Available online at www.sciencedirect.com
SCIENCE DIRECT®
Discrete Mathematics 289 (2004) 163–168
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Note

When n-cycles in n-partite tournaments are longest cycles

Gregory Gutin[1], Arash Rafiey

Department of Computer Science, Royal Holloway, University of London, Egham, Surrey TW20 0EX, UK

Received 24 June 2003; received in revised form 11 October 2004; accepted 26 October 2004

Abstract

An n-tournament is an orientation of a complete n-partite graph. It was proved by J.A. Bondy in 1976 that every strong n-partite tournament has an n-cycle. We characterize strong n-partite tournaments in which a longest cycle is of length n and, thus, settle a problem in Volkmann (Discrete Math. 199 (1999) 279).

Keywords: n-partite tournament; Longest cycles; Short cycles

1. Introduction

We use terminology and notation of [3]; all necessary notation and a large part of terminology used in this paper are provided in the next section.

A very informative paper [11] of Volkmann is the latest survey on cycles in an important class of digraphs, multipartite tournaments. Cycles in multipartite tournaments were earlier overviewed in [2,5,9]. Along with description of a large number of results on cycles in multipartite tournaments, Volkmann [11] formulates several open problems.

Bondy [4] proved that every strong n-partite tournament has a cycle of length n. Problem 3.4 in [11] is as follows:

E-mail addresses: gutin@cs.rhul.ac.uk (G. Gutin), arash@cs.rhul.ac.uk (A. Rafiey).
[1] Research was partially supported by the Leverhulme Trust.

0012-365X/$ - see front matter © 2004 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2004.10.007

Problem 1.1. *Characterize all strong n-partite tournaments in which a longest cycle is of length n.*

Notice that Problem 1.1 was first stated in [10]. This seemingly simple problem turns out to be fairly non-trivial. In this paper, we provide such a characterization in Theorems 3.3 and 3.11 and prove that our necessary and sufficient conditions are verifiable in polynomial time.

2. Terminology and notation

A digraph obtained from an undirected graph G by replacing every edge of G with a directed edge (arc) with the same end-vertices is called an *orientation* of G. An *oriented graph* is an orientation of some undirected graph. A *tournament* is an orientation of a complete graph and an *n-partite tournament* is an orientation of a complete n-partite graph. Partite sets of complete graphs become *partite sets* of n-partite tournaments. An *extended tournament* is an n-partite tournament obtained from a tournament on n vertices by replacing every vertex with an independent set of vertices. In an extended tournament all arcs between two partite sets are oriented in the same direction.

The terms *cycle* and *path* mean simple directed cycle and path. A cycle of length k is a *k-cycle*. For a cycle $C = v_1v_2 \ldots v_kv_1$, $C[v_i, v_j]$ denotes the path $v_iv_{i+1} \ldots v_j$ which is part of C. A *cycle subdigraph* of a digraph D is a collection of vertex-disjoint cycles of D. A digraph D is *strong* if for every ordered pair x, y of distinct vertices in D there exist paths from x to y. For a set X of vertices of a digraph D, $D\langle X\rangle$ denotes the subdigraph of D induced by X.

For sets T, S of vertices of a digraph $D = (V, A)$, $T \rightarrow S$ means that for every vertex $t \in T$ and for every vertex $s \in S$, we have $ts \in A$, and $T \Rightarrow S$ means that for no pair $s \in S, t \in T$, we have $st \in A$. While for oriented graphs $T \rightarrow S$ implies $T \Rightarrow S$, this is not always true for general digraphs. We also use the notation $T \rightleftharpoons S$, if neither $T \rightarrow S$ nor $S \rightarrow T$. If $u \rightarrow v$ (i.e., $uv \in A$), we say that u *dominates* v and v *is dominated* by u.

The following simple argument is called *directed duality*. Many properties of a given digraph D are preserved when we reverse all arcs of D and obtain a new digraph D'. For example, D has a k-cycle if and only if D' does.

3. Characterization

The following simple lemma first proved in [6] is very useful in our investigation. Similar, yet different results, can be found in [1,7]. We provide a proof for the sake of completeness and because of its usefulness for an algorithm described later on.

Lemma 3.1. *If a strong n-partite tournament, $n \geqslant 3$, has a k-cycle containing vertices from less than k partite sets, then D has an m-cycle with $m > n$.*

Proof. Let $Z = z_1z_2 \ldots z_sz_1$ be a longest cycle in D with at least two vertices from the same partite set. Assume that $s \leqslant n$. Consider the set S of vertices from partite sets not in Z.

If a vertex $x \in S$ has arcs to and from $V(Z)$, then there exists i such that $z_i \to x \to z_{i+1}$, and thus x can be inserted in Z to get a longer cycle with at least two vertices from the same partite set, a contradiction.

Thus, we may assume that either $S \to V(Z)$ or $V(Z) \to S$. Since both alternatives can be treated similarly, we consider only $V(Z) \to S$. Since D is strong, we can find a path P from a vertex x in S to Z. Let P be a shortest such path and let z_i be the terminal vertex of P. Then $PZ[z_{i+1}, z_{i-1}]x$ is a longer cycle with at least two vertices from the same partite set, a contradiction. $\square$

The following theorem allows us to settle Volkmann's problem for extended tournaments:

Theorem 3.2 ([8]). *The length of a longest cycle in a strong extended tournament D equals the maximal number of vertices in a cycle subdigraph of D. A longest cycle in D can be found in time $\mathrm{O}(p^3)$, where p is the number of vertices in D.*

As a special case, we immediately obtain the following:

Theorem 3.3. *In a strong extended tournament D with n-partite sets, the length of a longest cycle equals n if and only if the maximal number of vertices in a cycle subdigraph of D equals n. One can verify whether the length of a longest cycle in D is n in time $\mathrm{O}(p^3)$, where p is the number of vertices in D.*

There exist strong n-partite tournaments D that are not extended tournament, yet every longest cycle in D is of length n. Consider a strong 4-partite tournament H with partite sets $V_1 = \{v_1\}$, $V_2 = \{v_2, v_2'\}$, $V_3 = \{v_3\}$, $V_4 = \{v_4\}$ and such that $V_1 \to V_2 \to V_3 \to V_4 \to V_1 \to V_3$ and $v_2' \to v_4 \to v_2$. It is not difficult to check that H has no Hamilton cycle, but H contains an n-cycle.

Theorem 3.3 allows us, from now on, to consider only strong n-partite tournaments D, which are not extended tournaments. We know that D has an n-cycle C and we assume that D has no longer cycle. Let $V_1, V_2, \ldots, V_n$ be partite sets of D. By Lemma 3.1, we may assume that $C = v_1 v_2 \ldots v_n v_1$, $v_i \in V_i$, $i = 1, 2, \ldots, n$. Let $U[V_i, V_j]$ denote $V_i \cup V_{i+1} \cup \cdots \cup V_j$, where all indices are taken modulo n.

To study the structure of D we prove the following series of lemmas.

Lemma 3.4. *Let T (S) be the maximal subset of $D - V(C)$ such that $T \Rightarrow V(C)$ and $V(C) \Rightarrow S$. Then $T = S = \emptyset$.*

Proof. Assume that $T \neq \emptyset$. Let $U = V(D) - (V(C) \cup S \cup T)$. Since D is strong, there exists an arc xy from $S \cup U$ to T. There is a (shortest) path from a vertex $v_i \in C$ to x. Since y dominates either v_{i+1} or v_{i+2} or both, it is easy to see that D has a cycle of length more than n. Thus, $|T| = 0$, a contradiction. By directed duality, $|S| = 0$. $\square$

Lemma 3.5. *For every $i \in \{1, 2, \ldots, n\}$, $V_{i-1} \to V_i$, where $V_0 = V_n$.*

Proof. Clearly, the lemma holds if both V_{i-1} and V_i are singletons. By directed duality, we may assume that $|V_i|\geqslant 2$. Let $V_{i-1}=\{v_{i-1}\}$ and $z\in V_i-v_i$. If $z\to v_{i-1}$ then $z\to v_{i-2}$, since otherwise the cycle $zC[v_{i-1},v_{i-2}]z$ has length more than n. By continuing this argument we conclude that $z\Rightarrow C$, which contradicts Lemma 3.4.

It remains to consider the case of $|V_{i-1}|\geqslant 2$. Let $y\in V_{i-1}-v_{i-1}$. Suppose that $z\to y$. By directed duality $V_{i-1}\to v_i$ and thus, in particular, $y\to v_i$. Hence, $yC[v_i,v_{i-1}]zy$ is an $(n+2)$-cycle, a contradiction. Thus, $V_{i-1}\to V_i$. □

This lemma implies immediately the following:

Corollary 3.6. *For every choice $w_i\in V_i$, $i=1,2,\ldots,n$, $w_1w_2\ldots w_nw_1$ is a cycle in D.*

Lemma 3.7. *For every pair of non-singletons V_i, V_j we have that either $V_i\to V_j$ or $V_j\to V_i$.*

Proof. Suppose that neither $V_i\to V_j$ nor $V_j\to V_i$ holds. Then, without loss of generality, we may assume that there are vertices $x\in V_i$ and $y,z\in V_j$ such that $z\to x\to y$. By Corollary 3.6, we may assume that $x\neq v_i$ (we may replace v_i in C by another vertex in V_i). By Lemma 3.5, we have that $|i-j|>1$ and $v_{j-1}\to\{y,z\}\to v_{j+1}$. Thus, $xyC[v_{j+1},v_{j-1}]zx$ is an $(n+1)$-cycle, a contradiction. □

Lemma 3.8. *For every triple v_i,v_j,v_k such that $v_j\in C[v_i,v_k]$,*

(a) *If $|V_i|>1$ and $x\leftarrow v_j$ for some $x\in V_i$, then $x\leftarrow V_k$,*
(b) *If $|V_k|>1$ and $z\to v_j$ for some $z\in V_k$, then $z\to V_i$.*

Proof. By directed duality, Claims a and b are equivalent. Thus, it suffices to prove only Claim a. Let $|V_i|>1$, $x\in V_i$ and $x\leftarrow v_j$. By Corollary 3.6, we may assume that $x\neq v_i$. We have $v_{j+1}\to x$ since otherwise the cycle $xC[v_{j+1},v_j]x$ has length more than n. Continuing this argument, we conclude that $x\leftarrow v_k$. Now by Lemma 3.7 if $|V_k|>1$ then $V_k\to V_i$ because $x\leftarrow v_k$. □

Lemma 3.9. *Let $|V_i|>1$ and $|V_j|=1$. If $V_i\rightleftharpoons V_j$, then $U[V_{i+1},V_{j-1}]\leftarrow U[V_{j+1},V_{i-1}]$.*

Proof. Let $x\in V_i-v_i$. As above we may assume that $x\to v_j$ and $v_i\leftarrow v_j$. According to Lemma 3.8, for every $v\in C[v_{i+1},v_j]$ we have $x\to v$ and for every $u\in C[v_{j+1},v_{i-1}]$ we have $u\to v_i$. Now consider arbitrary vertices $v_t\in C[v_{i+1},v_{j-1}]$, $v_l\in C[v_{j+1},v_{i-1}]$ and suppose that $v_t\to v_l$. However, the cycle

$$xC[v_{t+1},v_{l-1}]C[v_i,v_t]C[v_l,v_{i-1}]x$$

has length greater than n. This is a contradiction and we have $v_t\leftarrow v_l$. By Corollary 3.6, instead of C we may consider the cycle obtained from C by replacing v_t with a vertex from $U[V_{i+1},V_{j-1}]$ and v_l with a vertex from $U[V_{j+1},V_{i-1}]$. All arguments above remain valid, which proves the lemma. □

Lemma 3.10. *Let V_i, V_j be two partite sets such that $|V_i|>1$, $|V_j|=1$ and $V_i \rightleftharpoons V_j$. Let X be the maximal subset of V_i such that $X \to v_j$. Let D_{ij} be obtained from $D\langle U[V_i, V_j]\rangle$ by changing orientations of the arcs between X and v_j and let D_{ji} be obtained from $D\langle U[V_j, V_i]\rangle$ by changing orientations of the arcs between $V_i - X$ and v_j. Then D_{ij} and D_{ji} have no cycles of length greater than the number of their partite sets.*

Proof. Assume that $j>i$. Clearly, D_{ij} is strong and the number of partite sets in D_{ij} is $m=j+1-i$. Suppose that D_{ij} has a cycle C' of length greater than m. Let $\bar{S}$ be the set of arcs in $D\langle U[V_i, V_j]\rangle$ whose orientations have been changed to obtain D_{ij}.

If C' does not contain an arc from $\bar{S}$, then it follows from Lemma 3.1 that D has a cycle of length greater than n, a contradiction. Now let C' contain an arc v_jx such that $v_jx \in \bar{S}$, $x \in X$. By deleting v_jx we find a path P in $D\langle U[V_i, V_j]\rangle$ that starts at $x \in V_i$ and ends at v_j with length at least m. Then the cycle $PC[v_{j+1}, v_{i-1}]x$ is of length greater than n, a contradiction.

By direct duality, the claim on cycles in D_{ji} follows. □

Observe that if D is not an extended tournament, then there exist partite sets V_i, V_j such that $V_i \rightleftharpoons V_j$.

Theorem 3.11. *Let D be a strong n-partite tournament. Suppose D is not an extended tournament. Let $V_1, V_2, \ldots, V_n$ be partite sets of D and let D have a cycle $v_1v_2\ldots v_nv_1$, where $v_i \in V_i$, $i=1,2,\ldots,n$. Choose a pair V_i, V_j with the property $V_i \rightleftharpoons V_j$ and let $|V_j| \leqslant |V_i|$. Choose a pair $x, y \in V_i$ such that $y \to v_j \to x$. Then D has no cycle of length greater than n if and only if the following conditions hold*:

(a) *For every pair V_s, V_t with the property $V_s \rightleftharpoons V_t$, we have* $\min\{|V_s|, |V_t|\}=1$;
(b) $U[V_j, V_{i-1}] \to x$ *and* $y \to U[V_{i+1}, V_j]$;
(c) $U[V_{i+1}, V_{j-1}] \leftarrow U[V_{j+1}, V_{i-1}]$;
(d) *The digraphs D_{ij}, D_{ji} defined in Lemma* 3.10 *have no cycles of length greater than the number of their partite sets*.

Proof. Condition (a) is necessary by Lemma 3.7; (b) follows from Lemmas 3.5 and 3.8; (c) and (d) follow from Lemmas 3.9 and 3.10, respectively.

We will now prove that (a)–(d) are sufficient. By (a), $|V_j|=1$. Let $A=U[V_j, V_i]$, $B=U[V_i, V_j]$. By (c), every path that starts from $B-(V_i \cup V_j)$ and enters into A contains the singleton partite set V_j. This implies that no cycle in D can go through $B-V_i-V_j$ and A more than once.

Assume that D has a cycle C' of length greater than n. By (d), C' is entirely in neither $D\langle B\rangle$ nor $D\langle A\rangle$. Now let P' be the part of C' in $D\langle A\rangle$. Clearly, P' is a path whose first vertex is v_j. Observe that, by the first part of (b) ($U[V_j, V_{i-1}] \to x$), if the terminal vertex of P' is not in V_i, then P' does not contain x. If the terminal vertex of P' is in V_i, then, by (d), the length of P' is less than the number of partite sets in $D\langle A\rangle$. If the terminal vertex of P' is not in V_i, then $P''=P'x$ is a path by (b). By (d), the length of P'' and thus of P' is less than number of partite sets in $D\langle A\rangle$.

Thus, in either case, the length of P' is less than number of partite sets in $D\langle A\rangle$. Analogously, one can prove the corresponding result for $D\langle B\rangle$. The above arguments show that the length of C' is not greater than n, a contradiction. $\square$

Theorem 3.12. *One can check whether a strong n-partite tournament D on p vertices,* $n\geqslant 3$, *has a longest cycle of length n in time* $\mathrm{O}(np^3)$.

Proof. Let $V_1, V_2, \ldots, V_n$ be partite sets of D. One can easily check whether D is an extended tournament in time $\mathrm{O}(p^2)$. If D is an extended tournament, using Theorem 3.3, we can verify whether the length of a longest cycle in D is n in time $\mathrm{O}(p^3)$. So, we may assume that D is not an extended tournament.

The proof of Lemma 3.1 can be easily converted into a recursive procedure that either finds out that D has a cycle of length at least $n+1$ or constructs an n-cycle in D. The total time required by the procedure is at most $\mathrm{O}(p^3)$.

Now we may assume that, in time $\mathrm{O}(p^3)$, we have constructed an n-cycle $C=v_1v_2\ldots v_nv_1$ such that $v_i\in V_i,\ i=1,\ldots,n$, found a pair V_i, V_j with the property $V_i \rightleftharpoons V_j$ and $|V_j|=1$, and chosen a pair $x, y\in V_i$ such that $y\to v_j\to x$. By the previous theorem, it remains to be seen that the conditions (a)–(d) can be checked in time $\mathrm{O}(np^3)$. In fact, the conditions (a)–(c) can be verified in time $\mathrm{O}(p^2)$. To check (d), we can check whether some of the digraphs D_{ij} and D_{ji} are extended tournaments. For all extended tournaments we can use Theorem 3.3. For others, we find special pairs of partite sets and check the conditions (a)–(c) before ‘splitting’ the digraphs into smaller ones to verify (d) for each of them.

Due to $V_{i-1}\to V_i\to V_{i+1}$, each of D_{ij} and D_{ji} has less partite sets than D has and, thus, the number of levels (or parallel ‘splittings’) at which we need to verify the condition (d) is at most $\mathrm{O}(n)$. Prior to checking (d), we will have spent $\mathrm{O}(p^3)$ time, which means the total amount of time required is at most $\mathrm{O}(np^3)$. $\square$

References

[1] R. Balakrishnan, P. Paulraja, Note on the existence of directed $(k+1)$-cycles in disconnected complete k-partite digraphs, J. Graph Theory 8 (1984) 423–426.

[2] J. Bang-Jensen, G. Gutin, Generalizations of tournaments: a survey, J. Graph Theory 28 (1998) 171–202.

[3] J. Bang-Jensen, G. Gutin, Digraphs: Theory, Algorithms and Applications, Springer, London, 2000.

[4] J.A. Bondy, Disconnected orientation and a conjecture of Las Vergnas, J. London Math. Soc. 14 (1976) 277–282.

[5] Y. Guo, Semicomplete multipartite digraphs: a generalization of tournaments, Habilitation thesis, RWTH Aachen, Germany, 1998.

[6] Y. Guo, A. Pinkernell, L. Volkmann, On cycles through a given vertex in multipartite tournaments, Discrete Math. 164 (1997) 165–170.

[7] G. Gutin, On cycles in strong n-partite tournaments, Vestsi Acad. Navuk BSSR Ser. Fiz.-Mat. 5 (1984) 105–106.

[8] G. Gutin, Characterization of vertex pancyclic partly oriented k-partite tournaments, Vestsi Acad. Navuk BSSR Ser.Fiz.-Mat. 2 (1989) 41–46.

[9] G. Gutin, Cycles and paths in semicomplete multipartite digraphs, theorems and algorithms: a survey, J. Graph Theory 19 (1995) 481–505.

[10] L. Volkmann, Longest paths in semicomplete multipartite digraphs, Discrete Math. 199 (1999) 279–284.

[11] L. Volkmann, Cycles in multipartite tournaments: results and problems, Discrete Math. 245 (2002) 19–53.