}$ to D results in a minimal circuit $C = (x_0, x_1), (x_1, x_2), \dots, (x_{n-1}, x_n), (x_n, x_0)$ in D.

By Lemma 37, we have $n \leq 3$. First consider the case where $n = 3$. Assume that (x_0, x_1) and (x_2, x_3) are head pairs and (x_1, x_2) and (x_3, x_0) are tail pairs. By Lemma 29 we have $x_0x_2 \in E(G)$, and $c(x_1) = c(x_2)$ and $x_0x_1 \notin E(G)$. Furthermore, $c(x_3) = c(x_0)$ have the same color and $x_2x_3 \notin E(G)$ ($c(x_2) \neq c(x_3)$). Note that there is a_0 such that $a_0x_0 \in E(G)$, $x_0x_1 \notin E(G)$. There is also $x_2a_2 \in E(G)$ so that $x_1a_2 \notin E(G)$. Note that $x_0a_2, a_0a_2 \notin E(G)$.

Notice that $c(a_2) \geq \min\{c(x_0), c(a_0)\}$ otherwise, since a_0a_2, x_0a_0 are not edges of G, $S_{a_2x_0}$ is in a component and we should have selected (a_2, x_0) before (x_0, x_1), implying a shorter circuit. We show that $c(x_0) = c(a_2)$ or $c(a_0) = c(a_2)$. Otherwise, (x_0, a_2) is in a component, and therefore (x_0, a_2) has been selected. Now $(x_0, a_2) \to (x_0, x_2)$, and hence we get a shorter circuit.

Case 1: Suppose $a_2b_2 \in E(G)$ where $x_1b_2 \notin E(G)$ and $a_2b_2 \in E(G)$. This implies that $a_0b_2 \notin E(G)$. Now, since x_0a_0 and a_2b_2 are independent edges, (x_0, a_2) is part of a component. We also observe that $c(a_2), c(b_2) > c(x_1)$; otherwise, (a_2, x_1) should have been chosen earlier.

Subcase 1.1: $x_1a_0 \notin E(G)$ and $c(a_0) \neq c(x_1)$. Now (x_0, x_1) is in a component and hence $c(x_0) < c(x_1)$. Now (x_0, a_2) must have been selected before (x_1, a_2) and hence $(x_0, a_2) \to (x_0, a_2) \to (x_0, x_2)$ a shorter circuit.

Subcase 1.2: $x_1a_0 \notin E(G)$, and $c(a_0) = c(x_1)$. Now (a_0, x_1) is in a component and hence, there is x_1b_1 so that $a_0b_1 \notin E(G)$. Note that we must have $c(a_2) = c(x_0)$ and $c(b_2) = c(x_0)$ but this is a contradiction, as it would imply that we should have chosen (a_2, x_1) because $c(x_1) > c(a_2)$ or $c(x_1) > c(b_2)$.

Case 2. There are $a_2b_2, x_1a_1 \in E(G)$ so that $x_1a_2, a_1b_2 \notin E(G)$. Note that $(x_0, x_1) \to (x_0, a_1)$. Now, $x_0b_2 \notin E(G)$, otherwise, $(a_1, b_2) \to (a_1, x_0)$ a shorter circuit.

Note that again using the arguments in Case 1, we conclude that there are $x_1b_1 \in E(G)$ so that $a_0b_1 \notin E(G)$. Since x_0x_2 is an edge, x_2b_1 is an edge of G, otherwise $(x_0, b_1) \to (x_2, b_1) \to (x_2, x_1)$.

Analogously, for pairs $(x_2, x_3), (x_3, x_0)$ we conclude that there are independent edges x_2c_2, x_3a_3, uv of G where $x_2x_3, c_2a_3, x_2u, a_3u, c_2v, x_3v \notin E(G)$. Furthermore, we have $vx_0 \in E(G)$. Now it is easy to see that $va_1, va_2, a_3a_1, a_3a_2$ are edges of G, otherwise we get a shorter circuit. However, an exobiclique appears on $G[\{x_0, x_1, x_2, x_3, a_0, a_1, c_2, b_2, a_3, u, v\}]$.

For $n < 3$, we conclude $n = 2$, where (x_0, x_1) belongs to a component, and (x_1, x_2) is implied by a component. Independent edges $x_0a_0, x_1a_1 \in E(G)$ exist such that $x_0x_1, a_0a_1 \notin E(G)$ and $c(x_0) \neq c(x_1)$. Moreover, edges $x_1b_1, c_2b_2 \in E(G)$ exist such that $x_1c_2, b_1b_2 \notin E(G)$ and $x_2b_1, x_2c_2, x_2a_1, x_2x_0 \in E(G)$. Suppose $(x_2, u) \to (x_2, x_0)$ in D. Then, $x_2, u_2 \notin E(G)$ and $x_0u_2 \in E(G)$. Here, $c(x_2) \neq c(u_2)$. Using similar arguments, we show that G contains a forbidden structure as illustrated in Figure 8, confirming that G is not an interval k-graph. □

Now suppose $(x_2, u) \to (x_2, x_0)$ in D. Thus, we should have $x_2, u_2 \notin E(G)$, and $x_0u_2 \in E(G)$ (assuming that (x_2, x_0) is a tail pair). Here $c(x_2) \neq c(u_2)$. Notice that $a_1u_2 \in E(G)$, otherwise, $(x_0, a_1) \to (u_2, a_1) \to (u_2, x_2)$, a contradiction as $(x_2, u_2) \in D$. By similar argument, we conclude that $c_2u_2 \in E(G)$. Now $(x_0, x_1) \to (x_0, b_1) \to (u_2, b_1) \to (u_2, x_2)$ if $b_1u_2 \notin E(G)$. Thus, $x_0u_2, a_1u_2, b_1u_2, c_2u_2 \in E(G)$. However, $G[\{x_0, x_1, x_2, a_0, a_1, b_1, b_2, c_2, u_2\}]$ is isomorphic to Figure 5 which is an obstruction to interval k-graphs. □ □

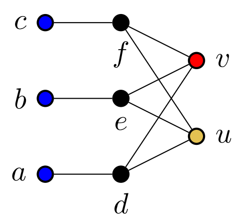

Figure 8: Smaller obstruction

Lemma 39. *Let $C = (x_0, x_1), (x_1, x_2), \dots, (x_{n-1}, x_n), (x_n, x_0)$ be a minimal circuit formed in $N^*[D]$ in Stage 2 of the algorithm. Then $n = 3$, and x_0, x_3 have the same color and x_1, x_2 have the same color and different from the color of x_0. Furthermore, $x_0x_2 \in E(G)$.*

Proof. Suppose (x_n, x_0) is the last pair added to $N^*[D]$. Now $(x_0, x_1), (x_1, x_2), \dots, (x_{n-1}, x_n)$ is a minimal chain between x_0, x_n. Thus, by Lemma 36, either $n = 2$ and $(x_0, x_1), (x_1, x_2), (x_2, x_0)$ are all components, resulting in a circuit in Stage 1 of the algorithm, or we have $n = 3$, and x_1, x_2 have the same color and different from the color of x_0. Furthermore, $x_0x_2 \in E(G)$. □ □

From Lemma 36 we derived the following Lemma.

Lemma 40. *Let (x, y) be a pair in D after Stage 1 of the algorithm, and assume that the current D has no circuit.*

- *Suppose x and y have the same color and $(x, w)(x, y) \in A(G^+)$ such that (x, w) is by transitivity with a minimal chain $(x, w_1), (w_1, w_2), \dots, (w_m, w)$. Then $m = 2$ and vertices x and w_1 have the same color and opposite to the color of w_2 and w.*
- *Suppose x and y have different colors and $(w, y)(x, y) \in A(G^+)$ such that (w, y) is in a trivial component. Then(w, y) is by transitivity with a minimal chain $(w, w_1), (w_1, w_2), (w_2, y)$ where w_1 and w_2 have the same color opposite to the color of w and y.*

Lemma 41. *Let $C : (x_0, x_1), (x_1, x_2), (x_2, x_3), (x_3, x_0)$ be a circuit formed at Stage 2 of Algorithm 1. Then there is a component $S \in D$, so for any complete set D_1 containing $N^*[D_1]$ contains a circuit.*

Proof. By Lemma 39, assume that x_0, x_3 has the same color and is opposite to the color of x_1, x_2. We first show that if one of the pairs (x_1, x_2) is not in $N^+[S]$ for any component S, then it is implied by (x_1, w) where (x_1, w) is implied transitivity $(x_1, w_1), (w_1, w_2), (w_2, w)$ where x_1, w_1 have the same color and are opposite to the color of w_2, x_0. The same happens for the other pairs of the circuit. In other worlds, the pairs involved in creating such a circuit are a subset of $V_i \times V_j$ for some fixed $1 \leq i < j \leq k$. Suppose $(x_1, w) \in N_l^*[D] \to (x_1, x_2)$ with $wx_2 \in E(G)$, and $x_1 w \notin E(G)$. We observe that $wx_3 \notin E(G)$, else $(x_1, w) \to (x_1, x_3)$, contradicting the minimality of the circuit. This implies that if w, x_3 has different colors, then $(x_2, x_3) \to (w, x_3) \to (x_2, x_3)$ in $N_1^*[D]$. Now we get a circuit $(x_0, x_1), (x_1, w), (w, x_3), (x_3, x_0)$, which contradicts the minimality of the circuit and Lemma 39. Therefore, w, x_0, x_3 have the same color. Now, by Lemma 40, x_1, w_1 have the same color and different from the color of w_2, x_3. Using the same argument, we can show that the vertices involved in creating circuit C belong to $V_i \cup V_{i+1}$ (two different color classes). Therefore, we can apply Lemmas 6.14, 6.15, 6.16, 6.17, 6.18, and 6.19 from [22] the lemma is proved. □

5 Generalization

For a graph H and a coloring $c \colon V(G) \to V(H)$, the graph G equipped with intervals I_v for each $v \in V(G)$ is an *interval H-graph* if, for different $u, v \in V(G)$, there is an edge $uv \in E(G)$ if and only if $I_u \cap I_v \neq \emptyset$ and $c(u)c(v) \in E(H)$. Since interval k-graphs are interval K_k-graphs, the concept of interval H-graphs generalizes interval k-graphs. Let $\chi(G)$ and $\omega(G)$ denote the chromatic number of G and the maximum size of a clique in G. For all intervals H-graphs we have $\chi(G) \leq \chi(H)$ and $\omega(G) \leq \omega(H)$. Moreover, every graph H is an interval H-graph where all intervals I_v coincide. In order to decide if G is an interval H-graph we construct the auxiliary digraph G^+, as follows. The vertex set of G^+ consists of ordered pairs (u, v) where $u \neq v \in V(G)$. The arc set of G^+ is defined as follows:

- $(u, v)(u, v')$ is an arc if $uv \notin E(G)$, $c(u) \neq c(v)$, $c(v)c(v') \in E(H)$, and $vv' \in E(G)$.
- $(u, v)(u', v)$ is an arc if $c(u)c(u') \in E(H)$, $uu' \in E(G)$, and $vu' \notin E(G)$.

Clearly, if a component of G^+ contains a circuit, then G is not an interval H-graph. Assuming that no strong component of G^+ contains a circuit, the key challenge is to select, from each pair of components S and S', one component to be included in the set D, ensuring that D remains closed under reachability and transitivity. We believe that our approach for interval k-graphs could be extended to this setting. However, we may end up having a new set of obstructions.

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