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Communication

Minimum cost and list homomorphisms to semicomplete digraphs

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Abstract

For digraphs D and H, a mapping $f : V(D) \to V(H)$ is a *homomorphism of D to H* if $uv \in A(D)$ implies $f(u)f(v) \in A(H)$. Let H be a fixed directed or undirected graph. The *homomorphism problem* for H asks whether a directed or undirected input graph D admits a homomorphism to H. The *list homomorphism problem* for H is a generalization of the homomorphism problem for H, where every vertex $x \in V(D)$ is assigned a set L_x of possible colors (vertices of H).

The following optimization version of these decision problems generalizes the list homomorphism problem and was introduced in Gutin et al. [Level of repair analysis and minimum cost homomorphisms of graphs, Discrete Appl. Math., to appear], where it was motivated by a real-world problem in defence logistics. Suppose we are given a pair of digraphs D, H and a positive integral cost $c_i(u)$ for each $u \in V(D)$ and $i \in V(H)$. The cost of a homomorphism f of D to H is $\sum_{u\in V(D)} c_{f(u)}(u)$. For a fixed digraph H, the *minimum cost homomorphism problem* for H is stated as follows: for an input digraph D and costs $c_i(u)$ for each $u \in V(D)$ and $i \in V(H)$, verify whether there is a homomorphism of D to H and, if one exists, find such a homomorphism of minimum cost.

We obtain dichotomy classifications of the computational complexity of the list homomorphism and minimum cost homomorphism problems, when H is a semicomplete digraph (digraph in which there is at least one arc between any two vertices). Our dichotomy for the list homomorphism problem coincides with the one obtained by Bang-Jensen, Hell and MacGillivray in 1988 for the homomorphism problem when H is a semicomplete digraph: both problems are polynomial solvable if H has at most one cycle; otherwise, both problems are NP-complete. The dichotomy for the minimum cost homomorphism problem is different: the problem is polynomial time solvable if H is acyclic or H is a cycle of length 2 or 3; otherwise, the problem is NP-hard.

Keywords: Homomorphisms; Digraphs; Semicomplete digraphs

1. Introduction

For excellent introductions to homomorphisms in directed and undirected graphs, see [20,22]. In this paper, directed (undirected) graphs have no parallel arcs (edges) or loops. The vertex (arc) set of a digraph G is denoted by $V(G)$ ($A(G)$). The vertex (edge) set of an undirected graph G is denoted by $V(G)$ ($E(G)$). For a digraph G, if $xy \in A(G)$, we say that x *dominates* y and y is *dominated* by x. A k-*cycle*, denoted by $\vec{C}_k$, is a directed simple cycle with k vertices. A digraph is *acyclic* if it has no cycle. A digraph D is *semicomplete* if, for each pair x, y of distinct vertices either x

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dominates y or y dominates x or both. A *tournament* is a semicomplete digraph with no 2-cycle. Semicomplete digraphs and, in particular, tournaments are well-studied in graph theory and algorithms [4]. A digraph G' is the *dual* of a digraph G if G' is obtained from G by reversing the orientation of all arcs.

For digraphs D and H, a mapping $f : V(D) \to V(H)$ is a *homomorphism of* D *to* H if $uv \in A(D)$ implies $f(u)f(v) \in A(H)$. A homomorphism f of D to H is also called an H-*coloring* of D, and $f(x)$ is called the *color* of the vertex x in D. We denote the set of all homomorphisms from D to H by $HOM(D, H)$. Let H be a fixed digraph. The *homomorphism problem for* H, HOMP(H), asks whether there is a homomorphism of an input digraph D to H (i.e., whether $HOM(D, H) \neq \emptyset$). In the *list homomorphism problem for* H, LHOMP(H), we given an input digraph D and a set (called a *list*) $L_v \subseteq V(H)$ for each $v \in V(D)$. Our aim is to check whether there is a homomorphism $f \in HOM(D, H)$ such that $f(v) \in L_v$ for each $v \in V(D)$.

The problems HOMP(H) and LHOMP(H) have been studied for several families of directed and undirected graphs H, see, e.g., [20,22]. A well-known result of Hell and Nešetřil [21] asserts that HOMP(H) for undirected graphs is polynomial time solvable if H is bipartite and it is NP-complete, otherwise. Feder et al. [11] proved that LHOMP(H) for undirected graphs is polynomial time solvable if H is a bipartite graph whose complement is a circular arc graph (a graph isomorphic to the intersection graph of arcs on a circle), and LHOMP(H) is NP-complete, otherwise. Such a dichotomy classification is not known for the homomorphism problems HOMP(H) when H is a digraph and only partial classifications have been obtained, see [22]. For example, Bang-Jensen et al. [5] showed that HOMP(H) for semicomplete digraphs H is polynomial time solvable if H has at most one cycle and HOMP(H) is NP-complete, otherwise. Nevertheless, Bulatov [7] managed to prove that each list homomorphism problem LHOMP(H) is polynomial time solvable or NP-complete. Such a dichotomy result for HOMP(H) has been conjectured, see, e.g., [20,22]. If this conjecture holds, it will imply that the well-known Constraint Satisfaction Problem Dichotomy Conjecture of Feder and Vardi also holds [12].

The authors of [16] introduced an optimization problem, MinHOMP(H), on H-colorings of undirected graphs H. The problem is motivated by a problem in defence logistics. Suppose we are given a pair of digraphs D, H and a positive integral cost $c_i(u)$ for each $u \in V(D)$ and $i \in V(H)$. The *cost* of a homomorphism f of D to H is $\sum_{u \in V(D)} c_{f(u)}(u)$. For a fixed digraph H, the *minimum cost homomorphism problem* MinHOMP(H) is stated as follows: for an input digraph D and costs $c_i(u)$ for each $u \in V(D)$ and $i \in V(H)$, verify whether $HOM(D, H) \neq \emptyset$ and, if $HOM(D, H) \neq \emptyset$, find a homomorphism in $HOM(D, H)$ of minimum cost. The problem MinHOMP(H) generalizes LHOMP(H) (and, thus, HOMP(H)): assign $c_i(u) = 1$ if $i \in L_u$ and $c_i(u) = 2$, otherwise. Then a list homomorphism with respect to lists L_u, $u \in V(D)$, exists if and only if there is a homomorphism of D to H of cost $|V(D)|$.

In this paper, we obtain dichotomy classifications for the time complexity of LHOMP(H) and MinHOMP(H) when H is a semicomplete digraph. Our classification for LHOMP(H) coincides with that for HOMP(H) [5] described earlier. However, for MinHOMP(H) the classification is different: the problem is polynomial time solvable when H is either an acyclic tournament or a 2-cycle or a 3-cycle. Otherwise, MinHOMP(H) is NP-hard. This implies that even when H is a unicyclic semicomplete digraph on at least four vertices, MinHOMP(H) is NP-hard (unlike HOMP(H) and LHOMP(H)).

Cohen et al. [8,9] considered an optimization version of the well-known constraint satisfaction problem (CSP), the valued CSP (VCSP). Special cases of VCSP were studied in several other papers including [10], where weighted Max CSP is investigated. The problem VCSP and some of its special cases generalize MinHOMP(H). We consider VCSP in the next section and demonstrate that an important result on VCSP describing some polynomial cases can be applied to MinHOMP(H). However, since VCSP is a proper generalization of MinHOMP(H) we could not possibly use NP-hardness results proved for VCSP. Moreover, many of these NP-hardness results are for some special cases of VCSP that do not generalize MinHOMP(H).

VCSP extends another optimization problem on H-colorings, the minimum graph homomorphism problem, introduced in [1]. However, the authors of [1] considered only reflexive undirected graphs H, i.e., graphs in which every vertex of H has a loop, and the costs are assigned only to edges of H. Thus, MinHOMP(H) and the minimum graph homomorphism problem from [1] are rather different problems. Another related but different homomorphism problem on weighted graphs is investigated in [15].

The *maximum cost homomorphism problem* MaxHOMP(H) is the same problem as MinHOMP(H), but instead of minimization we consider maximization. Let M be a constant larger than any cost $c_i(u)$, $u \in V(D)$, $i \in V(H)$. Then the cost $c'_i(u) = M - c_i(u)$ is positive for each $u \in V(D)$, $i \in V(H)$. Due to this transformation, the problems MinHOMP(H) and MaxHOMP(H) are equivalent. Note that allowing negative or zero costs would not make MinHOMP(H) and

MaxHOMP(H) more difficult: we can easily transform this more general case to the positive costs one by adding a large constant M' to each cost. This transformation does not change optimal solutions.

The rest of the paper is organized as follows. In Section 2, we consider two approaches that can be used for proving that MinHOMP(H) is polynomial time solvable for some digraphs H. Using the approaches we give two proofs that MinHOMP(H) is polynomial time solvable when H is an acyclic tournament. The dichotomy classifications LHOMP(H) and MinHOMP(H) when H is a semicomplete digraph are proved in Sections 3 and 4, respectively. We conclude the paper by posing some open problems.

2. Polynomial solvable cases of MinHOMP(H)

In this section, we consider two approaches for proving that MinHOMP(H) is polynomial time solvable for certain digraphs H. Using the approaches, we give two proofs that MinHOMP(H) is polynomial time solvable for acyclic tournaments.

The first approach was developed recently within the framework of valued constraint satisfaction, see [8,9] and our short description of the framework below. It makes use of submodular function minimization. The second approach is an extension of an approach developed in [16]. For H belonging to a special family $\mathscr{H}$ of digraphs, we can transform MaxHOMP(H) into the problem of finding a maximum cost independent set in a special family $\mathscr{F}(\mathscr{H})$ of undirected graphs. If the last problem is polynomial time solvable (when, for example, $\mathscr{F}(\mathscr{H})$ consists of perfect graphs, $2P_2$-free graphs, claw-free graphs or graphs of other special classes, see [2,3,6,14,24]), then the second approach is useful. The proof of Theorem 2.6 using the first approach is significantly shorter than that using the second approach. However, we present both approaches as we know of cases when only the second approach applies. Moreover, the second approach may lead to faster algorithms than the first approach, see Remark 2.7.

The first approach is based on some results for the *valued constraint satisfaction problem* (*VCSP*) [8,9]. Let Z be the set consisting of all nonnegative integers and ∞, and let Φ be a set of functions $\phi : W^{r(\phi)} \to Z$, where $r(\phi)$ is the arity of ϕ. An instance $\mathscr{I}$ of VCSP(Φ) is a triple (V, W, C), where V is a finite set of *variables*, which are to be assigned values from the set W, and C is a set of (valued) constraints. Each element of C is a pair $c = (\sigma, \phi)$, where σ is a $|\sigma|$-tuple of variables and $\phi : W^{|\sigma|} \to Z$ is a (cost) function, $\phi \in \Phi$. An *assignment* for $\mathscr{I}$ is a mapping s from V to W. The *cost* of s is defined as follows:

$$c_{\mathscr{I}}(s) = \sum_{((v_1,\ldots,v_m),\phi)\in C} \phi(s(v_1), \ldots, s(v_m)).$$

An *optimal solution* of $\mathscr{I}$ is an assignment s of minimum cost.

Let W be a totally ordered set. A binary function $\phi : W^2 \to Z$ is called *submodular* if, for all $x, y, u, v \in W$, we have

$$\phi(\min\{x, u\}, \min\{y, v\}) + \phi(\max\{x, u\}, \max\{y, v\}) \leqslant \phi(x, y) + \phi(u, v).$$

The following theorem is the main 'positive' result in [9].

Theorem 2.1. *For each Φ consisting of some unary functions and some binary submodular functions, VCSP(Φ) can be solved in time* $O(|V|^3|W|^3)$.

We will use this theorem to provide the basic result of our first approach.

Theorem 2.2. *Let H be a digraph and let there exist a labeling* $1, 2, \ldots, p$ *of the vertices of H satisfying the following property (SM): For any pair* $(i, k), (j, s)$ *of arcs in H, we have* $(\min\{i, j\}, \min\{k, s\}) \in A(H)$ *and* $(\max\{i, j\}, \max\{k, s\}) \in A(H)$. *Then MinHOMP(H) is polynomial time solvable.*

Proof. Let $1, 2, \ldots, p$ be a labeling of vertices of H satisfying the property (SM). The property ensures that the binary function ϕ, defined by $\phi(i, j)=0$ if $ij \in A(H)$ and $\phi(i, j)=\infty$ otherwise, is submodular. We will reduce MinHOMP(H) to VCSP(Φ), where Φ satisfies the conditions of Theorem 2.1. Let $\phi_u(i) = c_i(u)$ for all $u \in V(D)$ and $i \in V(H)$. Let $V = V(D)$ and $W = V(H)$. An assignment is an arbitrary function f from $V(D)$ to $V(H)$. Let $C = C' \cup C''$, where

$C' = \{(u, \phi_u) : u \in V(D)\}$ (for a fixed u, ϕ_u is a unary function from $V(H)$ to Z) and $C'' = \{((u, v), \phi_{uv}) : uv \in A(D)\}$, where each $\phi_{uv} = \phi$. Since each ϕ_{uv} is submodular, $\Phi = \{\phi_u : u \in V(D)\} \cup \{\phi_{uv} : uv \in A(D)\}$ satisfies the conditions of Theorem 2.1.

Let $\mathscr{I}$ be an instance of the above-constructed VCSP(Φ). It remains to observe that, if an assignment f is an H-coloring of D, then

$$c_{\mathscr{I}}(f) = \sum_{u \in V(D)} \phi_u(f(u)) + \sum_{uv \in A(D)} \phi_{uv}(f(u), f(v)) = \sum_{u \in V(D)} c_{f(u)}(u),$$

which is the cost of f in MinHOMP(H) (an integer), and if f is not an H-coloring, then $c_{\mathscr{I}}(f) = \infty$. Thus, by solving VCSP($\Phi$) we will determine whether $HOM(H) \neq \emptyset$, and find an optimal $h \in HOM(H)$, if $HOM(H) \neq \emptyset$. $\square$

A labeling $1, 2, \ldots, p$ of the vertices of H satisfies the *X-underbar property* if for any pair $(i, k), (j, s)$ of arcs in H, we have $(\min\{i, j\}, \min\{k, s\}) \in A(H)$. This property was introduced in [18] where it was used to prove that HOMP(H) is polynomial time solvable when H is an oriented path. So, it would be natural to call the property (SM) the X-bar & X-underbar property.

The second approach is based on Theorem 2.3 below. This idea (part (i)) can be traced back at least as far as [19], see [22, Exercise 7, Chapter 2]. It appears that Theorem 2.6 is the first nontrivial application of Theorem 2.3.

The *homomorphic product* of digraphs D and H is an undirected graph $D \otimes H$ defined as follows: $V(D \otimes H) = \{u_i : u \in V(D), i \in V(H)\}$, $E(D \otimes H) = \{u_i v_j : uv \in A(D), ij \notin A(H)\} \cup \{u_i u_j : u \in V(D), i \neq j \in V(H)\}$. Let $\mu = \max\{c_j(v) : v \in V(D), j \in V(H)\}$. We define the cost of u_i, $c(u_i) = c_i(u) + \mu|V(D)|$. For a set $X \subseteq V(D \otimes H)$, we define $c(X) = \sum_{x \in X} c(x)$.

Theorem 2.3. *Let D and H be digraphs.*

(i) *There is a homomorphism of D to H if and only if the number of vertices in a largest independent set of $D \otimes H$ equals $|V(D)|$.*
(ii) *If $HOM(D, H) \neq \emptyset$, then a homomorphism $h \in HOM(D, H)$ is of maximum cost if and only if $I = \{x_{h(x)} : x \in V(D)\}$ is an independent set of maximum cost in $D \otimes H$.*

Proof. Let $h : D \to H$ be a homomorphism. Consider $I = \{x_{h(x)} : x \in V(D)\}$. Suppose that $x_{h(x)} y_{h(y)}$ is an edge in $D \otimes H$. Then either $xy \in A(D)$ and $h(x)h(y) \notin A(H)$ or $yx \in A(D)$ and $h(y)h(x) \notin A(H)$. Either case contradicts the fact that h is a homomorphism. Thus, I is an independent set in $D \otimes H$.

Observe that each independent set in $D \otimes H$ contains at most one vertex in each set $S_x = \{x_i : i \in V(H)\}, x \in V(D)$. Let $I = \{x_{f(x)} : x \in V(D)\}$ be an independent set in $D \otimes H$ with $|V(D)|$ vertices. Consider the mapping $f : x \mapsto f(x)$. Assume $xy \in A(D)$. Since I is independent, $f(x)f(y) \in A(H)$. Thus, $f \in HOM(D, H)$.

Let $HOM(D, H) \neq \emptyset$ and let $n = |V(D)|$. Let X and Y be subsets of $V(D \otimes H)$ and $|X| = |Y| + 1 \leqslant n$. Then

$$c(X) - c(Y) \geqslant |X|n\mu - (|X| - 1)(n + 1)\mu \geqslant \mu > 0.$$

Thus, in particular, every maximum cost independent set of $D \otimes H$ is a largest independent set. Observe that the cost of the homomorphism f defined above equals the cost of vertices in the independent set I minus $n^2\mu$, which is a constant. Thus, every maximum cost independent set of $D \otimes H$ corresponds to a maximum cost homomorphism of D to H and vice versa. $\square$

Remark 2.4. In applications of Theorem 2.3, we may need to replace a pair D, H by another pair D', H' such that $HOM(D, H) = HOM(D', H')$ and the costs of the homomorphisms remain the same.

A digraph D is *transitive* if $xy, yz \in A(D)$ implies $xz \in A(D)$ for all pairs xy, yz of arcs in D. A graph is a *comparability graph* if it has an orientation, which is transitive. In the second proof of Theorem 2.6, we will use the following result proved in [23].

Theorem 2.5. *Let G be a comparability graph with n vertices and m edges and let every vertex of G be assigned a positive integer weight. We can compute a maximum weight independent set in G in time* $O(nm \log(n^2/m))$.

Bang-Jensen et al. [5] proved that if H is an acyclic tournament, then HOMP(H) is polynomial time solvable. We extend this result to MinHOMP(H). We provide two proofs using both approaches above.

Theorem 2.6. *If H is an acyclic tournament, then MinHOMP(H) is polynomial time solvable.*

First Proof. Let H be an acyclic tournament with $V(H) = \{1, 2, \ldots, p\}$ and $A(H) = \{ij : 1 \leqslant i < j \leqslant p\}$. Let (i, k) and (j, s) be arcs in H. Since $i < k$ and $j < s$, we conclude that $(\min\{i, j\}, \min\{k, s\})$ and $(\max\{i, j\}, \max\{k, s\})$ are also arcs in H. Thus, our theorem follows from Theorem 2.2.

Second Proof. Let H be an acyclic tournament with $V(H) = \{1, 2, \ldots, p\}$ and $A(H) = \{ij : 1 \leqslant i < j \leqslant p\}$. Observe that H is transitive. Also observe that $HOM(D, H) = \emptyset$ unless D is acyclic. Since we can verify that D is acyclic in time $O(|V(D)| + |A(D)|)$ [4], we may assume that D is acyclic. Since H is transitive, we have $HOM(D, H) = HOM(D^+, H)$, where D^+ is the transitive closure of D, i.e., if there is a path from x to y in D, then $xy \in D^+$. One can find the transitive closure of a digraph in polynomial time [4], so we may assume that D is transitive.

Let $G = D \otimes H$. Let G' be an orientation of G such that

$$A(G') = \{x_i y_j : j \leqslant i, xy \in A(D)\} \cup \{x_i x_j : x \in V(D), j < i\}.$$

We will prove that G' is a transitive digraph. Let $x_i y_j, y_j z_k \in A(G')$. Observe that $i \geqslant j \geqslant k$ and consider three cases covering all possibilities.

Case 1: $x = y = z$. Then $x_i x_j, x_j x_k \in A(G')$ and, thus, $i > j > k$ and $x_i z_k = x_i x_k \in A(G')$.

Case 2: $x = y = z$ does not hold, but not all vertices x, y, z are distinct. Without loss of generality, assume that $x = y \neq z$. Then $x_i x_j, x_j z_k \in A(G')$ and, thus, $i > k$ and $x_i z_k \in A(G')$.

Case 3: x, y, z are all distinct. Then $xy, yz \in A(D^+)$ and, thus, $xz \in A(D^+)$. Since $i \geqslant k$, we conclude that $x_i z_k \in A(G')$.

So, we have proved that G is a comparability graph. Therefore, by Theorem 2.5, a maximum cost independent set in $D \otimes H$ can be found in polynomial time. It remains to apply Theorem 2.3. If $D \otimes H$ has an independent set with $|V(D)|$ vertices, $HOM(D, H) \neq \emptyset$ and a maximum cost independent set corresponds to a maximum cost H-coloring. □

Remark 2.7. Let $n = |V(D)|$, $m = |A(D)|$. The first proof of Theorem 2.6 can be converted to an algorithm of complexity $O(n^3)$ (see Theorem 2.1). The second proof allows one to obtain an algorithm of complexity $O(n(n + m)\log(n^2/(n + m)) + n^{2.376})$ (by Theorem 2.5 and the fact that the transitive closure of digraph with n vertices can be found in time $O(n^{2.376})$ [4]). Observe that $O(n(n + m)\log(n^2/(n + m)) + n^{2.376}) = O(n^3)$ and the second proof leads to an asymptotically faster algorithm for $m = o(n^2)$.

Corollary 2.8. *If H is an acyclic tournament, then LHOMP(H) is polynomial time solvable.*

3. Dichotomy for LHOMP(H)

Recall that $\vec{C}_k$ denotes a directed cycle on k vertices, $k \geqslant 2$; let $V(\vec{C}_k) = \{1, 2, \ldots, k\}$. One can check whether $HOM(D, \vec{C}_k) \neq \emptyset$ using the following algorithm $\mathscr{A}$ from Section 1.4 of [22]. First, we may assume that D is connected (i.e., its underlying undirected graph is connected) as otherwise $\mathscr{A}$ can be applied to each component of D separately. Choose a vertex x of D and assign it color 1. Assign every out-neighbor of x color 2 and each in-neighbor of x color k. For every vertex y with color i, we assign every out-neighbor of y color $i + 1$ modulo k and every in-neighbor of y color $i - 1$ modulo k. We have $HOM(D, \vec{C}_k) \neq \emptyset$ if and only if no vertex is assigned different colors.

A special case of the following theorem was first proved by Green [13], who has shown that unicyclic tournaments admit a majority polymorphism (defined in, e.g., [7]). Our proof below is elementary, and does not rely on the machinery of polymorphisms.

Theorem 3.1. *Let H be a semicomplete digraph with a unique cycle, then LHOMP(H) is polynomial time solvable.*

Proof. It is well-known [4] that a semicomplete digraph with a unique cycle contains a cycle with two or three vertices. We assume that H has a cycle with three vertices (the case of 2-cycle can be treated similarly) and we prove this theorem by induction $|V(H)|$. If $|V(H)| = 3$, then we can use the algorithm $\mathscr{A}$ described earlier. Otherwise, there must exist a vertex $i \in V(H)$ with either in-degree or out-degree 0. Without loss of generality, let the out-degree of i be 0. Let R_i be the set of vertices in D that have out-degree 0 and have i in their list. Observe that a list homomorphism of D to H exists if and only if there exists a list homomorphism of D to H that maps all vertices in R_i to i. Since vertices that do not have out-degree 0 cannot map to i, we can reduce the problem to LHOMP$(H - i)$ with input $D - R_i$. By the induction hypothesis, the last problem admits a polynomial time algorithm. □

Recall that HOMP(H) is NP-complete when H is a semicomplete digraph with at least two cycles. This result, Corollary 2.8 and Theorem 3.1 imply the following:

Theorem 3.2. *Let H be a semicomplete digraph. Then LHOMP(H) is polynomial time solvable if H has at most one cycle, and LHOMP(H) is* NP-*complete, otherwise.*

4. Dichotomy for MinHOMP(*H*)

To solve MinHOMP(H) for $H = \vec{C}_k$, choose an initial vertex x in each component D' of D (a component of its underlying undirected graph). Using the algorithm $\mathscr{A}$ from the previous section, we can check whether each D' admits an $\vec{C}_k$-coloring. If the coloring of D' exists, we compute the cost of this coloring and compute the costs of the other $k - 1$, $\vec{C}_k$-colorings when x is colored $2, 3, \ldots, k$, respectively. Thus, we can find a minimum cost homomorphism in $HOM(D', \vec{C}_k)$. Thus, in polynomial time, we can obtain a $\vec{C}_k$-coloring of the whole digraph D of minimum cost. In other words, we have the following:

Lemma 4.1. *For $H = \vec{C}_k$, MinHOMP(H) is polynomial time solvable.*

Addition of an extra vertex to a cycle may well change the complexity of MinHOMP(H).

Lemma 4.2. *Let H' be a digraph obtained from $\vec{C}_k$, $k \geqslant 2$, by adding an extra vertex dominated by the vertices of the cycle, and let H be H' or its dual. Then MinHOMP(H) is* NP-*hard.*

Proof. Without loss of generality we may assume that $H = H'$ and that $V(H) = \{1, 2, 3, \ldots, k, k+1\}$, $123\ldots k1$ is a k-cycle, and the vertex $k + 1$ is dominated by the vertices of the cycle.

We will reduce the maximum independent set problem to MinHOMP(H). Let G be a graph. Construct a digraph D as follows:

$$V(D) = V(G) \cup \{v_i^e : e \in E(G) i \in V(H)\}, \quad A(D) = A_1 \cup A_2,$$

where

$$A_1 = \{v_1^e v_2^e, v_2^e v_3^e, \ldots, v_{k-1}^e v_k^e, v_k^e v_1^e : e \in E(G)\}$$

and

$$A_2 = \{v_1^{uv} u, v_{k+1}^{uv} u, v_2^{uv} v, v_{k+1}^{uv} v : uv \in E(G)\}.$$

Let all costs $c_i(t) = 1$ for $t \in V(D)$ apart from $c_{k+1}(p) = 2$ for all $p \in V(G)$.

Consider a minimum cost homomorphism $f \in HOM(D, H)$. By the choice of the costs, f assigns the maximum possible number of vertices of G (in D) a color different from $k + 1$. However, if pq is an edge in G, by the definition of D, f cannot assign colors different from $k + 1$ to both p and q. Indeed, if both p and q are assigned colors different from $k + 1$, then the existence of v_{k+1}^{pq} implies that they are assigned the same color, which however is impossible by the existence of $\{v_i^{pq} : i \in \{1, 2, \ldots, k\}\}$. Observe that f may assign exactly one of the vertices p, q color $k+1$ and the other a color different from $k + 1$. Also f may assign both of them color $k + 1$. Thus, G has a maximum independent set with α vertices if and only if D has a minimum cost H-coloring of cost $|E(G)|\cdot|V(H)|+2|V(G)|-\alpha$. This reduces the maximum independent set problem to MinHOMP(H). $\square$

Interestingly, the problem HOMP(H') for H' (especially, with $k=3$) defined in Lemma 4.2 is well known to be polynomial time solvable (see, e.g., [5,17,22]). The following lemma allows us to prove that MinHOMP(H) is NP-hard when MinHOMP(H') is NP-hard for an induced subdigraph H' of H.

Lemma 4.3. *Let H' be an induced subdigraph of a digraph H. If MinHOMP(H') is* NP-*hard, then MinHOMP(H) is also* NP-*hard.*

Proof. Let D be an input digraph with n vertices and let $c_i(u)$ be the costs, $u \in V(D), i \in V(H')$. Let all costs $c_i(u)$ be bounded from above by $\beta(n)$. For each $i \in V(H) - V(H')$ and each $u \in V(D)$, set costs $c_i(u) := n\beta(n)+1$. Observe that there is an H-coloring of D of cost at most $n\beta(n)$ if and only if $HOM(D, H') \neq \emptyset$ and if $HOM(D, H') \neq \emptyset$, then the cost of minimum cost H-coloring equals to that of minimum cost H'-coloring. $\square$

As a corollary of Theorem 2.6 and Lemmas 4.1–4.3, we obtain the following theorem.

Theorem 4.4. *For a semicomplete digraph H, MinHOMP(H) is polynomial time solvable if H is acyclic or $H=\vec{C}_k$ for $k=2$ or 3, and* NP-*hard, otherwise.*

Proof. By Theorem 2.6 and since HOMP(H) is NP-complete when a semicomplete digraph H has at least two cycles [5], we may restrict ourselves to the case when H has a unique cycle. Observe that this cycle has two or three vertices. If no other vertices are in H, MinHOMP(H) is polynomial time solvable by Lemma 4.1. Assume that H has a vertex i not in the cycle. Observe that i is dominated by or dominates all vertices of the cycle, i.e., H contains, as an induced subdigraph one of the digraphs of Lemma 4.2. So, we are done by Lemmas 4.2 and 4.3. $\square$

5. Discussion

In this paper we have obtained dichotomy classifications for the time complexity of the list and minimum cost H-coloring problems when H is a semicomplete digraph. It would be interesting to find out whether there exists a dichotomy classification for the minimum cost H-coloring problem (for an arbitrary digraph H) and if it does exist, to obtain such a classification. Since these problems seem to be far from trivial, one could concentrate on establishing dichotomy classifications for special classes of digraphs such as semicomplete multipartite digraphs (digraphs obtained from complete multipartite graphs by replacing every edge with an arc or the pair of mutually opposite arcs).

We have recently obtained some partial results on MinHOMP(H) for semicomplete multipartite digraphs H. To find a complete dichotomy for the case of semicomplete bipartite digraphs, one would need, among other things, to solve an open problem from [16]: establish a dichotomy classification for the complexity of MinHOMP(H) when H is a bipartite (undirected) graph. Indeed, let B be a semicomplete bipartite digraph with partite sets U, V and arc set $A(B)=A_1\cup A_2$, where $A_1=U\times V$ and $A_2\subseteq V\times U$. Let B' be a bipartite graph with partite sets U, V and edge set $E(B')=\{uv : vu \in A_2\}$. Observe that MinHOMP($B$) is equivalent to MinHOMP(B').

It was proved in [16] that MinHOMP(H) is polynomial time solvable when H is a bipartite graph whose complement is an interval graph. It follows from the main result of [11] that MinHOMP(H) is NP-hard when H is a bipartite graph whose complement is not a circular arc graph. This leaves the obvious gap in the classification for MinHOMP(H) when H is a bipartite graph.

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