Characterization of Edge-Colored Complete Graphs with Properly Colored Hamilton Paths

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Abstract: An edge-colored graph H is properly colored if no two adjacent edges of H have the same color. In 1997, J. Bang-Jensen and G. Gutin conjectured that an edge-colored complete graph G has a properly colored Hamilton path if and only if G has a spanning subgraph consisting of a properly colored path C_0 and a (possibly empty) collection of properly colored cycles $C_1, C_2, \ldots, C_d$ such that $V(C_i) \cap V(C_j) = \emptyset$ provided $0 \leq i < j \leq d$. We prove this conjecture. © 2006 Wiley Periodicals, Inc. J Graph Theory 53: 333–346, 2006

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1. INTRODUCTION

Let $G = (V, E)$ be a complete graph, and let $c : E \to \{1, 2, \ldots, \chi\}$ be a fixed (not necessarily proper) edge-coloring of G with χ colors, $\chi \geq 2$. With given c, G is called a *χ-edge-colored* (or, *edge-colored*) complete graph. A subgraph $H \subseteq G$ is called *properly colored* if c defines a proper edge-coloring of H.

The existence of properly colored Hamilton paths and cycles has been studied in several articles; this topic was surveyed in [2] and later in Chapter 11 of [3]. While there are characterizations [6,11] (see also Chapter 11 of [3]) of 2-edge-colored complete graphs with properly colored Hamilton cycles, no such characterization is known for χ-edge-colored complete graphs with $\chi \geq 3$, and it is still an open question to determine the computational complexity of this problem [8].

The most studied *possibly* sufficient condition for an edge-colored complete graph with n vertices to have a properly colored Hamilton cycle is $\Delta_{mon} < \lfloor n/2 \rfloor$, where Δ_{mon} is the maximal number of edges of the same color incident to the same vertex. This was conjectured by B. Bollobás and P. Erdős [9] in 1976, but remains unsolved. The best result so far for 'small' values of n is by J. Shearer [12]: $7\Delta_{mon} < \lfloor n/2 \rfloor$ guarantees the existence of a properly colored Hamilton cycle. The best result so far for large values of n is due to N. Alon and G. Gutin [1]: For every $\epsilon > 0$ and $n = n_\epsilon$ large enough, $\Delta_{mon} \leq (1 - (1/\sqrt{2}) - \epsilon)\lfloor n/2 \rfloor$ implies the existence of a properly colored Hamilton cycle.

For the case of properly colored Hamilton paths, the situation is somewhat different. Let the abbreviation PCHP stand for "properly colored Hamilton path." Let G be an edge-colored graph. A *properly colored cycle factor* of G is a spanning subgraph of G consisting of properly colored cycles $C_1, C_2, \ldots, C_d$ such that $V(C_i) \cap V(C_j) = \emptyset$ provided $1 \leq i < j \leq d$. A *properly colored 1-path-cycle factor* of G is a spanning subgraph of G consisting of a properly colored path C_0 and a (possibly empty) collection of properly colored cycles $C_1, C_2, \ldots, C_d$ such that $V(C_i) \cap V(C_j) = \emptyset$ provided $0 \leq i < j \leq d$.

The following theorem gives a PCHP characterization for the case of just two colors:

Theorem 1.1 [2]. *A 2-edge-colored complete graph G has a PCHP if and only if G contains a properly colored 1-path-cycle factor.*

It is conjectured in [2] that the above theorem holds for any number of colors. We call it the BJG conjecture. In support of the BJG conjecture, the following result was proved in [5]: If a χ-edge-colored complete graph G ($\chi \geq 2$) contains a properly colored cycle factor, then G contains a PCHP.

It is easy to see that the BJG conjecture in [2] can be reduced to the following:

Conjecture 1.2 (PCHP Conjecture). *Let $\chi \geq 3$ and let G be a χ-edge-colored complete graph. Assume that there exist $C, P \subseteq G$, where C is a properly colored cycle and P a properly colored path, such that $V(C) \cap V(P) = \emptyset$ and $V(C) \cup V(P) = V(G)$. Then G contains a PCHP.*

In this article, we prove the PCHP conjecture and, thus, the BJG conjecture. Since it takes polynomial time to check whether an edge-colored graph has a properly colored 1-path-cycle factor [2], our result implies that the PCHP problem is polynomial time solvable for edge-colored complete graphs. The proof of Theorem 2.1 is constructive and can be turned into a polynomial time algorithm for transforming a properly colored 1-path-cycle factor into a properly colored Hamilton path.

This gives, in particular, some indication that the problem of the existence of a properly colored Hamilton cycle in an edge-colored graph may be polynomial time solvable after all. The situation may remind one of that with the existence of Hamilton paths and cycles in semicomplete multipartite digraphs (SMDs) [4] (see also Chapter 5 in [3]). Both Hamilton path and cycle problems for SMDs are polynomial time solvable, but only for the Hamilton path problem we have a nice characterization (see, e.g., [10] or Chapter 5 in [3]) so far.

In passing we mention a simple sufficient condition proved in [7] for the existence of a PCHP in an edge-colored $K_n : K_n$ has no monochromatic triangles.

2. RESULTS

If H is connected, the *distance in H* between two vertices $u, v \in V(H)$ is the length of a shortest path in H from u to v, and we denote it by $\mathrm{dist}_H(u, v)$.

Theorem 2.1. *The PCHP Conjecture holds.*

Proof. Let $C = v_1 \dots v_n v_1$ $(n \geq 3)$ and $P = u_1 \dots u_m$ $(m \geq 1)$. Throughout we will perform addition and subtraction in the indices of the vertices $v_j \in C$ modulo n.

Let $j \in \{1, 2, \dots, n\}$. If $m \geq 2$ and $c(u_1 v_j) \neq c(u_1 u_2)$, then at least one of the paths $u_m u_{m-1} \dots u_1 v_j v_{j+1} \dots v_{j-1}$ and $u_m u_{m-1} \dots u_1 v_j v_{j-1} \dots v_{j+1}$ is a PCHP. Similarly there exists a PCHP if $c(u_m v_j) \neq c(u_{m-1} u_{m-2})$. So we may assume the following:

(1) If $m \geq 2$, then $c(u_1 v_j) = c(u_1 u_2)$ and $c(u_m v_j) = c(u_{m-1} u_m)$ for every $j = 1, 2, \dots, n$.

Thus, to complete the proof of this theorem it suffices to prove the following claim:

Claim A. *If* (1) *is satisfied, then there exists a PCHP H in G with u_1 as its first vertex, such that the initial edge of H is either $u_1 u_2$ or one of the edges $u_1 v_j$ $(1 \leq j \leq n)$, and such that if $m \geq 2$, then u_m is the last vertex of H and the last edge of H is either $u_{m-1} u_m$ or one of the edges $v_j u_m$ $(1 \leq j \leq n)$.*

Let $b(P, C) = 2(n - 3) + m$; we notice that $b(P, C) \geq 1$. Suppose that Claim A is false, and let (G, P, C, c) be a counterexample with a minimal value of $b(P, C)$.

If $m = 1$, then either $u_1 v_1 v_2 \dots v_n$ or $u_1 v_1 v_n \dots v_2$ is a PCHP as desired. Thus, we have established $m \geq 2$ and $b(P, C) \geq 2$.

Now we prove

(2) $m \geq 3$.

Suppose that $m = 2$. With $x = c(u_1u_2)$, we have $c(u_1v_j) = c(u_2v_j) = x$ for all $j = 1, 2, \ldots, n$, by (1). Choose r such that $c(v_{r-1}v_r) \neq x \neq c(v_{r+1}v_{r+2})$; this is possible if r can be chosen with $c(v_rv_{r+1}) = x$, and otherwise it is trivial. But then the path $u_1v_rv_{r-1} \ldots v_{r+2}v_{r+1}u_2$ yields a contradiction.

We continue to prove further properties of the coloring c.

(3) For every i, $1 < i < m$, there exist r and s ($1 \leq r, s \leq n$), such that $c(u_iu_{i+1}) \neq c(u_iv_r)$, $c(u_{i-1}u_i) \neq c(u_iv_s)$.

Otherwise the path $P' = P - u_1 - u_2 - \cdots - u_{i-1}$ satisfies (1) in $G' = G - u_1 - u_2 - \cdots - u_{i-1}$ (with u_i in place of u_1). Since $b(P', C) < b(P, C)$, there exists a PCHP H' in G' that starts from u_i with one of the edges u_iu_{i+1} or u_iv_j, $1 \leq j \leq n$ and finishes in u_m with $u_{m-1}u_m$ or with an edge v_ku_m, $1 \leq k \leq n$. Then $u_1 \ldots u_{i-1}H'$ is a PCHP in G of the desired type having u_1u_2 as its initial edge, a contradiction. A similar argument shows the existence of s.

Suppose that $c(v_{i-1}v_i) = c(v_jv_{j+1}) = x$, where $\mathrm{dist}_C(v_i, v_{j+1}) \geq 3$. If $c(v_iv_j) = x$, then define $G' = G - v_i - v_{i+1} - \cdots - v_j$ and define $c' : E(G') \to \{1, 2, \ldots, \chi\}$ by

$$c'(e) = \begin{cases} x & \text{if } e = v_{i-1}v_{j+1} \\ c(e) & \text{otherwise.} \end{cases}$$

Both P and the cycle $C' = v_{j+1}v_{j+2} \ldots v_nv_1 \ldots v_{i-1}v_{j+1}$ are properly colored by c'. Moreover, P clearly satisfies (1) with respect to C'. Since $b(P, C') < b(P, C)$, there exists a PCHP P_1 in G' with initial edge u_1u_2 or u_1v_r for some $r \in \{j + 1, j + 2, \ldots, n, 1, \ldots, i - 1\}$. If $v_{i-1}v_{j+1} \in P_1$, then we find the desired PCHP H in G by replacing the edge $v_{i-1}v_{j+1}$ by the path $v_{i-1}v_i \ldots v_jv_{j+1}$. Otherwise, if $v_{i-1}v_{j+1} \notin P_1$, then P_1 is properly colored in G and satisfies (1) with respect to $C'' = v_iv_{i+1} \ldots v_jv_i$, which is a properly colored cycle. By $b(P_1, C'') < b(P, C)$, the desired PCHP exists in G, a contradiction.

Thus, we have the following:

(4) Assume that $\mathrm{dist}_C(v_s, v_{t+1}) \geq 3$. If $c(v_{s-1}v_s) = c(v_tv_{t+1}) = x$, then $c(v_sv_t) \neq x$.

Consider the path $u_1 \ldots u_{p-2}u_{p-1}v_{q-1}v_{q-2} \ldots v_{q+1}v_qu_pu_{p+1} \ldots u_m$. As it cannot be a PCHP we conclude the following (Figs. 1 and 2).

(5) Let $2 \leq p \leq m$ and $1 \leq q \leq n$. Then at least one of the following holds.

(a) $p \geq 3$ and $c(u_{p-2}u_{p-1}) = c(u_{p-1}v_{q-1})$
(b) $c(u_{p-1}v_{q-1}) = c(v_{q-2}v_{q-1})$
(c) $c(u_pv_q) = c(v_qv_{q+1})$
(d) $p < m$ and $c(u_pv_q) = c(u_pu_{p+1})$.

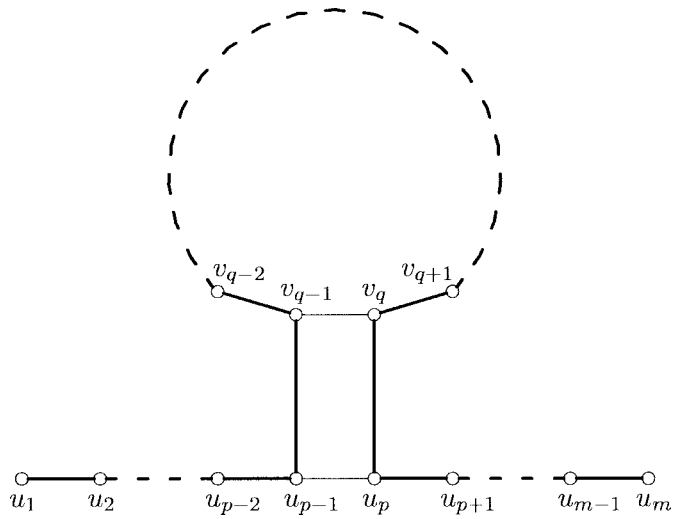

FIGURE 1. The path used to prove (5).

Considering the path $u_1 \ldots u_{p-2}u_{p-1}v_{q+1}v_{q+2} \ldots v_{q-1}v_q u_p u_{p+1} \ldots u_m$ similarly leads to:

(6) Let $2 \leq p \leq m$ and $1 \leq q \leq n$. Then at least one of the following holds.

(a) $p \geq 3$ and $c(u_{p-2}u_{p-1}) = c(u_{p-1}v_{q+1})$
(b) $c(u_{p-1}v_{q+1}) = c(v_{q+1}v_{q+2})$
(c) $c(u_p v_q) = c(v_{q-1}v_q)$
(d) $p < m$ and $c(u_p v_q) = c(u_p u_{p+1})$.

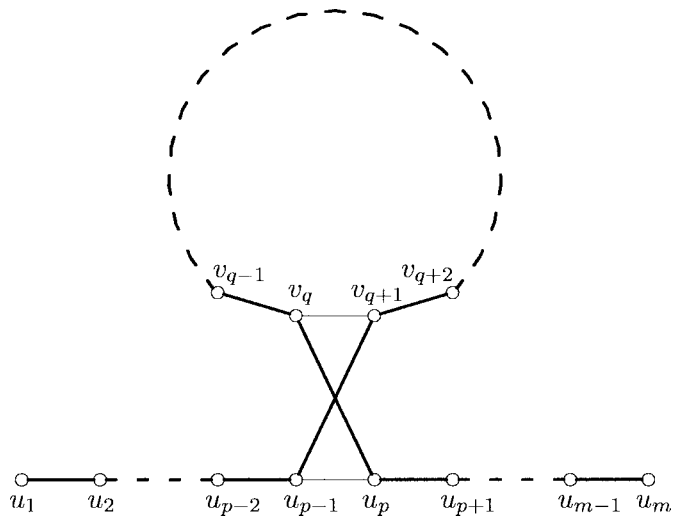

FIGURE 2. The path used to prove (6).

In several of the following applications of (5) and (6), it will be useful to note that (5c) and (6c) are mutually exclusive statements for any values of p and q, since C is properly colored, and that (5d) and (6d) are identical statements.

For the remaining part of the article, we define $x = c(u_1u_2)$ and $y = c(u_{m-1}u_m)$.

(7) Assume $c(u_2v_j) = z \neq c(u_2u_3)$ for some $j \in \{1, 2, \ldots, n\}$.

(a) If $c(v_jv_{j+1}) \neq z$ then $c(v_{j-2}v_{j-1}) = x$, and if $c(v_{j-1}v_j) \neq z$ then $c(v_{j+1}v_{j+2}) = x$.
(b) If $z = x$ or $c(v_{j-1}v_j) \neq z \neq c(v_jv_{j+1})$, then $c(v_{j-2}v_{j-1}) = c(v_{j+1}v_{j+2}) = c(v_{j-1}v_{j+1}) = x$ and $n \in \{3, 5\}$.

If $c(v_{j-1}v_j) \neq z$, then $c(v_{j+1}v_{j+2}) = x$ follows from (6) with $p = 2$ and $q = j$ (only (b) of (6) is not necessarily false). If $c(v_jv_{j+1}) \neq z$, then $c(v_{j-2}v_{j-1}) = x$ similarly follows from (5). This shows (a).

Now assume $c(v_{j-1}v_j) \neq z \neq c(v_jv_{j+1})$ or $z = x$. In the case $z = x$ the fact that c is a proper coloring of C together with (a) implies $c(v_{j-2}v_{j-1}) = c(v_{j+1}v_{j+2}) = x$. The same conclusion follows directly from (a) when $c(v_{j-1}v_j) \neq z \neq c(v_jv_{j+1})$. By symmetry, and since $c(v_{j-1}v_j) \neq c(v_jv_{j+1})$, we may assume $c(v_{j-1}v_{j+1}) \neq c(v_jv_{j+1})$. Since the Hamilton path $u_1v_{j+2}v_{j+3}\ldots v_{j-2}v_{j-1}v_{j+1}v_ju_2\ldots u_m$ fails to be a PCHP, it follows that $c(v_{j-1}v_{j+1}) = x$.

Consider (4) for $s = j - 1$ and $t = j + 1$. The conclusion $c(v_sv_t) \neq x$ of (4) implies $\mathrm{dist}_C(v_{j-1}, v_{j+2}) \leq 2$, which is only possible for $n \leq 5$. Moreover, the edges $v_{j-2}v_{j-1}$ and $v_{j+1}v_{j+2}$, both of color x, are not adjacent on C, which implies $n \neq 4$. Thus (b) is proved.

(8) Assume $c(u_{m-1}v_k) = z \neq c(u_{m-2}u_{m-1})$ for some $k \in \{1, 2, \ldots, n\}$.

(a) If $c(v_{k-1}v_k) \neq z$ then $c(v_{k+1}v_{k+2}) = y$, and if $c(v_kv_{k+1}) \neq z$ then $c(v_{k-2}v_{k-1}) = y$.
(b) If $z = y$ or $c(v_{k-1}v_k) \neq z \neq c(v_kv_{k+1})$, then $c(v_{k-2}v_{k-1}) = c(v_{k+1}v_{k+2}) = c(v_{k-1}v_{k+1}) = y$ and $n \in \{3, 5\}$.

The proof of (8) is similar to that of (7).

(9) Assume $c(u_2v_j) \neq x$ for all $j = 1, 2, \ldots, n$. Then there exists $j \in \{1, 2, \ldots, n\}$ such that

(a) $c(u_2v_{j-1}) = c(u_2v_j) \neq c(u_2u_3)$, and
(b) $c(v_{j-2}v_{j-1}) = c(v_jv_{j+1}) = x$.

By (3) there exists $j \in \{1, 2, \ldots, n\}$ such that $c(u_2v_j) = z \neq c(u_2u_3)$. We may assume $c(v_jv_{j+1}) \neq z$ (if not, then $c(v_{j-1}v_j) \neq z$ holds, and we may renumber the vertices on C so that $v_{j+\ell}$ becomes $v_{j-\ell}$ for all $\ell = 0, 1, 2, \ldots$ without change of the conclusion). Then (7a) implies $c(v_{j-2}v_{j-1}) = x$.

Again by (3) there exists $r \in \{1, 2, \ldots, n\}$ such that if $m \geq 4$, then $c(u_3v_r) \neq c(u_3u_4)$ holds.

Suppose $c(u_2v_{j-1}) \neq z$. Since $u_1v_{r+1}v_{r+2}\ldots v_{j-2}v_{j-1}u_2v_jv_{j+1}\ldots v_{r-1}v_ru_3u_4 \ldots u_m$ is not a PCHP, at least one of the following holds:

(i) $c(u_1v_{r+1}) = c(v_{r+1}v_{r+2})$,
(ii) $c(u_3v_r) = c(v_{r-1}v_r)$,
(iii) $r = j$ and $c(u_2v_r) = c(u_3v_r)$.

Since $u_1v_{r-1}v_{r-2}\ldots v_{j+1}v_ju_2v_{j-1}v_{j-2}\ldots v_{r+1}v_ru_3u_4\ldots u_m$ is not a PCHP, at least one of the following holds:

(iv) $c(u_1v_{r-1}) = c(v_{r-2}v_{r-1})$,
(v) $c(u_3v_r) = c(v_rv_{r+1})$,
(vi) $r = j - 1$ and $c(u_2v_r) = c(u_3v_r)$.

Let $p = 3$ and $q = r$, and observe that neither of (5a), (5d), (6a), or (6d) holds.

We will now show that (iii) and (vi) do not hold. If $r = j$, then $c(v_{r-2}v_{r-1}) = x \neq c(u_2v_{r-1})$, hence also (5b) does not hold, and (5c) must be satisfied, that is, $c(u_3v_r) = c(v_rv_{r+1})$. In particular, if $r = j$, then $c(u_3v_r) = c(v_jv_{j+1}) \neq z = c(u_2v_j)$, contrary to (iii).

If $r = j - 1$, then $c(u_2v_{r+1}) = c(u_2v_j) = z$ and $c(v_{r+1}v_{r+2}) = c(v_jv_{j+1}) \neq z$, so (6b) does not hold. Then (6c) implies $c(u_3v_r) = c(v_{r-1}v_r) = c(v_{j-2}v_{j-1}) = x$. In particular, if $r = j - 1$, then $c(u_2v_r) \neq c(u_3v_r)$ follows, hence also (vi) does not hold.

We deduce that (i) or (ii) is true, and that (iv) or (v) is true. Now (6c) is equivalent to (ii), and (6b) and (i) both are not true (by (1) and our assumption), therefore (ii) holds. Similarly (5c) is equivalent to (v) and (5b) contradicts (iv), so also (v) holds. But (ii) contradicts (v), since C is properly colored. This establishes $c(u_2v_{j-1}) = z$.

Finally, $c(v_jv_{j+1}) = x$ follows from (7a).

(10) Assume $c(u_{m-1}v_j) \neq y$ for all $j = 1, 2, \ldots, n$. Then there exists $k \in \{1, 2, \ldots, n\}$ such that

(a) $c(u_{m-1}v_{k-1}) = c(u_{m-1}v_k) \neq c(u_{m-2}u_{m-1})$, and
(b) $c(v_{k-2}v_{k-1}) = c(v_kv_{k+1}) = y$.

The proof is similar to the proof of (9).

(11) Assume $z = c(u_2v_j) = c(u_2v_{j-1}) \notin \{x, c(u_2u_3)\}$ for some $j \in \{1, 2, \ldots, n\}$. Furthermore assume $c(v_jv_{j+1}) \neq z$. Then one of the following holds:

(a) n is an even number, and $n/2$ edges of C have color x.
(b) If $c(v_{j'}v_{j'+1}) = x$, then $v_{j'} \in S = \{v_{j-4}, v_{j-2}, v_j, v_{j+2}\}$.

First $c(v_{j-2}v_{j-1}) = c(v_jv_{j+1}) = x$ follows from two applications of (7a), applying (7) to u_2v_j and u_2v_{j-1} in this order. Since C is properly colored, we conclude that $c(v_{j'}v_{j'+1}) = x$ implies $v_{j'} \notin \{v_{j-3}, v_{j-1}, v_{j+1}\}$. Assume that (b) does not hold, and choose any $v_{j'} \in V(C) \setminus S$ satisfying $c(v_{j'}v_{j'+1}) = x$. Then, $n \geq 8$ follows from $v_{j'} \notin S \cup \{v_{j-3}, v_{j-1}, v_{j+1}\}$. We will prove the following statement.

$(*)$ $c(v_{j'-2}v_{j'-1}) = c(v_{j'+2}v_{j'+3}) = x$.

First we suppose $c(v_{j'-2}v_{j'-1}) \neq x$. Then, $v_{j'} \notin S \cup \{v_{j-3}, v_{j-1}, v_{j+1}\}$ and $n \geq 8$ imply $\mathrm{dist}_C(v_{j-1}, v_{j'+1}) \geq 3$. By (4) with $s = j-1$ and $t = j'$, we have $c(v_{j-1}v_{j'}) \neq x$. We consider the path $P_1 = u_1v_{j'-1}v_{j'-2}\dots v_{j+1}v_ju_2u_3\dots u_m$ and the cycle $C_1 = v_{j'}v_{j'+1}\dots v_{j-2}v_{j-1}v_{j'}$, which are properly colored and satisfy (1). Since $b(C_1, P_1) < b(C, P)$ holds, our minimality assumption yields a PCHP as in Claim A, which is a contradiction. So $c(v_{j'-2}v_{j'-1}) = x$ holds. Now suppose $c(v_{j'+2}v_{j'+3}) \neq x$. Then by (4) with $s = j$ and $t = j'+1$ we similarly have $c(v_jv_{j'+1}) \neq x$, and we consider the path $P_2 = u_1v_{j'+2}v_{j'+3}\dots v_{j-2}v_{j-1}u_2u_3\dots u_m$ and the cycle $C_2 = v_{j'+1}v_{j'}\dots v_{j+1}v_jv_{j'+1}$ instead, again with a contradiction. Thus we have also $c(v_{j'+2}v_{j'+3}) = x$, which finishes the proof of $(*)$.

Applying $(*)$ recursively, it follows that $c(v_{j'+2\ell}v_{j'+2\ell+1}) = x$ holds for every $\ell \in \mathbb{N}$ with $v_{j'+2\ell} \notin S \cup \{v_{j-3}, v_{j-1}, v_{j+1}\}$. In particular, either $c(v_{j+3}v_{j+4}) = x$ or $c(v_{j+4}v_{j+5}) = x$ holds ($v_{j+3} \notin S$ follows from $n \geq 8$, and $v_{j+4} \in S$ only occurs if $n = 8$ and $v_{j'} = v_{j+3}$, in which case $c(v_{j+3}v_{j+4}) = x$ follows). However, applying $(*)$, $c(v_{j+3}v_{j+4}) = x$ would imply $c(v_{j+1}v_{j+2}) = x$, contradicting the fact that C is properly colored. So $c(v_{j+4}v_{j+5}) = x$ holds. Similar reasoning leads to $c(v_{j-6}v_{j-5}) = x$. It follows from $(*)$ that all of $v_{j+4}v_{j+5}, v_{j+6}v_{j+7}, \dots, v_{j-6}v_{j-5}$ are colored x.

Since both $v_{j+4}v_{j+5}$ and $v_{j-6}v_{j-5}$ are colored x, it follows from $(*)$ that $v_{j+2}v_{j+3}$ and $v_{j-4}v_{j-3}$ are also colored x. Combining this with the fact that v_jv_{j+1} and $v_{j-2}v_{j-1}$ are both colored x, we have shown that (a) is true, which proves (11).

(12) There is an index j, $1 \leq j \leq n$, such that $c(u_2v_j) = x$ or $c(u_{m-1}v_j) = y$.

Suppose $c(u_2v_j) \neq x$ and $c(u_{m-1}v_j) \neq y$ for all $j = 1, 2, \dots, n$. By (9) and (10) we may choose $j, k \in \{1, 2, \dots, n\}$ such that j satisfies (9a) and (9b), and k satisfies (10a) and (10b).

By (9b) we have $c(v_{j-2}v_{j-1}) = c(v_jv_{j+1}) = x$, from which $m \geq 4$ follows, as otherwise $u_1u_2v_jv_{j+1}\dots v_{j-2}v_{j-1}u_3$ would be a PCHP. Now the path

$$u_1v_{k-1}v_{k-2}\dots v_{j+1}v_ju_2u_3\dots u_{m-2}u_{m-1}v_kv_{k+1}\dots v_{j-2}v_{j-1}u_m$$

is not a PCHP, so $y = x$ follows.

We will show that (11a) holds. So suppose not; then it follows from (11) that (11b) holds. By (10b) we have $c(v_{k-2}v_{k-1}) = c(v_kv_{k+1}) = x$, which by (11b) implies $v_k \in \{v_{j-2}, v_j, v_{j+2}\}$.

The case $v_k = v_j$ would lead to a contradiction, since the path

$$u_1 v_{j-2} v_{j-3} \dots v_{j+1} v_j u_2 u_3 \dots u_{m-2} u_{m-1} v_{j-1} u_m$$

would be a PCHP.

Suppose $v_k = v_{j-2}$. By (10b) we then have $c(v_{j-4}v_{j-3}) = x$. Then $n \notin \{3, 5\}$ follows from the fact that C is properly colored, and $n \notin \{4, 6\}$ holds since (11a) is not satisfied, so we deduce $n \geq 7$. Applying (4) to $s = j - 3$ and $t = j$ we have $c(v_{j-3}v_j) \neq x$. However, the path $u_1 v_{j-1} u_2 u_3 \dots u_{m-2} u_{m-1} v_{j-2} u_m$ and the cycle $v_j v_{j+1} \dots v_{j-4} v_{j-3} v_j$ are properly colored and satisfy (1), which contradicts our minimality assumption.

For $v_k = v_{j+2}$ we similarly conclude by (10b) and (4), with $s = j - 1$ and $t = j + 2$, that $c(v_{j-1}v_{j+2}) \neq x$. Examination of the path $u_1 v_j u_2 u_3 \dots u_{m-2} u_{m-1} v_{j+1} u_m$ and the cycle $v_{j-1} v_{j+2} v_{j+3} \dots v_{j-2} v_{j-1}$ again leads to contradiction, which shows that (11a) does hold.

We have that n is an even number, and the edges of C are alternately colored x. Let $r \in \{1, 2, \dots, n\}$ be chosen so that $c(u_3 v_r) \neq c(u_3 u_4)$; this is possible by (3). We apply (5) and (6) with $p = 3$ and $q = r$. Then (5d) and (6d) both fail by the choice of r. Neither of the edges $u_2 v_{r-1}$ and $u_2 v_{r+1}$ have color x, due to our assumption, hence (5a) and (6a) both fail. For the same reason one of (5b) and (6b) fails, because either $v_{r-2}v_{r-1}$ or $v_{r+1}v_{r+2}$ is colored x. If $v_{r-2}v_{r-1}$ is colored x, then (5b) fails, so (5c) holds and gives $c(u_3 v_r) = c(v_r v_{r+1})$, and now

$$u_1 u_2 v_k v_{k+1} \dots v_{r-1} v_r u_3 u_4 \dots u_{m-2} u_{m-1} v_{k-1} v_{k-2} \dots v_{r+2} v_{r+1} u_m$$

is a PCHP, a contradiction. If $v_{r+1}v_{r+2}$ is colored x, then (6b) fails, and $c(u_3 v_r) = c(v_{r-1} v_r)$ follows similarly. But

$$u_1 u_2 v_{k-1} v_{k-2} \dots v_{r+1} v_r u_3 u_4 \dots u_{m-2} u_{m-1} v_k v_{k+1} \dots v_{r-2} v_{r-1} u_m$$

is a PCHP, again with contradiction. This finishes the proof of (12).

(13) Assume $c(u_2 v_j) = x$ for some $j \in \{1, 2, \dots, n\}$, and let $w = c(v_{j-1}v_j)$ and $z = c(v_j v_{j+1})$. Then

(a) $c(v_{j-2}v_{j-1}) = c(v_{j+1}v_{j+2}) = c(v_{j-1}v_{j+1}) = x$ and $n \in \{3, 5\}$
(b) $c(u_2 v_k) \neq x$ for all $v_k \neq v_j$
(c) $c(u_2 v_{j-1}) = c(u_2 v_{j+1}) = c(u_2 u_3) \in \{w, z\}$
(d) $m \geq 4$
(e) if $c(u_2 u_3) = w$, then $c(u_3 v_k) = c(u_3 u_4) \neq c(u_3 v_{j-1}) = w$ for all $v_k \neq v_{j-1}$.
(f) if $c(u_2 u_3) = z$, then $c(u_3 v_k) = c(u_3 u_4) \neq c(u_3 v_{j+1}) = z$ for all $v_k \neq v_{j+1}$.

We may assume $j = 1$, so that $c(u_2 v_1) = x$, $c(v_n v_1) = w$, and $c(v_1 v_2) = z$. Then $c(u_2 v_1) = x \neq c(u_2 u_3)$ and (7b) directly imply (a). Since $n \in \{3, 5\}$ and C is properly colored, $c(u_2 v_k) = x$ now implies $v_k = v_1$ by (a) and (7b), so (b) holds.

Now $c(u_2v_2) = c(u_2u_3)$ follows from (5) and (6) with $p = 2$ and $q = 2$, since (5a), (5b), (6a), (6b) all fail, and (5c) contradicts (6c), so that (5d) and the equivalent (6d) hold. Similarly we deduce $c(u_2v_n) = c(u_2u_3)$ from (5) and (6) with $p = 2$ and $q = n$.

Now (d) and $c(u_3v_1) = c(u_3u_4)$ follow from (5) and (6) with $p = 3$ and $q = 1$, as (5d) or (6d) is again satisfied. Moreover for $n = 5$ the identity $c(u_3v_3) = c(u_3u_4)$ follows, using $c(v_2v_3) \neq c(v_3v_4)$, from the fact that neither $u_1u_2v_4v_5v_1v_2v_3u_3u_4 \ldots u_m$ nor $u_1v_2v_1u_2v_5v_4v_3u_3u_4 \ldots u_m$ is a PCHP. Similarly we have $c(u_3v_4) = c(u_3u_4)$ when $n = 5$.

We deduce $c(u_3v_n) \in \{w, c(u_3u_4)\}$, since $u_1v_{n-1}v_{n-2} \ldots v_3v_2u_2v_1v_nu_3u_4 \ldots u_m$ is not a PCHP, Moreover $c(u_3v_n) \in \{c(u_2u_3), c(u_3u_4)\}$ follows, using $c(u_2v_n) = c(u_2u_3)$, by examining the path $u_1v_{n-1}v_{n-2} \ldots v_2v_1u_2v_nu_3u_4 \ldots u_m$, and we have $c(u_3v_n) \in \{w, c(u_3u_4)\} \cap \{c(u_2u_3), c(u_3u_4)\}$. A similar argument shows $c(u_3v_2) \in \{z, c(u_3u_4)\} \cap \{c(u_2u_3), c(u_3u_4)\}$. At least one of $c(u_3v_n)$ and $c(u_3v_2)$ is not equal to $c(u_3u_4)$, by (3), so it follows that either $c(u_3v_n) = c(u_2u_3) = w$ or $c(u_3v_2) = c(u_2u_3) = z$ holds. This shows the remaining parts of (c), (e), and (f), and (13) is proved.

(14) Assume $c(u_{m-1}v_j) = y$ for some $j \in \{1, 2, \ldots, n\}$, and let $w' = c(v_{j-1}v_j)$ and $z' = c(v_jv_{j+1})$. Then

(a) $c(v_{j-2}v_{j-1}) = c(v_{j+1}v_{j+2}) = c(v_{j-1}v_{j+1}) = y$ and $n \in \{3, 5\}$
(b) $c(u_{m-1}v_k) \neq y$ for all $v_k \neq v_j$
(c) $c(u_{m-1}v_{j-1}) = c(u_{m-1}v_{j+1}) = c(u_{m-2}u_{m-1}) \in \{w', z'\}$
(d) $m \geq 4$
(e) if $c(u_{m-2}u_{m-1}) = w'$, then $c(u_{m-2}v_k) = c(u_{m-3}u_{m-2}) \neq c(u_{m-2}v_{j-1}) = w'$ for all $v_k \neq v_{j-1}$.
(f) if $c(u_{m-2}u_{m-1}) = z'$, then $c(u_{m-2}v_k) = c(u_{m-3}u_{m-2}) \neq c(u_{m-2}v_{j+1}) = z'$ for all $v_k \neq v_{j+1}$.

(14) is proved similarly to (13).

By (12) we may assume $c(u_2v_1) = x$ without loss of generality. Let $w = c(v_nv_1)$ and $z = c(v_1v_2)$. We will further assume $c(u_2v_n) = c(u_2v_2) = c(u_2u_3) = w$, which is admissible by (13c) without loss of generality. Then, $m \geq 4$ holds by (13d) and

$$c(u_3v_k) = c(u_3u_4) \neq c(u_3v_n) = w \text{ for all } v_k \neq v_n$$

by (13e). These facts will be used frequently throughout the remaining part of the proof.

(15) $n = 3$.

Suppose $n \neq 3$; from (13a) it follows that n is equal to 5. Further we then have $c(v_4v_5) = c(v_2v_3) = c(v_2v_5) = x$ by (13a), and $c(u_2v_k) \neq x$ for $v_k \in \{v_3, v_4\}$ by (13b).

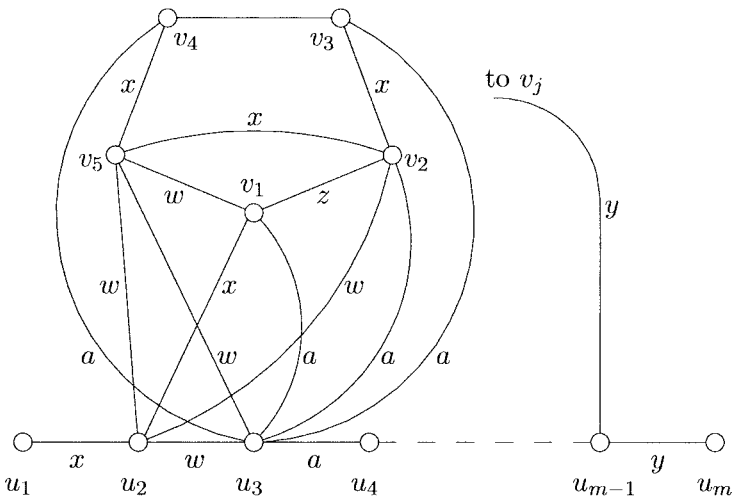

FIGURE 3. The situation in the proof of (15), where $a = c(u_3u_4)$.

We will first show that there exists $j \in \{1, 2, \ldots, 5\}$ with $c(u_{m-1}v_j) = y$. So suppose not. Then by (10) there is a k such that $c(u_{m-1}v_{k-1}) = c(v_{m-1}v_k) \neq c(u_{m-2}u_{m-1})$ and $c(v_{k-2}v_{k-1}) = c(v_kv_{k+1}) = y$. The latter implies $k \notin \{1, 2\}$, using $c(v_4v_5) \neq c(v_1v_2)$ and $c(v_5v_1) \neq c(v_2v_3)$. For $k \in \{3, 4\}$ the path $u_1v_4u_2v_1v_2v_5u_3u_4 \ldots u_{m-2}u_{m-1}v_3u_m$ is a PCHP, and for $k = 5$ the path $u_1v_3u_2v_1v_2v_5u_3u_4 \ldots u_{m-2}u_{m-1}v_4u_m$ is a PCHP, giving a contradiction in each case. Thus we have shown that j exists as desired. Figure 3 summarizes what has been shown so far about the colors of various edges.

Now (14a) implies $y \in \{x, z, w\}$, and the value of j is uniquely determined, by (14b). Moreover, (14c) implies $c(u_{m-2}u_{m-1}) \in \{x, z, w\} \setminus \{y\}$.

Case 1. $y = x$. In this case $j = 1$ and $c(u_{m-1}v_1) = x \neq c(u_{m-2}u_{m-1})$. However, $u_1u_2v_2v_1u_{m-1}u_{m-2} \ldots u_4u_3v_5v_4v_3u_m$ is a PCHP, a contradiction.

Case 2. $y = z$. In this case $j = 5$, $c(u_{m-1}v_5) = z$, and $c(u_{m-2}u_{m-1}) \in \{x, w\}$. Supposing that $c(u_{m-2}u_{m-1}) = x$ holds, the path $u_1u_2 \ldots u_{m-3}u_{m-2}v_4v_3u_{m-1}u_m$ and the cycle $v_1v_5v_2v_1$ would both be properly colored, contradicting our minimality assumption. So we have $c(u_{m-2}u_{m-1}) = w$.

By (13e) we have $c(u_3v_1) = c(u_3u_4)$, and $c(u_{m-1}v_1) = c(u_{m-2}u_{m-1})$ follows from (14c), so we deduce $m \neq 4$. Since $c(u_2u_3) = w = c(u_{m-2}u_{m-1})$ it is clear that $m \neq 5$ holds, hence $m \geq 6$. By (14e) we have $c(u_{m-2}v_1) = w$. Now

$$u_1u_2v_2v_1u_{m-2}u_{m-3} \ldots u_4u_3v_5v_4v_3u_{m-1}u_m$$

is a PCHP, a contradiction.

Case 3. $y = w$. In this case $j = 2$ and $c(u_{m-1}v_2) = w$. We note by (14a) that $c(v_1v_3) = w$ holds. However $u_1u_2v_4v_5u_3u_4 \ldots u_m$ and $v_1v_2v_3v_1$ are now properly

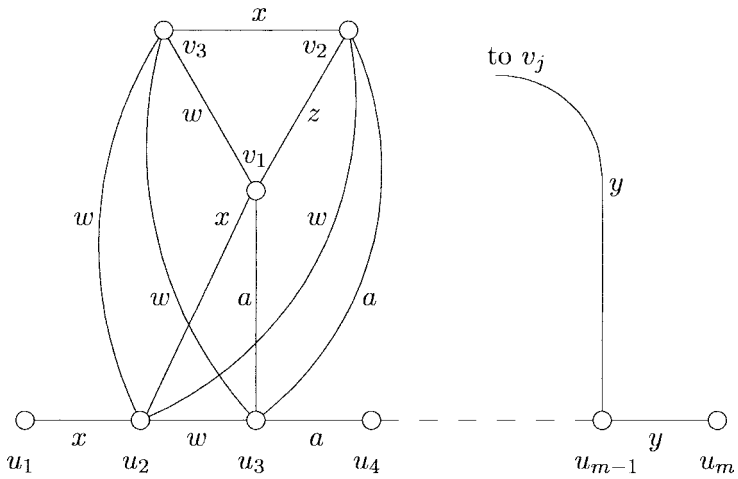

FIGURE 4. The situation in the final steps of the proof of Claim A, where $a = c(u_3u_4)$.

colored, again contradicting our minimality assumption. This finishes the last case and (15) is proved.

We choose $j \in \{1, 2, 3\}$ with $c(u_{m-1}v_j) = y$, which is possible since (10b) fails for $n = 3$, implying that the assumption of (10) does not hold. Then j satisfies the assumption of (8), hence $c(v_{j+1}v_{j+2}) = y$ follows from (8b). We deduce $y \in \{x, z, w\}$ and proceed to divide into the three respective cases (see also Fig. 4).

Case 1. $y = x$. Then $j = 1$ follows from $c(v_2v_3) = x$, and the choice of j implies $c(u_{m-1}v_1) = x$. But now $u_1u_2v_2v_1u_{m-1}u_{m-2}\ldots u_4u_3v_3u_m$ is a PCHP, a contradiction.

Case 2. $y = z$. In this case $j = 3$ follows from $c(v_1v_2) = y$, implying $c(u_{m-1}v_3) = z$. By (13e) $c(u_3v_1) = c(u_3v_2) = c(u_3u_4) \neq c(u_3v_3) = w$. For $m = 4$ all of (6a,b,c,d) would fail for $p = 4$ and $q = 1$ (in particular (6b) fails by $c(u_3v_2) = c(u_3u_4) = y \neq x = c(v_2v_3)$), so we deduce $m \geq 5$.

Suppose $c(u_4v_1) \neq c(u_4u_5)$. Then $c(u_3v_1) = c(u_4v_1)$ holds, as otherwise the path $u_1u_2v_2v_3u_3v_1u_4u_5\ldots u_m$ would be a PCHP. Now (6) with $p = 4$ and $q = 1$ implies $c(u_3v_2) = x$ or $c(u_4v_1) = w$ (i.e., (6b) or (6c)). We have $c(u_4v_1) = c(u_3v_1) = c(u_3u_4) \neq c(u_2u_3) = w$, so $c(u_3v_2) = x$ follows. But then $u_1u_2v_2u_3v_3u_{m-1}u_{m-2}\ldots u_5u_4v_1u_m$ is a PCHP, which is a contradiction. We conclude that $c(u_4v_1) = c(u_4u_5)$ holds.

Suppose $c(u_4v_2) \neq c(u_4u_5)$. Then (5c) for $p = 4$ and $q = 2$ holds, that is, $c(u_4v_2) = x$ and $u_1u_2u_3v_1v_3u_{m-1}u_{m-2}\ldots u_5u_4v_2u_m$ is a PCHP, a contradiction. Hence $c(u_4v_2) = c(u_4u_5)$ follows. Now by (3) we have $c(u_4v_3) \neq c(u_4u_5)$. With $p = 4$ and $q = 3$ either (5b) or (5c) holds, hence $c(u_3v_2) = z$ or $c(u_4v_3) = w$. Examination of the path $u_1v_2u_2v_1u_3v_3u_4u_5\ldots u_m$ allows us to deduce $c(u_3v_1) = x$ or

$c(u_4v_3) = w$. Then $c(u_4v_3) = w$ follows, since the two alternatives conflict due to $c(u_3v_1) = c(u_3v_2)$.

Let i be the largest number such that $i \leq m$ and such that $c(u_{i'}v_1) = c(u_{i'}v_2) = c(u_{i'}u_{i'+1}) \neq c(u_{i'}v_3) = w$ holds for every $i' = 3, 4, \ldots, i-1$. We note that i exists and satisfies $i \geq 5$. It is useful to observe that (5a) and (6a) fail for $p = i$ and every $q = 1, 2, 3$; for (5a) this follows from $c(u_{i-2}u_{i-1}) \neq w$ for $q = 1$, and it follows from $c(u_{i-1}v_{q-1}) = c(u_{i-1}u_i) \neq c(u_{i-2}u_{i-1})$ for $q \neq 1$. Similarly for (6a).

For $i = m$ all of (6a,b,c,d) fail for $p = i$ and $q = 1$ (in particular (6b) fails by $c(u_{i-1}v_2) = c(u_{i-1}u_i) = y \neq x = c(v_2v_3)$), so we deduce $i < m$.

Suppose $c(u_iv_1) \neq c(u_iu_{i+1})$. For $p = i$ and $q = 1$ (5) and (6) imply $c(u_iv_1) = z$ and $c(u_{i-1}v_2) = x$ (respectively (5c) and (6b)). Using $c(u_{i-1}v_1) = c(u_{i-1}v_2) = x$ and $c(u_{i-2}v_2) = c(u_{i-2}u_{i-1}) \neq c(u_{i-1}u_i) = c(u_{i-1}v_2) = x$, the path

$$u_1 \ldots u_{i-3}u_{i-2}v_2v_3u_{i-1}v_1u_iu_{i+1} \ldots u_m$$

is a PCHP. This contradiction shows $c(u_iv_1) = c(u_iu_{i+1})$. Suppose $c(u_iv_2) \neq c(u_iu_{i+1})$. Then (5) with $p = i$ and $q = 2$ implies $c(u_iv_2) = x$, but

$$u_1u_2 \ldots u_{i-2}u_{i-1}v_1v_3u_{m-1}u_{m-2} \ldots u_{i+1}u_iv_2u_m$$

is a PCHP, a contradiction. Therefore $c(u_iv_2) = c(u_iu_{i+1})$. By (3) we have $c(u_iv_3) \neq c(u_iu_{i+1})$. Then $c(u_iv_3) \neq w$ follows from the choice of i, and

$$u_1v_2v_1u_2u_3 \ldots u_{i-2}u_{i-1}v_3u_iu_{i+1} \ldots u_m$$

is a PCHP, a contradiction.

Case 3. $y = w$. This implies $j = 2$ and $c(u_{m-1}v_2) = w$. By (14c) we have $c(u_{m-2}u_{m-1}) \in \{x, z\}$. The case $c(u_{m-2}u_{m-1}) = z$ is symmetric to Case 2, so only the case $c(u_{m-2}u_{m-1}) = x$ remains. Then $m \geq 5$ follows, as P is properly colored. We can assume $c(u_2u_{m-1}) \neq x$ without loss of generality. But $u_1u_2u_{m-1}u_{m-2} \ldots u_4u_3v_3v_2v_1u_m$ is a PCHP, with contradiction. This finishes the proof of Case 3, and of Claim A. ■

REFERENCES

[1] N. Alon and G. Gutin, Properly colored Hamilton cycles in edge colored complete graphs, Random Struct Algorithms 11 (1997), 179–186.

[2] J. Bang-Jensen and G. Gutin, Alternating cycles and paths in edge-coloured multigraphs: A survey, Discrete Math 165/166 (1997), 39–60.

[3] J. Bang-Jensen and G. Gutin, Digraphs: Theory, Algorithms and Applications, Springer-Verlag, London, 2000.

[4] J. Bang-Jensen, G. Gutin, and A. Yeo, A polynomial algorithm for the Hamiltonian cycle problem in semicomplete multipartite digraphs, J Graph Theory 29 (1998), 111–132.

[5] J. Bang-Jensen, G. Gutin, and A. Yeo, Properly coloured Hamiltonian paths in edge-coloured complete graphs, Discrete Appl Math 82 (1998), 247–250.

[6] M. Bankfalvi and Zs. Bankfalvi, Alternating hamiltonian circuit in two-coloured complete graphs, (Proc. Colloq. Tihany 1968), Academic Press, New York, 1968, 11–18.

[7] O. Barr, Properly coloured Hamiltonian paths in edge-coloured complete graphs without monochromatic triangles, Ars Combinatoria 50 (1998), 316–318.

[8] A. Benkouar, Y. Manoussakis, V. Paschos, and R. Saad, On the complexity of finding alternating Hamiltonian and Eulerian cycles in edge-coloured graphs, Lecture Notes in Comput Sci 557 (Springer, Berlin, 1991), 190–198.

[9] B. Bollobás and P. Erdős, Alternating Hamiltonian cycles, Israel J Math 23 (1976), 126–131.

[10] G. Gutin, Finding a longest path in a complete multipartite digraph, SIAM J Discrete Math 6 (1993), 270–273.

[11] R. Saad, Finding a longest alternating cycle in a 2-edge-coloured complete graph is in RP, Combin Probab Comp 5 (1996), 297–306.

[12] J. Shearer, A property of the colored complete graph, Discrete Math 25 (1979), 175–178.