Min orderings and list homomorphism dichotomies for signed and unsigned graphs*

Jan Bok[1,6][0000−0002−7973−1361], Richard C. Brewster[2][0000−0001−7237−4288], Pavol Hell[3][0000−0001−7609−9746], Nikola Jedličková[4][0000−0001−9518−6386], and Arash Rafiey[5]

[1] Computer Science Institute, Faculty of Mathematics and Physics, Charles University, Prague, Czech Republic, bok@iuuk.mff.cuni.cz
[2] Department of Mathematics and Statistics, Thompson Rivers University, Canada, rbrewster@tru.ca
[3] School of Computing Science, Simon Fraser University, Canada, pavol@sfu.ca
[4] Department of Applied Mathematics, Faculty of Mathematics and Physics, Charles University, Prague, Czech Republic, jedlickova@kam.mff.cuni.cz
[5] Mathematics and Computer Science, Indiana State University, Terre Haute, Indiana, USA, arash.rafiey@indstate.edu
[6] Université Clermont Auvergne, CNRS, Clermont Auvergne INP, Mines Saint-Étienne, LIMOS, 63000 Clermont-Ferrand, France

Abstract. Since the CSP dichotomy conjecture has been established, a number of other dichotomy questions have attracted interest, including one for list homomorphism problems of signed graphs. Signed graphs arise naturally in many contexts, including for instance nowhere-zero flows for graphs embedded in non-orientable surfaces. The dichotomy classification is known for homomorphisms without list restrictions, so it is surprising that it is not known, or even conjectured, if lists are present since this usually makes the classifications easier to obtain.

There is however a conjectured classification, due to Kim and Siggers, in the special case of "semi-balanced" signed graphs. These authors confirmed their conjecture for the class of reflexive signed graphs. As our main result we verify the conjecture for irreflexive signed graphs. For this purpose we prove an extension theorem for certain unsigned bipartite graphs of independent interest. These graphs are known as two-directional ray graphs, but they are also exactly the bipartite graphs that are the complements of circular arc graphs, and are exactly the containment interval bigraphs. Moreover, we offer an alternative proof for the class of reflexive signed graphs, and a direct polynomial time algorithm in the polynomial cases where the previous algorithms used algebraic methods of general CSP dichotomy theorems.

For both reflexive and irreflexive cases the dichotomy classification depends on a result linking the absence of certain structures to the existence of a special ordering. The structures are used to prove the NP-completeness and the ordering is used to design polynomial algorithms.

* Corresponding author: Jan Bok

1 Introduction

The CSP Dichotomy Theorem [8,25] guarantees that each homomorphism problem for a fixed template relational structure $\mathbf{H}$ ("does a corresponding input relational structure $\mathbf{G}$ admit a homomorphism to $\mathbf{H}$?") is either polynomial-time solvable or NP-complete, the distinction being whether or not the structure $\mathbf{H}$ admits a certain symmetry. In the context of undirected graphs $\mathbf{H} = H$, there is a more natural structural distinction, namely the tractable problems correspond to the graphs H that have a loop, or are bipartite [15].

A graph is called *reflexive* if each vertex has a loop, and *irreflexive* if no vertex has a loop.

For list homomorphisms (when each vertex $v \in V(G)$ has a list $L(v) \subseteq V(H)$), the distinction turns out to be whether or not H is a "bi-arc graph", a notion related to interval graphs [10]. In the special case of bipartite graphs H, the distinction is whether or not H has a min ordering. A *min ordering of a bipartite graph* with parts A, B is a pair of linear orders $<_A, <_B$ of A and B respectively, such that if there are edges $ab, a'b'$ with $a \in A, a' \in A, a < a'$ and $b \in B, b' \in B, b' < b$, then there is also the edge ab'. If a bipartite graph H has a min ordering, then the list homomorphism problem to H is polynomial-time solvable; otherwise it is NP-complete [9,14]. The bipartite graphs that admit a min ordering are an interesting graph class, as they are precisely those bipartite graphs whose complements are circular arc graphs, precisely the containment interval bigraphs, and precisely the intersection graphs of two-directional rays [14,9,17,22].

An analogous situation occurs for reflexive graphs (and digraphs), where the distinction is similar, although the definition of a min ordering is slightly different. A *min ordering of a reflexive graph* H is a linear order $<$ of $V(H)$, such that if there are edges $uv, u'v' \in E(H)$ with $u < u'$ and $v' < v$, then there is also the edge uv'. (It is possible to interpret the two kinds of min orderings as special cases of a general min ordering for digraphs, but it will be simpler for our purposes to use these two separate definitions.) If a reflexive graph H has a min ordering, then the list homomorphism problem to H is polynomial-time solvable; otherwise it is NP-complete [11].

In both cases, there is an obstruction characterization of the situation when a min ordering exists. An *invertible pair in a reflexive graph* H is a pair (u, u') of vertices of H, with a pair of walks $u = v_1, v_2, \ldots, v_k = u'$ and $u' = v'_1, v'_2, \ldots, v'_k = u$ of equal length, and another pair of walks $u' = w_1, w_2, \ldots, w_m = u$ and $u = w'_1, w'_2, \ldots, w'_m = u'$ of equal length, such that each v_i is non-adjacent to v'_{i+1} for all $i = 1, 2, \ldots, k-1$ and each w_j is non-adjacent to w'_{j+1}, for all $j = 1, 2, \ldots, m-1$. An *invertible pair in a bipartite graph* H with parts A, B is defined exactly in the same way, but with the condition that u, u' belong to the same part (A or B). It is easy to see that if an invertible pair exists, then there can be no min ordering (both for the reflexive and the bipartite cases). The converse also holds for both cases. For the reflexive case, this is shown in [11]. In fact, the proof in this case (see the proof of Theorem 3.2 in [11]) implies a stronger result — namely, if a set of ordered pairs of vertices does not violate transitivity, then it can be extended to a min ordering if and only if it contains no invertible pair. (A set of ordered pairs is said to violate transitivity if it contains some pairs $(t_0, t_1), (t_1, t_2), (t_2, t_3), \ldots, (t_{k-1}, t_k), (t_k, t_0)$ with $t_0 < t_1 < \cdots < t_k < t_0$.) For the bipartite case, the converse of the characterization is proved in [14]; however, this is done by a reduction to the reflexive case, and there is no analogue for extending a given set of ordered pairs. In fact, such a result was not known for bipartite graphs.

In this paper, we fill the gap and prove an analogous extension version of the min ordering characterization for bipartite graphs, Corollary 7. This result is then used in the following section to prove the bipartite case of the conjecture of Kim and Siggers. Moreover, we show how to use the extension result for reflexive graphs from [11] to give an analogous proof of the conjecture for reflexive graphs, providing an alternative proof of the result first claimed by Kim and Siggers [19].

A *signed graph* $\widehat{H}$ is a graph H together with an assignment of *signs* $+, -$ to the edges of H. There may be parallel edges with the same end vertices in which case we require there are only two edges and they have opposite signs. In this situation we say there is *an edge* with both signs, a concept which we make precise below. There may be edges that are loops, and there may also be two parallel loops of opposite signs at the same vertex. Edges with a $+$ sign are called *positive*, or *blue*, edges with a $-$ sign are called *negative*, or *red*. Edges with both signs are called *bicoloured*, while purely red or purely blue edges are called *unicoloured*. Two signed graphs are called *switch-equivalent* if one can be obtained from the other by a sequence of vertex switchings, where a *switching* at a vertex v flips the signs of all edges incident with v. (A bicoloured edge remains bicoloured.) Signed graphs arise in many contexts in mathematics and in applications. This includes knot theory, qualitative matrix theory, gain graphs, psychosociology, chemistry, and statistical physics [24]. In graph theory, they are of particular interest in nowhere-zero flows for graphs embedded in non-orientable surfaces [18].

A *sign-preserving homomorphism* of a signed graph $\widehat{G}$ to a signed graph $\widehat{H}$ is a mapping taking vertices of G to vertices of H, and edges of G to edges of H preserving both incidence and the sign of edges. A *homomorphism* of a signed graph $\widehat{G}$ to a signed graph $\widehat{H}$ is a sign-preserving homomorphism of $\widehat{G'}$ to $\widehat{H}$ for some signed graph $\widehat{G'}$ switch-equivalent to $\widehat{G}$. Equivalently, a homomorphism of a signed graph $\widehat{G}$ to a signed graph $\widehat{H}$ is a homomorphism f of the underlying graph G of $\widehat{G}$ to the underlying graph H of $\widehat{H}$, such that for any closed walk W in G, the sign of W (the product of the signs of all edges) is the same as the sign of $f(W)$ in H. We will use this definition in the last section, as it does not require switching in the input graph before mapping it. The equivalence of the two definitions follows from the theorem of Zaslavsky [23], and the actual switching required for $\widehat{G}$ before the mapping if one exists, as well as the two violating closed walks if such a mapping doesn't exist, can be found in polynomial time [20].

We remark that the equivalent definition for homomorphisms of signed graphs is well defined with our notion of bicoloured edges. Suppose f is a homomorphism of $\widehat{G}$ to $\widehat{H}$. Let e be an edge of G such that $f(e)$ is bicoloured. Assume by induction that f maps $G-e$ so that (i) for any edge mapping to a bicoloured edge,

f assigns one of the two parallel edges as the image, and (ii) all closed walks of G map to a closed walk of the same sign in H. We claim there is a choice for $f(e)$ (of the two possible edges in the bicoloured edge) so that for any closed walk W of G containing e, the image $f(W)$ has the same sign. Without loss of generality assume e is positive. Suppose W is positive closed walk containing e (the case when W is negative is analogous). Then $f(W-e)$ has sign s and we choose $f(e)$ to have the same sign s. Now suppose W' is a negative closed walk containing e (the case when W' is positive is similarly handled). Suppose $f(W'-e)$ has sign s'. Then the closed walk obtained from the union of $f(W-e)$ and $f(W'-e)$ has sign ss'. Further since e is positive, we have $W-e$ union $W'-e$ forms a negative closed walk in G. Thus ss' is negative. We have already assigned $f(e)$ to be the edge of sign s, so with that same choice $f(W')$ is negative as required. In other words, all closed walks containing e enforce the same choice for $f(e)$. Hence, we can simply say e is mapped to *the bicoloured edge* $f(e)$ and know that there is an explicit choice for $f(e)$ the ensures all closed walks containing e have the correct sign.

Thus a homomorphism of $\widehat{G}$ to $\widehat{H}$ is a homomorphism of the underlying graphs G to H which maps bicoloured edges of $\widehat{G}$ to bicoloured edges of $\widehat{H}$, and for which any unicoloured closed walk W in $\widehat{G}$ with unicoloured image $f(W)$ in $\widehat{H}$ has the same product of the signs of its edges. (In other words, closed walks with only unicoloured edges map to closed walks that either contain a bicoloured edge or have the same parity of the number of negative edges.)

The study of homomorphisms of signed graphs was pioneered by Guenin [13] and introduced more systematically by Naserasr, Rollová, and Sopena, see the survey [20].

The *homomorphism problem* for the signed graph $\widehat{H}$ asks whether an input signed graph $\widehat{G}$ admits a homomorphism to $\widehat{H}$. The *s-core* of a signed graph $\widehat{H}$ is the smallest homomorphic image of $\widehat{H}$ that is a subgraph of $\widehat{H}$. (The s-core is unique up to isomorphism [6].) It was conjectured in [6] that the homomorphism problem for $\widehat{H}$ is polynomial if the s-core of $\widehat{H}$ has at most two edges (a bicoloured edge counts as two edges), and is NP-complete otherwise. The conjecture was verified in [6] for all signed graphs that do not simultaneously contain a bicoloured edge and a unicoloured loop of each colour. Finally, the full conjecture was established in [7].

The *list homomorphism problem* for a signed graph $\widehat{H}$ asks whether an input signed graph $\widehat{G}$ with lists $L(v) \subseteq V(\widehat{H}), v \in V(\widehat{G})$, admits a homomorphism f to $\widehat{H}$ with all $f(v) \in L(v), v \in V(\widehat{G})$. The complexity classification for these list homomorphism problems appears to be difficult, and no structural classification conjecture has arisen. (Even though these are not directly CSP problems, the fact that dichotomy holds can be derived from the CSP Dichotomy Theorem.) Some special cases have been treated [2,3,5,19], including a full classification for signed trees [1].

In [19], H. Kim and M.H. Siggers focus on a special class of signed graphs: we say that a signed graph $\widehat{H}$ is *semi-balanced* if any closed walk of unicoloured edges has an even number of negative edges. Equivalently, there is a switch-equivalent signed graph $\widehat{H'}$ in which there are no purely red edges [1]. We note that this class has been called *pr-graphs* in [19], *uni-balanced graphs* in [3], and *weakly balanced graphs* in [1].

Kim and Siggers [19] conjectured a classification of the complexity of the list homomorphism problems for semi-balanced signed graphs $\widehat{H}$, and verified it in the special case of signed graphs that are reflexive. (In the last version of [19] they actually apply a result from this paper, cf. the footnote on page 4 of [19], version v4.) Their paper also highlights the importance of irreflexive signed graphs, by reducing parts of the problem for general signed graphs to their bipartite translations.

We note that non-bipartite irreflexive signed graphs are not relevant because their list homomorphism problems are NP-complete by [15]; it is also easy to see that they always contain an invertible pair.

The Kim-Siggers conjecture is particularly elegant when stated for irreflexive signed graphs. To be specific, we assume that $\widehat{H}$ is a bipartite signed graph without purely red edges, and define a *special min ordering* of $\widehat{H}$ to be a min ordering of the underlying graph H of $\widehat{H}$, such that at each vertex its bicoloured neighbours precede its unicoloured neighbours. The conjectured classification for semi-balanced signed graphs states that the list homomorphism problem for $\widehat{H}$ is polynomial-time solvable if $\widehat{H}$ has a special min ordering, and is NP-complete otherwise.

This implies that there are two natural obstructions to $\widehat{H}$ having a polynomial-time solvable list homomorphism problem – namely invertible pairs, which obstruct the existence of a min ordering, and chains, which obstruct a min ordering from being made special. *Invertible pairs* are defined above for unsigned bipartite graphs, and for signed bipartite graphs they are just invertible pairs in the underlying unsigned graph. A *chain* in a signed graph $\widehat{H}$ consists of two walks of equal length, a walk U with vertices $u = u_0, u_1, \ldots, u_k = v$ and a walk D, with vertices $u = d_0, d_1, \ldots, d_k = v$ such that the edges $uu_1, d_{k-1}v$ are unicoloured, and the edges $ud_1, u_{k-1}v$ are bicoloured, and for each $i, 1 \leq i \leq k-2$, we have both u_iu_{i+1} and d_id_{i+1} edges of H while d_iu_{i+1} is not an edge of H, or both u_iu_{i+1} and d_id_{i+1} bicoloured edges of H while d_iu_{i+1} is not a bicoloured edge of H. See Figure 1 for an example.

Kim and Siggers also conjectured that a semi-balanced signed graph $\widehat{H}$ has a special min ordering if and only if it has no invertible pairs and no chains. We prove both conjectures (cf. Theorem 3 below), in the case of irreflexive and reflexive signed graphs. The irreflexive result generalizes previous results on semi-balanced signed trees, and semi-balanced separable signed graphs [1,2].

In this journal version of our conference paper [4] we have added a discussion of the extension result for reflexive graphs, of its application to characterize reflexive signed graphs that admit a special min ordering, as well as a simple direct algorithm for the polynomial cases. Moreover, we also offer an application of our results to obtain the concrete structure (via forbidden subgraphs) of the polynomial cases, at least for certain special classes of bipartite semi-balanced signed graphs.

$W_1 = 1 - 2 - 3 - 4 - 5 - 6 - 7 - 6 - 7 - 6 - 5 - 4 - 8 - 9 - 10$

$W_2 = 10 - 9 - 10 - 9 - 10 - 9 - 8 - 4 - 3 - 2 - 1 - 2 - 1 - 2 - 1$

Fig. 1. An example of a signed graph (on the left) with a chain (on the right) and an invertible pair $(1, 10)$ certified by the pair of walks W_1, W_2 and the pair consisting of the reverse of both walks.

2 Min orderings of (unsigned) bipartite graphs

In this section we only deal with unsigned bipartite graphs H, with a fixed bicolouring A, B. The *pair digraph* H^+ has as vertices all ordered pairs of distinct equicoloured vertices of H, i.e., $V(H^+) = \{(a, a') : a, a' \in A, a \neq a'\} \cup \{(b, b') : b, b' \in B, b \neq b'\}$. There is in H^+ an arc from (a, a') to (b, b') precisely if $ab, a'b'$ are edges of H while ab' is not an edge of H. In that case we also say that (a, a') *dominates* (b, b'). We note that (a, a') dominates (b, b') if and only if (b', b) dominates (a', a), a property we call *skew symmetry* of H^+. A subset C of pairs of H^+ is a *strong component* if for two pairs (a, a') and (b, b') in C, there is a directed path from (a, a') to (b, b') and vice versa, and C is maximal with respect to this property. Note an invertible pair (u, u') of H is precisely a pair of H^+ belonging to the same strong component as its reverse pair (u', u).

A sequence $(x_0, x_1), (x_1, x_2), \ldots, (x_n, x_{n+1})$ of pairs of H^+ will be called a *thread* from x_0 to x_{n+1} if $x_0 \neq x_{n+1}$, and a *circuit* if $x_0 = x_{n+1}$. Note that the vertices (of H) in any thread or circuit are either all in A or all in B. A thread or circuit all of whose pairs belong to a subset X of $V(H^+)$ is called a *thread or circuit in* X. We say X contains the thread or circuit.

In this language, (x_0, x_1) is an invertible pair if and only if $(x_0, x_1), (x_1, x_0)$ is a circuit (with $n = 1$) in some strong component of H^+. Also note that H^+ does not contain circuits with $n = 0$, since such a circuit would consist of a *repeat pair* (x_0, x_0) and such pairs are not vertices of H^+ by definition.

Let P be any set of pairs. We say that P is *closed under reachability* if $(x', y') \in P$ whenever $(x, y) \in P$ and $(x, y)(x', y')$ is an edge in H^+. We say that P is *closed under transitivity* if $(x, z) \in P$ whenever $(x, y) \in P$ and $(y, z) \in P$. We note that a set of pairs P in H^+, which contains a circuit cannot be closed under transitivity, because such a set would contain a repeat pair.

We have the following result.

Theorem 1. *The following statements are equivalent for a bipartite graph H:*

1. *H has a min ordering.*
2. *H has no invertible pairs.*

3. The vertices of H^+ can be partitioned into sets D, D' such that

(a) $(x, y) \in D$ if and only if $(y, x) \in D'$,

(b) D is closed under reachability, and

(c) D is closed under transitivity.

Proof. We may assume that H is connected, in particular has no isolated vertices.

It is straightforward to see that 1 implies 2, and 3 implies 1 (by defining $x < y$ if $(x, y) \in D$). Thus it remains to show that 2 implies 3.

Therefore, we assume that H has no invertible pairs. Note that for each strong component C of H^+, there is a corresponding reversed (or *dual*) strong component C' whose pairs are precisely the reversed pairs of the pairs in C, i.e., $C' = \{(a, b) : (b, a) \in C\}$. We shall say that C, C' are *coupled* strong components. Note that a strong component C may be coupled with itself; it is easy to check that all pairs in a self-coupled strong component are invertible.

The partition of $V(H^+)$ into D, D' will consist of separating each pair of coupled strong components C, C' of H^+. The pairs of one strong component will be placed in the set D, their reversed pairs will go into D'.

We shall build these sets D, D' by iteratively adding a strong component of $H^+ - D - D'$ to D and its dual to D'. The detailed algorithm is described below. Initially the algorithm starts with any (possibly empty) sets D and D' such that (a-c) of Condition 3 in Theorem 1 are satisfied. In the remainder of this section we show that our algorithm will maintain these properties (a-c) until each pair (x, y) with $x \neq y$ belongs either to D or to D', proving 2 implies 3.

We note that properties (a,b) above imply that each strong component of H^+ belongs entirely to D, D', or to $V(H^+) - D - D'$, and that no pair in D dominates a pair in $V(H^+) - D$. A strong component C of H^+ is *trivial* if it consists of just one pair. Note that for any D satisfying (a,b), a trivial strong component of $H^+ - D - D'$ is also a trivial strong component of H^+.

We say that a pair (a, a') is a *sink pair* if $N(a)$ contains $N(a')$. If a pair (a, a') dominates (b, b'), then $a'b'$ is an edge, and ab' is not. Thus $N(a)$ does not contain $N(a')$. We conclude a sink pair does not dominate any pair of H^+, and hence a sink pair forms a trivial strong component of H^+ (regardless of what is in D). Conversely, if a pair (a, a') is not a sink pair, then it dominates some other pair (b, b'). Indeed, b' can be any vertex in $N(a') - N(a)$ and b can be any neighbour of a. By skew symmetry we have (a, a') is a sink pair if and only if (a', a) is not dominated by some pair.

The *reachability closure* P^R of a set P is the smallest set containing P and closed under reachability. The *transitivity closure* P^T of a set P is the smallest set containing P and closed under transitivity. The *closure* P^* of a set P is the smallest set containing P and closed under reachability and transitivity. It is easy to see that the transitivity closure P^T is obtained from P by setting initially $P^T = P$ and then performing the following operation as long as new pairs are added:

(i) if $(x, y) \in P^T$ and $(y, z) \in P^T$, then add (x, z) to P^T.

Similarly, the reachability closure P^R is obtained from P by setting initially $P^R = P$ and then performing the the following operation as long as new pairs are added:

(ii) if $(x, y) \in P^R$ and $(x, y)(x', y')$ is an edge in H^+, then add (x', y') to P^R.

Finally, the closure P^* is obtained from P by initially setting $P^* = P$ and then performing alternating transitivity and reachability closures until no new pairs are added.

We now describe the algorithm. As suggested earlier, we start initially with (possibly empty) sets D, D', that satisfy (a-c). Clearly empty sets satisfy (a-c), but we require the generality of initial non-empty D, D' for application in the next section where we will specify certain pairs that must be in the min order. In the iterative step, we shall have current sets D, D' satisfying (a-c), and select a strong component C of $H^+ - D - D'$ which can be used to enlarge the set D to $(C \cup D)^*$ (and also enlarge the set D' to consist of the reversed pairs of the new set D), so that (a-c) are again satisfied. The algorithm ends when $V(H^+) - D - D'$ is empty; at this point the pairs in D, together with repeat pairs $(a, a), a \in V(H)$, define a transitive, reflexive, and antisymmetric relation by properties (a-c), which is a linear ordering on $V(H)$, as $V(H^+) - D - D'$ is empty. In fact, it is a min ordering of H, by property (b).

It remains to explain how to select the next strong component C so that the updated D, D', as explained above, still satisfy (a-c). Since D' is updated to consist of the reversed pairs in D, (a) is automatically satisfied. Moreover, as D is updated to the closure $(C \cup D)^*$, transitivity, (c), and reachability, (b), are both always satisfied. Thus we need to verify the closures can be completed while respecting the current D, D'; that is, taking the closures never yields a repeat pair, which by definition do not belong to $V(H^+)$, or a pair previously assigned to D'. It is easy to see that either of these cases to occur, the set D would have to contain a circuit. Indeed, a repeat pair could only be obtained during a transitive closure, and the pairs involved in the closure would form a circuit. Similarly, if a pair (x, y) is placed in D' and some later iteration in D, then the set D contains both pairs (x, y) and (y, x) and hence a circuit with $n = 1$. Thus it suffices to be checking for the existence of circuits.

In selecting the strong component C we shall give preference to non-trivial strong components. This breaks the execution of the algorithm into two stages. In the first stage we process non-trivial strong components of H^+, moving each to either D or D' as it is processed, together with all strong components, and their duals, involved in computing the closure $(C \cup D)^*$. At this point all non-trivial strong components are in $D \cup D'$ and we process the remaining trivial strong components in $H^+ - D - D'$. Recall, trivial strong components of H^+ belonging to $H^+ - D - D'$ remain trivial strong components, independently of what has been added to D, so the processing of non-trivial strong components first is well-defined.

We call a strong component C *admissible* if the dual strong component C' is not reachable from C. Note that if C is not admissible, then $(C \cup D)^*$ would contain a circuit as for any $(a, b) \in C$ both (a, b) and (b, a) would belong to

$C^R \subseteq (C \cup D)^*$. Also note that at least one of C, C' must be admissible; otherwise, they are reachable from each other and $C = C'$ contains an invertible pair. Hence, we can (and will) always choose an admissible strong component to add to D at each iteration. Testing admissibility is not relevant in the second stage, where all trivial strong components are admissible because a trivial strong component cannot be reachable from another trivial strong component. However, we do not need this observation, so we will skip the easy proof.

In conclusion, here is the **statement of the algorithm**. Given sets D, D' satisfying (a-c) if there exists a non-trivial admissible strong component C of $H^+ - D - D'$, we update D to $(C \cup D)^*$ and update D' to contain the reverse pairs of $(C \cup D)^*$. This is *stage 1* of the algorithm. Otherwise we select any trivial admissible strong component $H^+ - D - D'$, and update D and D' the same way; this is *stage 2*.

We now show that 2 implies 3 in Theorem 1. At the end of the algorithm we will have placed each pair in either D or D', and hence we indeed will have a partition of $V(H^+)$. Moreover, (a) follows from the description of the algorithm. To prove (b,c), we observe that at each step of the algorithm we take the closure of D, thus D will indeed be closed under reachability and transitivity as long as no circuits are formed during the transitivity closure. We prove in Corollary 3 that no circuits are formed in the first stage of the algorithm, and prove in Corollary 5 that no circuits are formed in the second stage of the algorithm. This completes the proof of Theorem 1 □

Every pair in $(C \cup D)^*$ is obtained by some sequence of transitive and reachability closures starting from pairs in $C \cup D$, possibly several such sequences. For each pair $(x, y) \in (C \cup D)^*$ we define the *time stamp* recording when the pair appears in $(C \cup D)^*$ for the first time. Thus, the time stamp of every pair $(x, y) \in (C \cup D)^*$ is unique. Pairs in $C \cup D$ have time stamp 0, those in $(C \cup D)^T \cup (C \cup D)^R$ but not in $C \cup D$ have time stamp 1, and so on. Moreover, for each pair $(x, y) \in (C \cup D)^*$ we also define a *derivation sequence*, which is a sequence of operations (R for reachability closure and T for transitivity closure) that produces the pair within time equal to its time stamp. This sequence is also not necessarily unique, as there are two possible sequences for each positive time stamp.

Pairs in $C \cup D$, having time stamp 0, have the unique empty derivation sequence (-). Pairs with time stamp 1 consist of those in $(C \cup D)^T - (C \cup D)$ that have the derivation sequence (T), together with those in $(C \cup D)^R - (C \cup D)$ that have derivation sequence (R). Pairs with time stamp 2 consist of those in $((C \cup D)^R)^T - (C \cup D)^R$, having the derivation sequence (RT), as well as those in $((C \cup D)^T)^R - (C \cup D)^T$ with the derivation sequence (TR).

It is worth emphasizing that despite the similarity of the notation, for an alternating sequence (Z) of T'and R's, the set $(C \cup D)^Z$ consists not only of pairs with derivation sequence Z but also includes all sequences with derivation sequences corresponding to the prefixes of Z.

In general, we call a pair which admits a derivation sequence ending in R (that is a pair that can be placed in $(C \cup D)^*$ within its time stamp when applying the reachability closure as its final operation) a *reachability pair*, and call a pair which only admits a derivation sequence ending in T (that is a pair that can be placed in $(C \cup D)^*$ within its time stamp only when applying the transitivity closure as its final operation) a *transitivity pair*. Finally, a pair in $C \cup D$ is called an *original pair*. Thus, each pair in $(C \cup D)^*$ is either an original pair, or a reachability pair, or a transitivity pair. It will turn out that the only possible time stamps are $0, 1, 2$ or 3.

To improve readability we shall omit the parentheses and write expressions like $((C \cup D)^T)^R$ as $(C \cup D)^{TR}$; if $Z = z_1 z_2 \dots z_k$ is an alternating sequence of T's and R's, we write $(C \cup D)^Z$ for $(C \cup D)^{z_1 z_2 \dots z_k}$.

If (u, v) is a transitivity pair in $(C \cup D)^Z$, there exists a thread $(u_0, u_1), (u_1, u_2), \dots, (u_m, u_{m+1})$ from $u = u_0$ to $v = u_{m+1}$ with each pair (u_i, u_{i+1}) in $(C \cup D)^{Z'}$, where Z' is obtained from $Z = z_1 z_2 \dots z_k$ by deleting the last symbol $z_k = T$.

We say that a thread or circuit is *good* if each pair (u_i, u_{i+1}) is an original pair or a reachability pair. If there is a thread from u to v in $(C \cup D)^Z$, there is also a good thread from u to v in $(C \cup D)^Z$, as each transitivity pair, being obtained by transitivity from other pairs, can be replaced by those pairs and stay within $(C \cup D)^Z$. Similarly, if there is a circuit in $(C \cup D)^Z$, then there is also a good circuit in $(C \cup D)^Z$.

A good thread $(u_0, u_1), (u_1, u_2), \dots, (u_m, u_{m+1})$ in $(C \cup D)^Z$ is called *minimal* if no pair (u_i, u_j) with $j \neq i + 1$ is a reachability pair in $(C \cup D)^Z$. If a thread $(u_0, u_1), (u_1, u_2), \dots, (u_m, u_{m+1})$ admits a reachability pair (u_i, u_j) with $j > i+1$, we can use it to obtin a shorter thread. On the other hand, if (u_i, u_j) with $j < i$ is a reachability pair in $(C \cup D)^Z$, then $(C \cup D)^Z$ contains a circuit. Thus it is clear that if $(C \cup D)^Z$ contains no circuits, and there is in $(C \cup D)^Z$ a good thread from u to v, then there is in $(C \cup D)^Z$ also a minimal good thread from u to v. In particular, we note for future reference that **if $(C \cup D)^Z$ contains no circuits, then for any transitivity pair (u, v) in $(C \cup D)^Z$ there exists in $(C \cup D)^Z$ a minimal good thread $(u, u_1), (u_1, u_2), \dots, (u_m, v)$ from u to v.** Moreover, if $(u_0, u_1), (u_1, u_2), \dots, (u_m, u_0)$ is a shortest good circuit in $(C \cup D)^Z$, then $(u_1, u_2), \dots, (u_m, u_0)$ is a minimal good thread in $(C \cup D)^Z$, as is any other thread obtained from the shortest circuit by removing one pair.

Our first goal is to show that given a minimal good thread $(u_0, u_1), (u_1, u_2), \dots, (u_m, u_{m+1})$, under certain conditions we can find vertices $v_0, v_1, \dots, v_{m+1}$ so that the edges $u_j v_j$, $j = 0, \dots, m + 1$, form an independent matching.

We proceed to a sequence of lemmas. **For all these lemmas we are assuming that no strong component of H^+ has an invertible pairs (assumption 2 of Theorem 1), and that D, D' satisfy conditions (a-c)**. In these lemmas we also assume that Z is any derivation sequence, i.e., a (possibly empty) alternating sequence of R's and T's. To account for the empty derivation sequence $(Z) = (-)$, we define $(C \cup D)^Z = C \cup D$.

Lemma 1. *Suppose* $(u_0, u_1), (u_1, u_2), \dots, (u_m, u_{m+1})$ *is a minimal good thread from* u_0 *to* u_{m+1} *in* $(C \cup D)^Z$.

If some (u_i, u_{i+1}) *is dominated by* $(p_i, q_i) \in (C \cup D)^Z$*, then* $p_i u_k \notin E(H)$ *for all* $k \neq i$*, and* $q_i u_k \notin E(H)$ *for all* $k \neq i, i + 1$.

If additionally (u_{i+1}, u_{i+2}) is also dominated by $(p_{i+1}, q_{i+1}) \in (C \cup D)^Z$, then also $q_i u_i \notin E(H)$ and $q_{i+1} u_k \notin E(H)$ for all $k \neq i+2$.

Proof. Suppose to the contrary there is an edge between p_i and some $u_k, k \neq i$. Then (p_i, q_i) dominates (u_k, u_{i+1}), making (u_k, u_{i+1}) a reachability pair (or original pair) in $(C \cup D)^Z$. This is a contradiction to the minimality of the good thread $(u_0, u_1), (u_1, u_2), \ldots, (u_m, u_{m+1})$. (Note $k \neq i+1$ as (p_i, q_i) dominates (u_i, u_{i+1}).)

Thus p_i is not adjacent to any u_k, $k \neq i$, and this now implies that each q_i is non-adjacent to all $u_k, k \neq i, i+1$ by the same reasoning, using the fact that $p_i u_k$ is not an edge. (Note that the argument fails when $k = i$, and we do not claim anything about $q_i u_i$.)

If there are consecutive pairs (u_i, u_{i+1}) and (u_{i+1}, u_{i+2}) that are dominated by pairs in $(C \cup D)^Z$, we now prove that $q_i u_i$ and $q_{i+1} u_{i+1}$ are also non-edges. We have already proved that $p_i u_{i+2}$ and $u_i q_{i+1}$ are non-edges. If $q_{i+1} u_{i+1}$ was an edge, we would have (u_i, u_{i+1}) dominates (p_i, q_{i+1}) which dominates (u_i, u_{i+2}) (and thus also places (u_i, u_{i+2}) in $(C \cup D)^Z$). This contradicts the minimality of our good thread. Moreover, if $q_i u_i$ was an edge, then (u_{i+1}, u_{i+2}) dominates (q_i, q_{i+1}) which dominates (u_i, u_{i+2}), yielding a similar contradiction. □

Lemma 2. *Suppose $(u_0, u_1), (u_1, u_2), \ldots, (u_m, u_{m+1})$ is a minimal good thread from u_0 to u_{m+1} in $(C \cup D)^Z$.*

If (u_i, u_{i+1}) is dominated by $(p_i, q_i) \in (C \cup D)^Z$ and additionally $u_j, j \neq i$, has a neighbour p_j that is non-adjacent to all $u_k, k \neq j$, then all pairs (u_k, u_{k+1}) with $i \leq k \leq j-1$ are also dominated by some $(p_k, q_k) \in (C \cup D)^Z$.

Proof. We first assume that $j \geq i+2$. Suppose to the contrary that not all intermediate pairs are so dominated. That is for $i \leq i' < j' \leq j$ the pair$(u_{i'}, u_{i'+1})$ is dominated by a pair in $(C \cup D)^Z$ but none of the pairs $(u_{i'+1}, u_{i'+2}), \ldots, (u_{j'-1}, u_{j'})$ is. Note either $j' = j$ or $j' < j$ in which case $(u_{j'}, u_{j'+1})$ is dominated by $(p_{j'}, q_{j'}) \in (C \cup D)^Z$.

First consider a neighbour r of $u_{i'+2}$. If r is not a neighbour of $u_{i'+1}$, then $(u_{i'+1}, u_{i'+2})$ dominates $(q_{i'}, r)$ placing $(q_{i'}, r) \in (C \cup D)^Z$. However, by Lemma 1, $q_{i'}$ is not a neighbour of $u_{i'+2}$ and hence $(q_{i'}, r)$ dominates $(u_{i'+1}, u_{i'+2})$, a contradiction. We conclude that all neighbours of $u_{i'+2}$ are also neighbours of $u_{i'+1}$. Next, consider s a neighbour of $u_{i'+3}$. If s not a neighbour of $u_{i'+2}$, then $(u_{i'+2}, u_{i'+3})$ dominates (r, s), implying $(r, s) \in (C \cup D)^Z$. If the edge from r to $u_{i'+3}$ is absent, then (r, s) dominates $(u_{i'+2}, u_{i'+3})$ contrary to our assumption. Thus r is adjacent to $u_{i'+3}$. Again, by Lemma 1, $q_{i'}$ is not adjacent to $u_{i'+3}$. Thus $(q_{i'}, r)$ dominates $(u_{i'+1}, u_{i'+3})$ making the latter a reachability pair. This contradicts the assumption the good thread is minimal. Hence, s is a neighbour of $u_{i'+2}$ and by the above also neighbour of $u_{i'+1}$.

Continuing in this vein we conclude every neighbour of $u_{j'}$ is adjacent to $u_{i'+1}$. If $j' = j$, this contradicts our assumption about p_j and if $j' < j$ this contradicts Lemma 1 (which states $p_{j'}$ is non-adjacent to $u_{i'+1}$).

The case $j \leq i-2$ is handled by an analogous argument started by showing any neighbour of $u_{j'+1}$ is a neighbour of $u_{j'}$, ultimately implying p_i is a neighbour of $u_{j'}$ contrary to Lemma 1. □

From these two lemmas we conclude that if (u_i, u_{i+1}) and $(u_j, u_{j+1}), j > i$ are dominated by $(p_i, q_i) \in (C \cup D)^Z$ and $(p_j, q_j) \in (C \cup D)^Z$ respectively, then all intermediate pairs are also so dominated and we have an independent matching $u_k v_k$, $i \leq k \leq j$. Indeed, each v_k can be chosen to be the corresponding p_k or q_{k-1}. In particular, if all pairs (u_k, u_{k+1}) are so dominated we obtain a full independent matching.

Corollary 1. *Suppose* $(u_0, u_1), (u_1, u_2), \ldots, (u_m, u_{m+1})$ *is a minimal good thread from* u_0 *to* u_{m+1} *in* $(C \cup D)^Z$.

If each pair (u_j, u_{j+1}) *is dominated by some pair in* $(C \cup D)^Z$*, then there exist vertices* v_j *such that the edges* $u_j v_j, j = 0, 1, \ldots, m+1$*, form an independent matching in* H.

This situation – a minimal good thread and a corresponding independent matching using each vertex involved in the thread – gives us a lot of structure we can use.

Lemma 3. *Suppose* $(u_0, u_1), (u_1, u_2), \ldots, (u_m, u_{m+1})$ *is a minimal good thread in* $(C \cup D)^Z$ *and* $u_0 v_0, u_1 v_1, \ldots, u_m v_m, u_{m+1} v_{m+1}$ *is an independent matching in* H.

A vertex of H *which is adjacent to at least two of the vertices* $u_0, u_1, \ldots, u_m, u_{m+1}$ *is adjacent to all of them, and a vertex of* H *adjacent to at least two of the vertices* $v_0, v_1, \ldots, v_m, v_{m+1}$ *is adjacent to all of them.*

Proof. If w is adjacent to u_j and u_k with $j < k$, but not adjacent to u_{j-1}, then the pair (v_{j-1}, w) is dominated by the pair $(u_{j-1}, u_j) \in (C \cup D)^Z$, and dominates the pair (u_{j-1}, u_k), thus $(u_{j-1}, u_k) \in (C \cup D)^Z$, contradicting the minimality of our thread. On the other hand, if w is adjacent to u_j and u_k with $j < k$, but not adjacent to u_{k-1} then $(u_{k-1}, u_k) \in (C \cup D)^Z$ dominates (v_{k-1}, w) which dominates (u_{k-1}, u_j), a similar contradiction. Finally if w is adjacent to u_j and u_k with $j < k$, but not adjacent to u_{k+1} we have (u_k, u_{k+1}) dominating (w, v_{k+1}) which dominates (u_j, u_{k+1}). Observing that $(v_0, v_1), (v_1, v_2), \ldots, (v_m, v_{m+1})$ is also a minimal good thread in $(C \cup D)^Z$ equipped with a corresponding independent matching $v_0 u_0, v_1 u_1, \ldots, v_m u_m, v_{m+1} u_{m+1}$, we conclude that the same holds for w adjacent to two of the v_i's. □

We denote by K the set of all vertices adjacent to all $u_0, u_1, \ldots, u_m, u_{m+1}$ and by K' the set of all vertices adjacent to all $v_0, v_1, \ldots, v_m, v_{m+1}$. We first observe that $K \cup K'$ induces a complete bipartite graph.

Lemma 4. *Suppose* $(u_0, u_1), (u_1, u_2), \ldots, (u_m, u_{m+1})$ *is a minimal good thread in* $(C \cup D)^Z$ *and* $u_0 v_0, u_1 v_1, \ldots, u_m v_m, u_{m+1} v_{m+1}$ *is an independent matching in* H.

If K is the set of all vertices adjacent to all $u_0, u_1, \ldots, u_m, u_{m+1}$ and K' the set of all vertices adjacent to all $v_0, v_1, \ldots, v_m, v_{m+1}$ then each vertex of K is adjacent to each vertex of K'.

Proof. If wz is not an edge for some $w \in K, z \in K'$, then $(u_0, u_1), (u_1, u_0)$ is an invertible pair, because the pairs $(u_0, u_1), (w, v_1), (u_1, z), (v_1, v_0), (u_1, u_0)$, $(w, v_0), (u_0, z), (v_0, v_1), (u_0, u_1)$ form a directed eight-cycle in H^+, implying (u_0, u_1), (u_1, u_0) are in the same non-trivial strong component of H^+. □

Lemma 5. *Suppose $(u_0, u_1), (u_1, u_2), \ldots, (u_m, u_{m+1})$ is a minimal good thread in $(C \cup D)^Z$ with $m \geq 1$, $u_0v_0, u_1v_1, \ldots, u_mv_m, u_{m+1}v_{m+1}$ is an independent matching in H, and K, K' are defined as above.*

Then any two distinct vertices $u_i, u_j, i \neq j$, belong to different components of the graph $H \setminus (K \cup K')$.

Proof. The definitions of K and K' imply that any vertex of $H \setminus (K \cup K')$ has at most one neighbour amongst $u_0, u_1, \ldots, u_{m+1}$ and at most one neighbour amongst $v_0, v_1, \ldots, v_{m+1}$. In the arguments that follow, we repeatedly appeal to this fact.

It suffices to show that any path joining two different vertices u_i, u_j must contain a vertex of $K \cup K'$. Let $u_i, b_1, a_2, \ldots, a_t, b_t, u_j$ be a path in H for some $i \neq j$. By the preceding observation, if b_1 is not in K, it is not adjacent to any $u_r, r \neq i$. Consider now the thread $(v_0, v_1), (v_1, v_2), (v_{i-1}, b_1)$, $(b_1, v_{i+1}), \ldots, (v_m, v_{m+1})$; we say that this thread was obtained from the minimal good thread $(v_0, v_1), (v_1, v_2), \ldots, (v_m, v_{m+1})$ by *replacing* v_i with b_1. The pairs in this new thread are again all in $(C \cup D)^Z$, because $(v_{i-1}, b_1), (v_{i-1}, v_i)$ are in the same strong component, and similarly for $(v_i, v_{i+1}), (b_1, v_{i+1})$. Moreover, the same argument shows it is again a minimal good thread. Note also that $v_0u_0, \ldots, v_{i-1}u_{i-1}, b_1u_i, v_{i+1}u_{i+1}, \ldots, v_{m+1}u_{m+1}$ is a corresponding independent matching in H containing an edge for each vertex involved in the new thread. Finally, each vertex k' of K' is adjacent to b_1 by Lemma 3 applied to the new thread, as k' is adjacent to all $v_j, j \neq i$ and $m + 1 \geq 2$. We have a new minimal good thread and a new corresponding matching, while keeping the same K, K'.

Therefore, we can continue with the modified thread $(v_0, v_1), (v_1, v_2), (v_{i-1}, b_1)$, $(b_1, v_{i+1}), \ldots, (v_m, v_{m+1})$ and matching $v_0u_0, \ldots, v_{i-1}u_{i-1}, b_1u_i, v_{i+1}u_{i+1}, \ldots$, $v_{m+1}u_{m+1}$ and replace u_{i+1} by a_2, similarly obtaining another modified minimal good thread $(u_0, u_1), \ldots, (u_i, a_2), (a_2, u_{i+2}), \ldots, (u_m, u_{m+1})$ and independent matching $v_0u_0, \ldots, v_{i-1}u_{i-1}, b_1u_i, v_{i+1}a_2, \ldots, v_{m+1}u_{m+1}$. We can continue replacing the vertices along the path $u_i, b_1, a_2, \ldots, a_t, b_t, u_j$, until we obtain the minimal good thread $(u_0, u_1), \ldots, (a_{t-1}, a_t), (a_t, u_j), \ldots, (u_m, u_{m+1})$ and independent matching $v_0u_0, \ldots, v_{i-1}u_{i-1}, \ldots, b_ta_t, v_ju_j, \ldots, v_{m+1}u_{m+1}$. Since b_t is adjacent to both u_j and a_t, we must have $b_t \in K$. □

We conclude from Lemma 5 that the graph $H \setminus (K \cup K')$ consists of distinct components $S_0, S_1, \ldots, S_{m+1}, \ldots S_n$, where each $S_i, i = 0, 1, \ldots, m+1$ contains the edge u_iv_i. (There may be other components $S_{m+2}, \ldots, S_n$.) Let C_i denote the strong component of H^+ containing the pair (u_i, u_{i+1}). We aim to prove that $C_i \neq C_j$ when $i \neq j$. For this purpose we analyze the relationship between the strong components C_i of H^+ and the components S_i of H.

Lemma 6. *Suppose* $(u_0, u_1), (u_1, u_2), \ldots, (u_m, u_{m+1})$ *is a minimal good thread in* $(C \cup D)^Z$ *with* $m \geq 1$*, and* $u_0v_0, u_1v_1, \ldots, u_mv_m, u_{m+1}v_{m+1}$ *is an independent matching in* H*.*

The strong component of H^+ *containing the pair* (u_i, u_{i+1}) *consists precisely of all those pairs* (a, b) *where* $a \in S_i, b \in S_{i+1}$*.*

Proof. Suppose first that $(a, b) \in C_i$, i.e., that there is a directed path P from (u_i, u_{i+1}) to (a, b) and a directed path P' from (a, b) to (u_i, u_{i+1}). If (p, q) is the second vertex of P, then $u_ip, u_{i+1}q$ are edges of H hence $p \in S_i \cup K, q \in S_{i+1} \cup K$. However, $q \notin K$, since u_iq is not an edge. Moreover, if $p \in K$ then (p, q) does not dominate any other pair and hence P ends in $(a, b) = (p, q)$; so in this case, there can be no directed path from (p, q) to (u_i, u_{i+1}). Therefore we also have $p \notin K$ and thus $p \in S_i, q \in S_{i+1}$ and the same holds for all other vertices on the path P, including (a, b).

On the other hand, for any pair (a, b) with $a \in S_i, b \in S_{i+1}$, we easily construct paths P, P' as above by using paths in S_i from u_i to a and in S_{i+1} from u_{i+1} to b. □

Corollary 2. *Suppose* $(u_0, u_1), (u_1, u_2), \ldots, (u_m, u_{m+1})$ *is a minimal good thread in* $(C \cup D)^Z$ *with* $m \geq 1$*, and* $u_0v_0, u_1v_1, \ldots, u_mv_m, u_{m+1}v_{m+1}$ *is an independent matching in* H*. Let* $C_i, i = 0, 1, \ldots, m$*, be the strong component of* H^+ *containing the pair* (u_i, u_{i+1})*.*

Then $C_i \neq C_j$ *if* $i \neq j$*. Thus there is no directed path in* H^+ *from* (u_i, u_{i+1}) *to* (u_j, u_{j+1}) *if* $i \neq j$*.*

We now consider **the first stage of the algorithm**, when non-trivial strong components are processed. It turns out that all reachability pairs have time stamp 1 in this case. The *time stamp of a thread or circuit* is understood to be the maximum time stamp of its pairs.

Suppose $Z = z_1 \ldots z_{t-1}z_t$ is an alternating sequence of T's and R's with $z_t = R$, corresponding to time stamp $t \geq 2$, and denote $Z' = z_1 \ldots z_{t-1}$ and $Z'' = z_1 \ldots z_{t-2}$. (Note that Z'' could be empty.)

Lemma 7. *If* C *is a non-trivial strong component, and* $(C \cup D)^{Z'}$ *contains no circuit, then each reachability pair in* $(C \cup D)^Z$ *belongs to* C^R*.*

Proof. It is enough to prove the time stamp of each reachability pair is 1, since $(C \cup D)^R = C^R \cup D$ and a reachability pair is not in D by definition. Thus for contradiction, assume that (x, y) is a reachability pair with time stamp $t \geq 2$. This means there is a sequence Z as described above the lemma, with $z_t = R$ and $z_{t-1} = T$ such that $(x, y) \in (C \cup D)^Z$. There is a directed path P in H^+ to (x, y) from some transitivity pair $(a, b) \in (C \cup D)^{Z'}$, i.e., where (a, b) has time stamp $t - 1$. Since $(C \cup D)^{Z'}$ contains no circuit, there is a minimal good thread from a to b, with all pairs $(a,a_1),\ldots,(a_m,b)$ in $(C\cup D)^{Z''}$, i.e., having time stamp at most $t-2$. Assume that of all transitivity pairs $(a,b)\in(C\cup D)^{Z'}$, all directed paths P from (a,b) to (x,y), and all minimal good threads from a to b in $(C\cup D)^{Z''}$, we have chosen those that minimize the length m of the thread. For convenience, we shall write $a=a_0$, $b=a_{m+1}$.

If $t=2$, the time stamp of the thread $(a,a_1),\ldots,(a_m,b)$ is $t-2=0$, so they are all original pairs. At least one of the pairs (a_i,a_{i+1}) must be in C, otherwise $(a,b)\in D$ and hence $(x,y)\in D$ is an original pair, not a reachability pair. Since C is non-trivial, (a_i,a_{i+1}) is dominated by some pair $(p,q)\in C$ with time stamp $t-2=0$. Similarly, if $t>2$, then at least one of the pairs (a_i,a_{i+1}) is a reachability pair in $(C\cup D)^{Z''}$, i.e., with time stamp at most $t-2$, so it is also dominated by some pair (p,q) in $(C\cup D)^{Z''}$. Assume that P has consecutive pairs $(a,b),(u,v),\ldots,(x,y)$. We claim that both u and v are non-adjacent to all a_j. If a_jv was an edge, then $(a,a_j)\in(C\cup D)^{Z'}$ with time stamp at most $t-1$, would dominate (u,v), since av is a non-edge; this would contradicts the minimality of m. Similarly, any edge a_ju would allow us to replace (a,b) by (a_j,b) with a shorter thread. Now we can apply Lemma 2 to the minimal good thread $(a,a_1),\ldots,(a_m,b)$ in $(C\cup D)^{Z''}$, and deduce that all pairs (a_i,a_{i+1}) are dominated in $(C\cup D)^{Z''}$ and so Corollary 1 implies that there is an independent matching $a_ju_j, j=0,1,\ldots,m+1$, and therefore (u,v) also admits a minimal good thread $(u,u_1),\ldots,(u_m,v)$. Since (a_i,a_{i+1}) and (u_i,u_{i+1}) are in the same strong component for each i, the time stamp of the thread $(u,u_1),\ldots,(u_m,v)$ is also $t-2$. Continuing this argument along the directed path P, we conclude that (x,y) is a pair with time stamp $t-1$, which is a contradiction. □

This lemma allows us to prove that the algorithm doesn't create circuits in the first stage, when adding non-trivial strong components.

Corollary 3. *If C is a non-trivial strong component, then $(C\cup D)^*$ does not contain a circuit.*

Proof. If there is a circuit in $(C\cup D)^*$, then suppose a first circuit appears with time stamp t, i.e., in $(C\cup D)^Z$ where Z has t symbols. Let $X=(x_0,x_1),(x_1,x_2),\ldots,(x_m,x_0)$ is a shortest good circuit in $(C\cup D)^Z$. Then $(x_0,x_1),\ldots,(x_{m-1},x_m)$ is a minimal good thread, and so is $(x_1,x_2),\ldots,(x_m,x_0)$; therefore each pair of X belongs to some minimal good thread, and hence it is an original pair from $C\cup D$, or a reachability pair from C^R by Lemma 7. Hence each (x_i,x_{i+1}) is in $C^R\cup D$. If there are two pairs $(x_i,x_{i+1}),(x_j,x_{j+1}), i<j$ in C^R, then both are dominated by a pair in $C^R\cup D=(C\cup D)^R$ and hence by Lemma 2 all pairs between (x_i,x_{i+1}) and (x_j,x_{j+1}) are also dominated by a pair in $C^R\cup D$. Therefore, by Lemma 6 applied to the minimal good thread $(x_i,x_{i+1}),\ldots,(x_j,x_{j+1})$, we obtain subgraphs $S_i,S_{i+1},\ldots,S_j$ of H, such that the strong component of H^+ containing (x_k,x_{k+1}) consists of all pairs $(a,b), a\in S_k, b\in S_{k+1}$, for any $k, i\le k\le j$. There is a directed path in H^+ from a pair in C to (x_i,x_{i+1}). Considering an edge $(p,q)(r,s)$ of this path, we note that pr, qs are independent edges of H, and so (p,q) and (r,s) are in the same strong component of H^+. This means

that (x_i, x_{i+1}) and (x_j, x_{j+1}) are both actually in C. This contradicts Corollary 2.

Thus there can be at most one pair of X in C^R, and no two consecutive ones in D. It easily follows that $m = 1$, i.e., that X is a circuit with two pairs $(x_0, x_1), (x_1, x_0)$. Both (x_0, x_1) and (x_1, x_0) cannot be in D since D has no circuits. Moreover, if $(x_0, x_1) \in D$ and (x_1, x_0) is reachable from C, then C is reachable from (x_0, x_1) by skew symmetry and hence C was not chosen disjoint from D as required. It remains to consider the case when both (x_0, x_1) and (x_1, x_0) are in C^R. Thus suppose that $(a, b) \in C$ has a directed path to both (x_0, x_1) and (x_1, x_0). By skew symmetry, we have a directed path from (x_1, x_0) to (b, a) and hence a directed path from (a, b) to (b, a). This means the strong component C was not admissible, contradicting what the algorithm is doing. □

We now focus on **the second stage of the algorithm**, after all non-trivial strong components have been handled. This means that any non-trivial strong component of H^+ is now in $D \cup D'$; in particular, if a pair (x, y) is dominated by (a, b) and dominates (c, d), then (x, y) is in D, because it is in the same strong component as (a, d); thus also $(c, d) \in D$. Hence any reachability pair (x, y) with time stamp t is directly dominated by a pair with time stamp at most $t - 1$. Moreover, if $t = 1$ then (x, y) is dominated by a pair in C, as if it was dominated by a pair in D it would be in D and hence not a reachability pair.

In this case, it turns out that all reachability pairs have time stamp at most 2. Below we use the same notation for the sequences Z, Z' as described before Lemma 7.

Lemma 8. *If C is a trivial strong component, and $(C \cup D)^{Z'}$ contains no circuit, then each reachability pair in $(C \cup D)^Z$ is directly dominated by a pair $(a, b) \in (C \cup D)^T$. Moreover, any minimal good thread from a to b has at most three pairs.*

Proof. This proof is similar to the proof of Lemma 7. Suppose a reachability pair (x, y) has time stamp $t > 2$. The observation preceding the lemma implies that (x, y) is directly dominated by a transitivity pair (a, b), which must have time stamp $t - 1 > 1$, and since there are no circuits at that time, it admits a minimal good thread. Any thread from a to b must contain at least one reachability pair, and hence a pair dominated in $(C \cup D)^*$. We may again assume that we minimized the length of the minimal good thread from a to b over all pairs (a, b) that dominate (x, y). This means as before that x and y are non-adjacent to all vertices in the pairs on the minimal good thread from a to b and, as in the proof of Lemma 7, we conclude there is an independent matching $a_j x_j$, and a minimal good thread $(x, x_1), (x_1, x_2), \ldots$ with time stamp $t - 2$, contradicting the fact that (x, y) has time stamp t. □

Lemmas 7 and 8 imply that all reachability pairs of the closure $(C \cup D)^*$ have derivation sequences (R) or (TR). Of course, this implies that transitivity pairs can only have derivation sequences (T), (RT), or (TRT), implying that all time stamps are in fact at most 3. (Original pairs have time stamps 0.) Moreover, a minimal good thread or circuit has time stamp at most 2, since all pairs are reachability pairs or original pairs. Since reflexive and transitive closures of a set S include the pairs of S, we also conclude the following.

Corollary 4. $(C \cup D)^* = (C \cup D)^{TRT}$.

Now we can prove that the algorithm also doesn't create circuits in the second stage, when adding trivial strong components.

Corollary 5. *If C is a trivial strong component, then $(C \cup D)^*$ does not contain a circuit.*

Proof. Assume a circuits first appears in $(C \cup D)^*$ with time stamp t and X is a shortest good circuit with time stamp t. The deletion of any pair from X results in a minimal good thread, thus each pair of X lies in some minimal good thread and hence either an original pair, or a reachability pair, which by Lemma 8 is dominated by some $(a, b) \in (C \cup D)^T$. Only one pair can be in C because C is trivial, and two consecutive pairs cannot be in D because D is closed under transitivity and X is minimal. We also claim that only one pair can be dominated by a pair in $(C \cup D)^T$. Indeed, if there are at least two such pairs, say $(a_i, a_{i+1}), (a_j, a_{j+1})$ then by Corollary 1 there are two consecutive pairs each in a non-trivial strong component and hence in D, contradicting the minimality of X.

From these constraints it follows that X consists of at most four pairs. If X is the circuit $(x_0, x_1), (x_1, x_2), (x_2, x_3), (x_3, x_0)$ then (up to relabeling) we may assume (x_0, x_1) is in C, $(x_1, x_2), (x_3, x_0)$ are in D, and (x_2, x_3) is dominated by a pair (a, b) in $(C \cup D)^*$, which admits a thread $(a, a_1), (a_1, a_2), \ldots, (a_{m-1}, b)$ in $C \cup D$. None of these pairs can be in C, as C has only one pair (x_0, x_1), and that pair consists of vertices of the opposite colour in the bipartition of H. Thus $m = 1$ and the pair (x_2, x_3) is actually in D, contradicting the minimality of the circuit X. The proof for the case when X has three pairs is similar.

It remains to consider the case when the circuit X has only two pairs, say $(x_0, x_1), (x_1, x_0)$. It is easy to see that both cannot be in $(C \cup D)$ as neither C nor D have circuits, and C is always chosen disjoint from D'. Thus one of the pairs, say (x_0, x_1), is dominated by some $(a, b) \in (C \cup D)^T$. Any minimal good thread from a to b must include a pair in C or else $(a, b) \in D$ and so we would have $(x_0, x_1) \in D$. Thus the other pair (x_1, x_0) cannot be in C because of the colour argument made when X has four pairs. If $(x_1, x_0) \in D$, then we would have $(b, a) \in D$ as (x_1, x_0) dominates (b, a) by skew symmetry, implying a circuit $(a, b), (b, a)$ with time stamp smaller than t. To see this, note that the time stamp of both (x_1, x_0) and (b, a) is 0; if the time stamp of (x_0, x_1) is 1 then the time stamp of (a, b) is 0, and if the time stamp of (x_0, x_1) is 2 then the time stamp of (a, b) is 1. This leaves the case that both (x_0, x_1) and (x_1, x_0) are dominated by pairs in $(C \cup D)^*$, say (x_0, x_1) is dominated by (u, v) and (x_1, x_0) is dominated by (w, y). Now the edges ux_1, wx_0 are independent and hence both (x_0, x_1) and (x_1, x_0) are in non-trivial strong components and hence in D, contradicting the fact that D has no circuits. $\square$

The preceding two corollaries provided the required results for the proof of Theorem 1. They also imply the first part of the following dichotomy for list homomorphisms of bipartite graphs (see [9,14]).

Corollary 6. *If a bipartite graph H has a min ordering, then the list homomorphism problem for a bipartite graph H is polynomial time solvable. Otherwise H contains an invertible pair and the problem is NP-complete.*

From the proof of Theorem 1 we derive the following Extension Theorem that will be used in the next section.

Corollary 7. *Suppose D is a set of ordered pairs of distinct vertices of a bipartite graph H that is closed under reachability and transitivity.*

Then there exists a bipartite min ordering $<$ of H such that $x < y$ for each $(x, y) \in D$ if and only if H has no invertible pair.

Given an arbitrary set D of pairs, we can apply the corollary to the closure of D. However, using the results of the next section, we are able to directly decide the existence of an extension for any set D of ordered pairs, without taking its closure.

A *D-inversion* consists of two pairs $(a, b), (c, d) \in D$ such that (d, c) is reachable from (a, b) in H^+.

Corollary 8. *Suppose D is a set of ordered pairs of distinct vertices of a bipartite graph H.*

There exists a bipartite min ordering $<$ of H such that $x < y$ for each $(x, y) \in D$ if and only if H has no invertible pairs and no D-inversions.

The proof of Corollary 8 will be presented at the end of the next section.

3 Obstructions to min orderings of semi-balanced bipartite signed graphs

Suppose $\widehat{H}$ is a semi-balanced signed graph and let us assume that it is switched to a signed graph without purely red edges. The underlying graph of $\widehat{H}$ is denoted by H. We assume H has no invertible pair. Define D_0 to consist of all pairs (x, y) in H^+ such that for some vertex z there is a bicoloured edge zx and a blue edge zy. Let D be the reachability and transitivity closure of D_0, i.e., the smallest set of pairs in H^+ containing all the pairs in D_0 and closed under reachability and transitivity. It is easy to see that a min ordering of H is a special min ordering of $\widehat{H}$ if and only if it extends D (in the sense that each pair $(x, y) \in D$ has $x < y$). Note that in bipartite graphs, for any $(x, y) \in D$ the vertices x and y are on the same side of any bipartition.

Theorem 2. *If $\widehat{H}$ has no chain, then the set D can be extended to a special min ordering.*

Proof. Clearly, the set D by its definition is closed under transitivity and reachability. It remains to show it has no repeat vertices, i.e., no circuits.

If zx is a bicoloured edge and a zy is a blue edge, then we call the three vertices z, x, y a *fork*. We then define a *petal* in $\widehat{H}$ recursively as follows:

1. A fork z, x, y is a petal of *length* 1 with *lower terminal* x and *upper terminal* y.
2. If P is a petal of length k with lower terminal l and upper terminal u, and P' is a petal of length k', with lower terminal $l' = u$ and upper terminal u', then $P \cup P'$ is a petal of length $\min(k, k')$ with lower terminal l and upper terminal u'.
3. If P is a petal of length k with lower terminal l and upper terminal u, and if ll', uu' are edges while lu' is not, then P together with l', u' is a petal of length $k+1$, with lower terminal l' and upper terminal u'.

Since petals are defined recursively, each is equipped with a sequence of steps in its construction. A petal which is not just a fork has as its last step either step 2, or step 3. We call the former *transitivity* petals, and the latter *reachability* petals.

We note that if P is a petal with lower terminal a and upper terminal b, then in any special min ordering we must have $a < b$.

A *flower* is a collection of petals $P_1, P_2, \ldots, P_n$ with the following structure. If each P_i has lower terminal l_i and upper terminal u_i, then $u_i = l_{i+1}$. (The petal indices are treated modulo n so that the lower terminal of P_1 equals the upper terminal of P_n.) We also note that the existence of a flower implies that there is no min ordering, as we have

$$l^{(1)} < u^{(1)} = l^{(2)} < \cdots < l^{(n)} < u^{(n)} = l^{(1)}.$$

It is clear that a flower yields a circuit in the set D (of H^+) defined at the start of this section, and conversely, each such circuit arises from a flower. Thus, it remains to prove that if $\widehat{H}$ contains a flower, then it also contains a chain. This is done using the three observations below together with Lemma 9 which completes the proof of Theorem 2. □

Observation 1. *Suppose F is a flower with petals $P_1, P_2, \ldots, P_n$, where P_1 is a transitivity petal obtained from petals P and P' as above (step 2). Then the sequence of petals $P, P', P_2, \ldots, P_n$ is also a flower F'.*

We will use this observation to reduce flowers to consist only of forks and reachability petals. Note that the new flower F' has the same number of forks as F and the minimum length of a petal in F and F' is the same.

Observation 2. *Suppose P is a petal of length k with lower terminal l and upper terminal u. Let v be a vertex such that uv is an edge and lv is not an edge, and let w be any neighbour of l. Then P together with v, w is again a reachability petal of length $k+1$ with lower terminal w and upper terminal v.*

Observation 3. *Suppose P' is a reachability petal of length $k+1$ with lower terminal l' and upper terminal u', obtained as in step 3 from a petal P with lower terminal l and upper terminal u, and let w be any neighbour of l. Then P'' obtained from P' by replacing l' by w is also a reachability petal of length $k+1$ with lower terminal w and upper terminal u'.*

We note that we can also replace the vertex x of a fork z, x, y by any w adjacent to z by a bicoloured edge.

Each petal in $\widehat{H}$ enforces an order on the pairs (l_i, u_i). Our aim is to prove that if (l_i, u_i) belongs to several petals, then all petals in $\widehat{H}$ enforce the same ordering, or we discover a chain in $\widehat{H}$.

We are now ready to prove the lemma needed.

Lemma 9. *Suppose $P_1, P_2, \ldots, P_n$ is a flower in $\widehat{H}$. Then $\widehat{H}$ contains a chain.*

Proof. As explained after Observation 1, we assume that each P_i is a reachability petal. We proceed by induction on the number of forks, say k, in the flower. Note we do not induct on the number of petals as an application of Observation 1 will increase the number of petals.

First note if $k = 2$, then the flower is precisely a chain and we are done. Thus assume $k > 2$ and consider a pedal of minimum length. We iteratively reduce this minimum length until it becomes length one, i.e., the petal is a fork, and then by eliminating the fork, we reduce the number of forks by one.

Without loss of generality suppose the length of P_2 is minimal over all petals. Assume P_2 has length at least two. Suppose the terminal pairs and their predecessors are labelled as in Figure 3 on the left. Recall, all petals are reachability petals consistent with the petals in the figure.

We first observe that if as is an edge, then by Observation 3 we can change the terminal pair of P_2 to be (s, e). Now $P_2, P_3, \ldots, P_n$ is a flower with fewer forks (each fork in P_1 is removed) and by induction $\widehat{H}$ has a chain. Hence, assume as is not an edge. By Observation 2 we can extend P_1 to $z, \ldots, (t, c), (s, b), (r, a)$. Using similar reasoning, we see that eu is not an edge and P_3 can be extended so its terminal pair is (d, u). Thus we remove the terminal pair from P_2 so that its terminal pair is (a, d). At this point, the modified P_1, P_2, P_3 are the first three petals of a flower where the length of P_2 has been reduced by one from its initial length. If the reduced P_2 is a transitivity petal (obtained through step 2), then

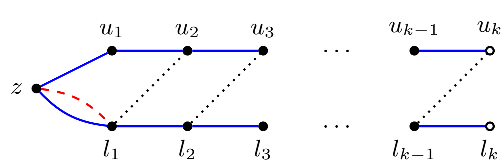

Fig. 2. A petal of length k with terminals (l_k, u_k). Dotted edges are missing.

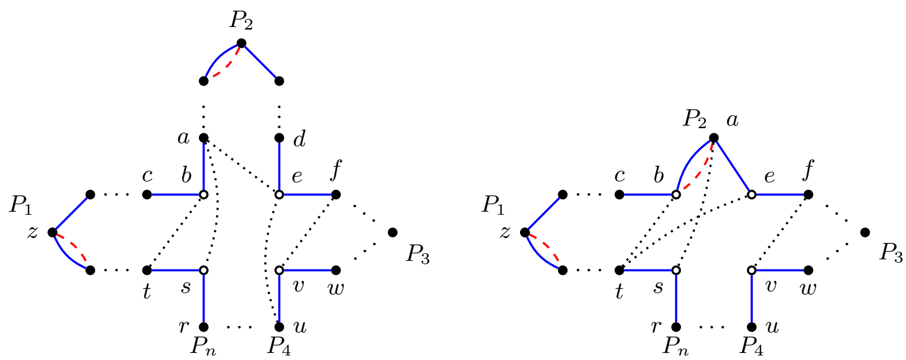

Fig. 3. The labellings used in Lemma 9. On the left is the case when P_2 has length greater than 1 and on the right when P_2 has length 1. Dotted edges are missing.

using Observation 3, modify the new flower to again consist of only reachability petals and forks without increasing the minimum length over all petals.

Thus, we may assume we have a flower where P_2 has length one, and hence is a fork. First assume the flower has $n > 2$ petals. If as is a unicoloured edge, then we modify the terminal pair of P_2 to be (b, s). Hence, P_1, P_2 is a flower with two petals and fewer forks (as the fork in P_3 is removed). If as is a bicoloured edge, then we modify P_2 to have terminal pair (s, e). Now $P_2, P_3, \ldots, P_n$ is a flower with fewer forks, and by induction $\widehat{H}$ contains a chain. Therefore, as is not an edge.

If et is an edge, then we can modify P_1 to have terminal pair (e, b) by Observation 3. Thus, P_1, P_2 is a flower with fewer forks. Hence, et is not an edge, and we can now extend P_1 by Observation 2 to be $z, \ldots, (t, c)$, (s, b), (t, a), (s, e) incorporating P_2 into P_1. Now we have a flower $P_1, P_3, \ldots, P_n$ with fewer forks, and by induction $\widehat{H}$ contains a chain.

The final case is when $n = 2$ but the number of forks $k > 2$. In this case (still assuming P_2 is reduced to a single fork), we have that P_1's derivation included an application of step 2 (transitivity). We can grow P_2 and shrink P_1 so that P_1 is a transitive petal. Applying Observation 1 allows us to change the flower to have 3 petals and the same number of forks. Thus, we can apply the argument above to shrink a petal to length 1 and apply induction as $n > 2$. □

Thus if a semi-balanced bipartite signed graph has no invertible pair and no chain, it has no flowers by Theorem 2, and hence by Corollary 7 it has a special min ordering.

Finally, we remark that the proofs are algorithmic, allowing us to construct the desired min ordering (if there is no invertible pair) or special min ordering (if there is no invertible pair and no chain).

We have proved our main theorem, which was conjectured by Kim and Siggers.

Theorem 3. *A semi-balanced bipartite signed graph $\widehat{H}$ has a special min ordering if and only if it has no chain and no invertible pair. If $\widehat{H}$ has a special min ordering, then the the list homomorphism problem for $\widehat{H}$ can be solved in polynomial time. Otherwise $\widehat{H}$ has a chain or an invertible pair and the list homomorphism problem for $\widehat{H}$ is NP-complete.*

The NP-completeness results are known [9,11,14], and the polynomial time algorithm is presented in the next section.

We complete this section with a proof of Corollary 8 from the previous section. Given a bipartite graph H, we form a signed bipartite graph $\widehat{H}$ whose vertices are all vertices of $V(H)$, together with special vertices $x_{ab}, (a, b) \in D$. The edges of H become blue edges of $\widehat{H}$, and for each x_{ab} we add a bicoloured edge to a and a blue edge to b. Note that a chain in $\widehat{H}$ corresponds precisely to a D-inversion in H. Therefore by Theorem 2 we conclude that if H has no invertible pairs and no D-inversions, D can be extended to a min ordering. This verifies Corollary 8.

4 A polynomial time algorithm for the bipartite case

Kim and Siggers have proved that the list homomorphism problem for semi-balanced bipartite or reflexive signed graphs with a special min ordering is polynomial time solvable. Their proof however depends on the dichotomy theorem [8,25], and is algebraic in nature. We provide simple direct low-degree algorithms that effectively use the special min ordering. In this section we describe the bipartite case, the next section deals with the reflexive case.

We begin by a review of the usual polynomial time algorithm to solve the list homomorphism problem to a bipartite graph H with a min ordering [12], cf. [16]. Recall that we assume H has a bipartition A, B. Futher for any input graph G with lists $L(v) \subseteq V(H), v \in V(G)$ we may assume G is also bipartite (or else there is no homomorphism at all), with a bipartition U, V, where lists of vertices in U are subsets of A, and lists of vertices in V are subsets of B.

Given such an input graph G, we first perform a consistency test, which reduces the lists $L(v)$ to $L'(v)$ by repeatedly removing from $L(v)$ any vertex x such that for some edge $vw \in E(G)$ no $y \in L(w)$ has $xy \in E(H)$. If at the end of the consistency check some list is empty, there is no list homomorphism. Otherwise it is easy to see that the min ordering property implies the mapping $f(v) = \min L(v)$, where the min is with respect to the min odering, is a homomorphism.

We will apply the same logic to a semi-balanced bipartite signed graph $\widehat{H}$; we assume that $\widehat{H}$ has been switched to have no purely red edges. If the input signed graph $\widehat{G}$ is not bipartite, we may again conclude that no homomorphism exists, regardless of lists. Otherwise, we refer to the alternate definition of a homomorphism of signed graphs, and seek a list homomorphism f of the underlying graph of $\widehat{G}$ to the underlying graph of $\widehat{H}$, that:

– maps bicoloured edges of $\widehat{G}$ to bicoloured edges of $\widehat{H}$, and

– maps unicoloured closed walks in $\widehat{G}$ that have an odd number of red edges to closed walks in $\widehat{H}$ that include bicoloured edges.

Indeed, as observed in the first section, this is equivalent to having a list homomorphism of $\widehat{G}$ to $\widehat{H}$, since $\widehat{H}$ does not have unicoloured closed walks with any purely red (i.e., negative) edges.

The above basic algorithm can now be applied to the underlying graphs; if it finds there is no list homomorphism, we conclude there is no list homomorphism of the signed graphs either. However, if the algorithm finds a list homomorphism of the underlying graphs which takes a closed walk R with odd number of red edges to a closed walk M with only purely blue edges edges, we need to adjust it. As noted in the introduction, Zaslavsky's algorithm will identify such a closed walk if one exists. Since the algorithm assigns to each vertex the smallest possible image, in the min ordering, we will remove all vertices of M from the list of each vertex of R, and repeat the algorithm. The following result ensures that vertices of M are not needed for the images of vertices of R.

Theorem 4. *Let $\widehat{H}$ be a semi-balanced bipartite signed graph with a special min ordering $\leq$.*

Suppose C is a closed walk in $\widehat{G}$ and f, f' are two homomorphisms of $\widehat{G}$ to $\widehat{H}$ such that $f(v) \leq f'(v)$ for all vertices v of $\widehat{G}$, and such that $f(C)$ contains only blue edges but $f'(C)$ contains a bicoloured edge.

Then the homomorphic images $f(C)$ and $f'(C)$ are disjoint.

Proof. We begin with three simple observations.

Observation 4. *There exists a blue edge $ab \in f(C)$ and a bicoloured edge $uv \in f'(C)$ such that $a < u, b < v$.*

Indeed, let u be the smallest vertex in A incident to a bicoloured edge in $f'(C)$, and let v be the smallest vertex in B joined to u by a bicoloured edge in $f'(C)$. Let xy be an edge of C for which $f'(x) = u, f'(y) = v$, and let $a = f(x), b = f(y)$. By assumption, $a = f(x) \leq f'(x) = u$ and $b = f(y) \leq f'(y) = v$. Moreover, $a \neq u$ and $b \neq v$ by the special property of min ordering.

Observation 5. *For every $r \in f'(C)$, there exists an $s \in f(C)$ with $s \leq r$.*

This follows from the fact that some x in $\widehat{G}$ has $s = f(x) \leq f'(x) = r$.

Observation 6. *There do not exist edges ab, bc, de with $a < d < c$ and $b < e$, such that ab is blue and de is bicoloured.*

Since $\leq$ is a min ordering, the existence of such edges would require db to be an edge and the special property of $\leq$ at d would require this edge to be bicoloured, contradicting the special property at b.

The following observation enhances Observation 6.

Observation 7. *There does not exist a walk $a_0b_0, b_0a_1, a_1b_1, \ldots, b_kc$ of blue edges, and a bicoloured edge de such that $a_0 < d < c$ and $b_0 < e$.*

This is proved by induction on the (even) length k. Observation 6 applies if $k = 0$. For $k > 0$, Observation 6 still applies if $a_0 < d < a_1$ (using the blue walk a_0b_0, b_0a_1 and the bicoloured edge de). If $d > a_1$, we can apply the induction hypothesis to $a_1 < d < c$ and de as long as $b_1 < e$. The special property of $<$ ensures that $b_1 \neq e$. Finally, if $e < b_1$, then Observation 6 applies to the edges b_0a_1, a_1b_1, ed.

Having these observations, we can now prove the conclusion. Indeed, suppose that $f(C)$ and $f'(C)$ have a common vertex g. Let us take the largest vertex g, and by symmetry assume it is in A, like a, u, where a, b, u, v are the vertices from Observation 4. Recall that we have chosen u to be the smallest vertex in A incident with a bicoloured edge of $f'(C)$, and v is smallest vertex in B adjacent to u by a bicoloured edge in $f'(C)$.

Suppose first that $g > u$. In $f(C)$ there is a path with edges $ab, ba_1, \ldots, hg$ which has $a < u < g$ and $b < v$, contradicting Observation 7.

If $g = u$ then the path with edges $ba, ab_1, b_1a_1, \ldots, a_kh, hg$ in $f(C)$ has all edges blue, and thus $h > v$ as $<$ is special. Therefore $b < v < h$ and $a < g$, also contradicting Observation 7.

Finally, suppose that $g < u$. Here we use the path in $f'(C)$ with edges gv_1, v_1u_1, u_1v_2, $\ldots, u_{k-1}v_k, v_ku, uv$. A small complication arises if $v_1 > v$, so we extend the path to also include ab by preceding it with the path in $f(C)$ with edges $ab, ba_1, a_1b_1, b_1a_2, \ldots, b_tg$. Of course the result is now a walk W, not necessarily a path. Note that the first edges of W are blue (being in $f(C)$), but the last edge uv is bicoloured.

If uv is the first bicoloured edge, then $v < v_k$ by the special property, and we have $b < v < v_k$ and $a < u$, a contradiction with Observation 7. Otherwise, the first bicoloured edge on the walk must be some u_jv_{j+1}, in case v_ju_j is unicoloured and $u_j \neq u$, or some v_ju_j, when $u_{j-1}v_j$ is unicoloured.

In the first case, where u_jv_{j+1} is the first bicoloured edge, $u_j > u$ by the definition of u. Then $a < u < u_j$ and $b < v$, implying again a contradiction with Observation 7. In the second case, where v_ju_j is the first bicoloured edge, we have again $a < u \leq u_j < u_{j-1}$, using the special property at v_j, and therefore we have $a < u < u_{j-1}$ and $b < v$ contrary to Observation 7. □

We observe that each phase removes at least one vertex from at least one list, and since $\widehat{H}$ is fixed, the algorithm consists of $O(n)$ phases of arc consistency, where n is the number of vertices (and m number of edges) of $\widehat{G}$. Since arc consistency admits an $O(m + n)$ time algorithm, our overall algorithm has complexity $O(n(m + n))$.

5 Semi-balanced reflexive signed graphs

We first briefly outline the proof in the reflexive case; it depends on the following extension result analogous to Corollary 7.

Corollary 9. *Suppose D is a set of pairs of vertices of a reflexive graph H, such that*

1. *if* $(x, y) \in D$ *and* xx', yy *are edges of* H *while* xy' *is not, then* $(x', y') \in D$,
2. *and* D *does not contain a set of pairs* $(x_0, x_1), (x_1, x_2), \dots, (x_n, x_0)$.

Then there exists a min ordering $<$ *of* H *such that* $x < y$ *for each* $(x, y) \in D$ *if and only if* H *has no invertible pair.*

This can be confirmed by a careful reading of the proof of Theorem 3.2 in [11]. That theorem and proof are stated in terms of reflexive digraphs, but if we view an undirected graphs as a symmetric digraph, the proof applies. In that proof, as in the proof of Theorem 1, we build the sets D, D' iteratively and in each step we only rely on the above properties 1, 2 of D.

Having this in hand, it only remains to show that Theorem 2 applies to reflexive signed graphs as well. In fact, the proof is unchanged. We again initialize D_0 to consist of all pairs (x, y) such that for some vertex z there is a bicoloured edge zx and a blue edge zy, and let D be the reachability closure of D_0. A min ordering of H is a special min ordering of $\widehat{H}$ if and only if each pair $(x, y) \in D$ has $x < y$. The proof of the fact that each flower contains a chain given in Section 3 applies in the reflexive case as well.

One can of course define the reflexive version of the auxiliary digraph H^+ in an obvious manner analogous to bipartite graphs; then condition 1 says D is closed under reachability and condition 2 says D has no circuits. (In this case we didn't need the fact that D is closed under transitivity because the algorithm we used was slightly different.)

In the reflexive case the definition of special min ordering is analogous to the bipartite case. Each vertex has its bicoloured neighbours appearing before its unicoloured neighbours.

Theorem 5. *A semi-balanced reflexive signed graph* $\widehat{H}$ *has a special min ordering if and only if it has no chain and no invertible pair. If* $\widehat{H}$ *has a special min ordering, then the list homomorphism problem for* $\widehat{H}$ *can be solved in polynomial time. Otherwise* $\widehat{H}$ *has a chain or an invertible pair and the list homomorphism problem for* $\widehat{H}$ *is NP-complete.*

We have the NP-complete cases from [9,11], so we focus on the polynomial algorithms.

As in the bipartite case, the polynomiality is known for the cases with special min ordering [19]. However, the algorithm of [19] is not direct and depends on the dichotomy theorem of [8,25], which uses deep results in universal algebra. We provide a simple direct polynomial algorithm along the lines of the bipartite case. The complexity of the algorithm is similar to the bipartite case, $O(n(m + n))$.

Theorem 6. *Let* $\widehat{H}$ *be a semi-balanced reflexive signed graph with a special min ordering* $\leq$. *Suppose* C *is a closed walk in* $\widehat{G}$ *and* f, f' *are two homomorphisms of* $\widehat{G}$ *to* $\widehat{H}$ *such that* $f(v) \leq f'(v)$ *for all vertices* v *of* $\widehat{G}$, *and such that* $f(C)$ *contains only blue edges but* $f'(C)$ *contains a bicoloured edge.*

Then the homomorphic images $f(C)$ *and* $f'(C)$ *are disjoint.*

Proof. We will first prove a couple of observations.

Observation 8. *There do not exist vertices $a \le c \le b \le d$ and edges ab, cd, such that ab is blue and cd is bicoloured.*

Suppose such vertices and edges did exist. By the property of min ordering, ac and bc must be edges. If ac is blue, c is not special. So ac is bicoloured. If now bc is blue, c is not special, and if bc is bicoloured, b is not special and we have a final contradiction. We note that this proof works even in the cases $a = c$, $c = b$, or $b = d$.

Observation 9. *There exists a blue edge $ab \in f(C)$ with $a \le b$, and a bicoloured edge $uv \in f'(C)$ with $u \le v$, such that $b < u$.*

Indeed, let u be the smallest vertex incident to a bicoloured edge in $f'(C)$, and let v be the smallest vertex joined to u by a bicoloured edge in $f'(C)$. Thus $u \le v$. Let xy be an edge of C for which $f'(x) = u, f'(y) = v$, and let $f(x) = a, f(y) = b$. By assumption, $a = f(x) \le f'(x) = u$ and $b = f(y) \le f'(y) = v$.

If $a = u$, the ordering $\le$ is not special. Suppose $a < u \le v$. If $b = u$, then u is not special. The same applies if $b = v$. If $u < b < v$, Observation 8 applies. Thus $b < u$ and we are done.

Observation 10. *If there is a blue edge ab and a bicoloured edge cd such that $a < c \le d < b$, then there is no blue edge ae with $a < e$ and $e < c$.*

By the definition of a min ordering, ac is an edge and by the definition of a special min ordering, it is bicoloured. Thus, ae contradicts the special property at a.

Observation 11. *Suppose that ab is a blue edge and de a bicoloured edge such that $a \le b < d \le e$. Then there cannot exist a blue walk from b to c, where $d \le c$.*

For a contradiction, suppose there exists such a walk. If the first edge of the walk ends in d, then d is not special; and if it ends at c with $d < c$, then we extend its beginning by edge ab. Denote by uv and vw the first two edges of the walk such that $u, v < d$ and $w \ge d$. If $w = d$ or $w = e$, then the ordering is not special. If $d < w < e$, then we have a contradiction with Observation 8. Finally, if $w > e$, we have a contradiction with Observation 10.

Having these observations, we can now prove the conclusion. Indeed, suppose that $f(C)$ and $f'(C)$ have a common vertex g. Let us take the largest vertex g and let a, b, u, v be the vertices from Observation 9. Recall that $a \le b$ and we have chosen u to be the smallest vertex incident with a bicoloured edge of $f'(C)$, and v is the smallest vertex adjacent to u by a bicoloured edge in $f'(C)$ (thus $u \le v$).

Suppose first that $g \ge u$. Then there is a blue path in $f(C)$ starting in b and ending in $g \ge u$, contradicting Observation 11.

Finally, suppose that $g < u$. Here we use the path in $f'(C)$ starting in g and ending in u. We extend the beginning of this path by a path from b to u in $f(C)$. Thus, this is a walk from b to some x with $u \le x$, contradicting Observation 11. □

As for bipartite graphs, we can simplify Corollary 9 as follows:

Corollary 10. *Suppose D is a set of ordered pairs of distinct vertices of a reflexive graph H.*

There exists a min ordering $<$ of H such that $x < y$ for each $(x, y) \in D$ if and only if H has no invertible pair and no D-inversion.

It is interesting to see the result stated for interval graphs, since min-orderable reflexive graphs are precisely interval graphs, and their min orderings correspond to the left-endpoint orderings of the intervals [9].

Corollary 11. *Suppose D is a set of ordered pairs of distinct vertices of a reflexive graph H.*

There exists an interval representation of H such that for each $(x, y) \in D$ the left endpoint of the interval representing x precedes the left endpoint of the interval representing y if and only if H has no invertible pairs and no D-inversions.

6 Refinements and special cases

In some cases one can be more specific about the dichotomy classification. In an earlier paper [2] Bok et al. described the detailed structure of the polynomial cases for semi-balanced bipartite signed graphs whose unicoloured edges form a hamiltonian path or cycle. The proofs of NP-completeness given there are all based on finding suitable chains and invertible pairs; and the polynomial algorithms given there all depend on finding a special min ordering. It is interesting to observe that, while Theorem 3 can be applied for this special class of signed graphs, this does not save much of the work presented in [2], which consists mostly of *finding* the chains and the min orderings.

We now restrict our attention to semi-balanced signed bipartite graphs whose underlying graphs have a min ordering. According to our Theorem 3, the polynomial cases are distinguished by the non-existence of a chain. It would be interesting to replace this condition by a list of forbidden induced subgraphs, as is the case for signed trees [1].

A *bipartite chain graph* is a bipartite graph in which the neighbourhoods of the vertices in each color class are linearly ordered by inclusion. This term is well established in the literature, and the word "chain" here refers to the ordering of neighbourhoods; it bears no relation to the obstructions defined earlier which we also called "chains", both here and in earlier papers.

According to [21], a bipartite graph has a min ordering if and only if it is the intersection of *two* bipartite chain graphs with the same bipartition. As a first step towards the above goal, we offer the following forbidden list characterization in the case of *one* bipartite chain graph. We will use the well-known fact that a bipartite graph is a bipartite chain graph if and only if it does not contain an induced $2K_2$.

Theorem 7. *Let $\widehat{H}$ be semi-balanced bipartite signed graph whose underlying unsigned graph is a bipartite chain graph. Then $\widehat{H}$ has a special min ordering*

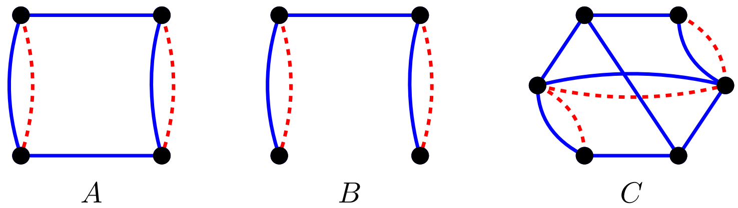

Fig. 4. Forbidden induced subgraphs of Theorem 7.

if and only if it does not have one of the three forbidden induced subgraphs in Figure 4.

Proof. Consider a chain in $\widehat{H}$ with the walk U being $a, b, d, f, \ldots$ and the walk D being $a, c, e, g, \ldots$. Without loss of generality, let us say that a is a black vertex.

We have $b \neq c$, since a is incident to b with unicoloured edge and to c with bicoloured edge. We also have $a \neq d$ because ac is bicoloured and cd is unicoloured or missing. Furthermore, b and c are white, while a and d are black. Thus all vertices a, b, c, d are different.

If bd is a bicoloured edge, then either cd is a unicoloured edge, and then we have the graph A present, or cd is a non-edge, and then we have the graph B present. Therefore, bd has to be unicoloured; moreover, cd is missing by the definition of chain.

Suppose that df is a unicoloured edge. From the definition of chain we have that e is not adjacent to f. Because of the edges incident to d, we have $f \neq c$. We also have $d \neq e$ as there is an edge between c and e but no edge between c and d. Note that c, f are both white, and d, e are both black. Thus, df is not the same edge as ce and there is an induced $2K_2$ in H. Therefore df is bicoloured; eg is also bicoloured and ef is unicoloured.

Recall that a, d, e are black and b, c, f are white. If ce is a bicoloured edge, then c, e, f, d would induce a copy of graph B. (Note that $c \neq f$ because of the adjacencies with e, and $d \neq e$ because of the adjacencies with f.) Thus ce is a unicoloured edge.

Observe that a, d, e are different because of adjacencies with c and b, c, f are different because of adjacencies with d. Since a, c, d, f do not induce a $2K_2$, the vertices a, f must be adjancet. If the edge af is unicoloured, then c, a, f, d induce a copy of graph B. Thus, af must be bicoloured. Also, be must be an edge, otherwise b, d and c, e would induce a $2K_2$. If be is bicoloured, then a, b, e, c is A. Therefore, be is unicoloured and a, b, c, d, e, f induce a copy of C. This concludes the proof. □

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Competing interests

The authors have no competing interests as defined by Springer, or other interests that might be perceived to influence the results and/or discussion reported in this paper.

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