The Dichotomy of List Homomorphisms for Digraphs

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Abstract

The Dichotomy Conjecture for Constraint Satisfaction Problems has been verified for conservative problems (or, equivalently, for list homomorphism problems) by Andrei Bulatov. An earlier case of this dichotomy, for list homomorphisms to undirected graphs, came with an elegant structural distinction between the tractable and intractable cases. Such structural characterization is absent in Bulatov's classification, and Bulatov asked whether one can be found. We provide an answer in the case of digraphs. In the process we give forbidden structure characterizations of the existence of certain polymorphisms relevant in Bulatov's dichotomy classification.

The key concept we introduce is that of a digraph asteroidal triple (DAT). The dichotomy then takes the following form. If a digraph H has a DAT, then the list homomorphism problem for H is NP-complete; and a DAT-free digraph H has a polynomial time solvable list homomorphism problem. DAT-free digraphs can be recognized in polynomial time. It follows from our results that the list homomorphism problem for a DAT-free digraph H can be solved by a local consistency algorithm (of width (2,3)).

1 Introduction.

The framework of constraint satisfaction problems (CSP's) allows a unification of many natural problems arising in applied computer science and artificial intelligence. In recent years, it has also become central in theoretical computer science, with most of the interest driven by the Dichotomy Conjecture formulated by T. Feder and M. Vardi in [19].

A general constraint satisfaction problem consists of a set of variables with values in a common domain, and a set of constraints limiting the values the variables can take. The theoretical investigations frequently focus on the so-called *non-uniform* CSP's, where the constraints are restricted by a certain finite *template*. The Dichotomy Conjecture simply says that for each such template the corresponding non-uniform problem is polynomial or NP-complete. Two original motivating examples for the Dichotomy Conjecture were Schaeffer's dichotomy classification of Boolean satisfiability problems [37], and Hell-Nešetřil's dichotomy classification of graph homomorphism problems [22]. The first case corresponds to the templates in which the common domain has only two values (say 0, 1); the second case corresponds to templates which are undirected graphs. Since that time, a number of other special cases have been established, e.g., [2, 3, 7, 13, 18], including the case of conservative problems [8] discussed below. In most of these cases, progress has been made possible by an algebraic approach pioneered by Jeavons, Cohen, and Gyssens [28]. In particular, Bulatov, Jeavons, and Krokhin [9] have established that the complexity of a non-uniform CSP only depends on the so-called polymorphisms of the template. This fundamentally affected the quest for the Dichotomy Conjecture, and in particular allowed more concrete statements of the expected distinction between tractable and intractable cases [9, 32, 35], cf. the survey [24].

The special case of conservative CSP's (or equivalently, CSP's with lists) is one of the early successes of the algebraic method; here dichotomy has been settled by Bulatov [8]. In his paper, Bulatov notes that his dichotomy lacks the combinatorial insights offered by the earlier special case of undirected graphs [16]. We provide such combinatorial insights in the case of directed graphs. This yields the first polynomial time distinction between the tractable and intractable cases for conservative CSP's in the case of digraphs. In the process, we also simplify Bulatov's classification in terms of polymorphisms, and give forbidden structure characterizations of digraphs which admit certain relevant polymorphisms.

Our technique is a combination of the polymorphism approach typical of the algebraic method, and of forbidden structure characterizations typical of structural graph theory. In particular, we introduce a class of digraphs that bears some similarity to the class of asteroidal-triple free graphs - of interest in the study of structured graphs [6, 12].

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2 Preliminaries.

In this paper we focus on templates that are digraphs. A *digraph* H is a finite set $V(H)$ of *vertices*, together with a binary relation $E(H)$ on the set $V(H)$; the elements of $E(H)$ are called *arcs* of H. A *homomorphism* of a digraph G to a digraph H is a mapping $f : V(G) \to V(H)$ which preserves arcs, i.e., such that $uv \in E(G)$ implies $f(u)f(v) \in E(H)$. The CSP with template H, also known as the *homomorphism problem for* H, is the decision problem in which the instance is a digraph G and the question is whether or not G admits a homomorphism to H. We note that we view an undirected graph H as a special case of a directed graph, in which the relation $E(H)$ is symmetric.

More general *relational structures* H are defined similarly, cf. e.g. [19, 23]. Feder and Vardi [19] have pioneered the view of non-uniform CSP's as homomorphism problems for a template H that is a relational structure. They have also identified the special case when H is a digraph as crucial for the Dichotomy Conjecture - if the conjecture holds for templates that are digraphs, then it holds in general [19].

For a fixed digraph (or more general relational structure) H, the *list homomorphism problem to* H, denoted $LHOM(H)$, asks whether or not an input digraph (or corresponding structure) G, equipped with lists $L(v), v \in V(G)$, admits a homomorphism $f : G \to H$, such that for each $v \in V(G)$ we have $f(v) \in L(v)$.

The problem LHOM(H) has been thoroughly studied for undirected graphs H [14, 15, 16]. For example, for *reflexive* graphs H (every vertex has a loop), the problem LHOM(H) is polynomial time solvable if H is an interval graph, and is NP-complete otherwise [14]. For *irreflexive* graphs (no vertex has a loop), the problem LHOM(H) is polynomial time solvable if H is a bipartite graph whose complement is a circular arc graph, and is NP-complete otherwise [15]. For general graphs, where some vertices may have loops and others don't, there is also a structural, albeit somewhat more technical, distinction [16]. The problem LHOM(H) has also been studied when H is a reflexive digraph, in [17].

These results have motivated a focus on CSP's with lists, leading to a full proof by Bulatov [8] of the dichotomy of LHOM(H) for all templates H. To explain the results, we first give the necessary definitions concerning polymorphisms. We again focus on digraphs, but similar definitions apply to general relational structures.

Let H be a digraph and k a positive integer. A mapping $f : V(H)^k \to V(H)$ is a *polymorphism* of H, of arity k, if it is compatible with the relation $E(H)$. (The mapping f is *compatible* with the relation $E(H)$ if $u_1^1u_2^1, \ldots, u_1^ku_2^k \in E(H)$ implies $f(u_1^1, \ldots, u_1^k)f(u_2^1, \ldots, u_2^k) \in E(H)$.) A polymorphism f is *conservative* if $f(u_1, u_2, \ldots, u_k)$ always is one of $u_1, u_2, \ldots, u_k$, and is *idempotent* if $f(u, u, \ldots, u) = u$, for all u. A polymorphism f of arity two is *commutative* if $f(u, v) = f(v, u)$ for all u, v, and is *associative* if $f(u, f(v, w)) = f(f(u, v), w)$ for all u, v, w. An idempotent commutative and associative polymorphism is called a *semi-lattice*. It is easy to see that a conservative semi-lattice polymorphism f on H defines a linear ordering $<$ on $V(H)$ such that $f(u, v) = \min(u, v)$. (It is enough to set $u < v$ exactly when $f(u, v) = u$.) Conversely, if $f(u, v) = \min(u, v)$ (with respect to some linear ordering $<$) is a polymorphism, then this polymorphism is clearly conservative, commutative and associative. Such a linear ordering $<$ is called a min-ordering. In other words, $<$ is a *min-ordering of* H just if it satisfies the following property: if $uv \in E(H)$ and $u'v' \in E(H)$, then $\min(u, u')\min(v, v') \in E(H)$. A polymorphism f of arity three is a *majority function* if $f(u, u, v) = f(u, v, u) = f(v, u, u) = u$ for any u and v. A polymorphism f of arity three is called *Maltsev* if $f(u, u, v) = f(v, u, u) = v$ for any u and v. It is known that if H admits a conservative majority function or a conservative Maltsev polymorphism, or a conservative semi-lattice polymorphism (i.e., a min-ordering), then the problem $LHOM(H)$ is polynomial time solvable [9, 19, 20]. Bulatov's result states that, locally, these are the only reasons for the polynomiality of LHOM(H). The result is formulated for general relational structures [8]; we state it for digraphs as follows.

THEOREM 2.1. *Let H be a digraph.*

If, for each pair of vertices u, v of H, there exists a conservative polymorphism f of H, that either is binary and $f|_{u,v}$ is a semi-lattice, or is ternary and $f|_{u,v}$ is majority, or Maltsev, then LHOM(H) is polynomial time solvable.

Otherwise, LHOM(H) is NP-complete.

This proves the dichotomy of LHOM(H) for digraphs, and provides a criterion for distinguishing the tractable and intractable cases. As pointed out in [8], the criterion lacks the combinatorial elegance and the structural information of the earlier results for undirected graphs.

We provide a simpler classification for digraphs. The characterization is similar to the spirit of the earlier combinatorial classifications for graphs [14, 15, 16], and gives structural information about obstructions that cause intractability. It provides for digraphs the first criterion that is polynomial in $V(H)$. As a byproduct, we will also conclude that in the case of digraphs, the statement of Bulatov's theorem can be simplified as follows.

COROLLARY 2.1. *Let H be a digraph.*

If, for each pair of vertices u, v of H, there exists a conservative polymorphism f of H, that either is binary and $f|_{u,v}$ is a semi-lattice, or is ternary and $f|_{u,v}$ is majority, then LHOM(H) is polynomial time solvable.

Otherwise, LHOM(H) is NP-complete.

We have recently learned that A. Kazda [29] has proved that if a digraph H admits a Maltsev polymorphism, it must also admit a majority polymorphism. Corollary 2.1 can be viewed as complementing Kazda's result, by showing that for conservative polymorphisms of digraphs, Maltsev polymorphisms are not needed even locally.

The fact that Maltsev polymorphisms are not needed in Corollary 2.1 has important algorithmic implications. As in [8, 9], a digraph H can be associated with a conservative algebra; if H satisfies the condition in the corollary then every two-element subalgebra of the associated algebra for H admits either a semi-lattice or a majority operation. According to Corollary 3.2 of [36], this property implies that the variety generated by the associated algebra of H omits types one and two. (The *types* of algebras are known to have a deep connection to the complexity of homomorphism problems cf. [27].) According to Theorem 9.10 of [27] this means the variety is semi-distributive. It then follows from Theorem 3.7 in [4] that, as conjectured in [33], a local consistency algorithm is applicable, and in fact the following result holds. (A more detailed discussion of the connection between algebras, varieties, types, and list homomorphism problems can be found in [31].)

COROLLARY 2.2. *If the list homomorphism problem for H is polynomial time solvable, then it has width $(2, 3)$.*

This means that the problem can be solved in polynomial time by a canonical local consistency algorithm which makes lists for pairs of vertices consistent over triples of vertices [4, 19, 23].

3 Asteroidal Triples.

Recall that for reflexive graphs H, the problem LHOM(H) is polynomial time solvable if H is an interval graph and is NP-complete otherwise [14]. According to the theorem of Lekkerkerker and Boland [34], a graph is an interval graph if and only if it does contain an induced cycle of length at least four, or an *asteroidal triple*, i.e., three vertices a, b, c any two of which are joined by a path not containing any neighbours of the third vertex. Since induced cycles of length at least six are easily seen to contain asteroidal triples themselves, we may view asteroidal triples as the principal structures in undirected graphs H that cause the NP-completeness of LHOM(H). We will introduce a digraph relative of an asteroidal triple that we call a *digraph asteroidal triple*, or DAT. Even though its somewhat technical definition makes a DAT only a distant relative of the simple concept of an asteroidal triple, DATs play the same pivotal role for digraphs as asteroidal triples (together with induced four- and five-cycles) play for undirected graphs - namely they are the only obstructions to polynomiality of the problem LHOM(H), cf. Theorem 3.3.

Let H be a digraph. We say that $uv \in E(H)$ is a *forward arc* of H (or just an *arc* of H); in that case we also say that vu is a *backward arc* of H. We define two walks $P = x_0, x_1, \ldots, x_n$ and $Q = y_0, y_1, \ldots, y_n$ in H to be *congruent*, if they follow the same pattern of forward and backward arcs. Specifically, by this we mean that $x_i x_{i+1}$ is a *forward arc* (respectively *backward arc*) of H if and only if $y_i y_{i+1}$ is a forward (respectively backward) arc of H. If P and Q as above are congruent walks in H, we say that P *avoids* Q, if there is no arc $x_i y_{i+1}$ in the same direction (forward or backward) as $x_i x_{i+1}$.

Note that a walk (or path) has a beginning and an end. A *reversal* of a walk $P = x_0, x_1, \ldots, x_n$ is the walk $P^{-1} = x_n, x_{n-1}, \ldots, x_0$.

An *invertible pair* in H is a pair of vertices u, v, such that

- there exist congruent walks P from u to v and Q from v to u, such that P avoids Q,
- and there exist congruent walks P' from v to u and Q' from u to v, such that P' avoids Q'.

Note that it is possible that P' is the reversal of P and Q' is the reversal of Q, as long as both P avoids Q and Q avoids P.

Let H be a digraph. We introduce the following auxiliary digraph H^+. The vertices of H^+ are all ordered pairs (u, v), where u, v are vertices of H. There is an arc from (u, v) to (u', v') in H^+ in one of the following situations:

- $uu' \in E(H), vv' \in E(H), uv' \notin E(H)$, or
- $u'u \in E(H), v'v \in E(H), v'u \notin E(H)$.

In the former case, we say the arc from (u, v) to (u', v') is a $+$ arc in H^+, in the latter case, we say it is a $-$ arc. Two directed walks in H^+ are *similar* if they have the same pattern of $+$ and $-$ arcs (thus, in particular, the same length).

We make the following observations:

1. u, v is an invertible pair if and only if $(u, v), (v, u)$ are in the same strong component of H^+;

2. H^+ contains an arc from (u, v) to (u', v') if and only if it contains an arc from (v', u') to (v, u);

3. if u, v is an invertible pair in H and (p, q) is in the same strong component of H^+ as (u, v), then p, q is also an invertible pair in H.

The observation 2 will be called the *skew-symmetry* of H^+. It is useful for proving many facts about H^+, including the observation 3 above.

THEOREM 3.1. *Suppose C is a strong component of H^+, and f a conservative polymorphism of H.*

If f is semi-lattice for a pair $(u, v) \in C$, then it is semi-lattice for all pairs in C.

If f is majority for a pair $(u, v) \in C$, then it is majority for all pairs in C.

If f is Maltsev for a pair $(u, v) \in C$, then it is Maltsev for all pairs in C.

Proof. If f is binary and semi-lattice for (u, v), then $f(u, v) = f(v, u)$. If (u, v) has an arc to (u', v') in H^+, then $f(u, v) = u$ implies $f(u', v') = u'$, and if it has an arc from (u', v'), then $f(u, v) = v$ implies $f(u', v') = v'$. Iterating over C, we conclude that f is semi-lattice over all pairs in C. (Only commutativity needs checking.) The proofs for the other polymorphisms are similar.

A *permutable triple* in H is a triple of vertices u, v, w together with six vertices $s(u), b(u)$, $s(v), b(v)$, $s(w), b(w)$, which satisfy the following condition.

- For any vertex x from u, v, w, there exists a walk $P(x, s(x))$ from x to $s(x)$ and two walks $P(y, b(x))$ (from y to $b(x)$), and $P(z, b(x))$ (from z to $b(x)$), congruent to $P(x, s(x))$, such that $P(x, s(x))$ avoids both $P(y, b(x))$ and $P(z, b(x))$ (here y and z are the other two vertices from u, v, w).

Note that since $P(x, s(x))$ avoids both $P(y, b(x))$, and $P(z, b(x))$, a permutable triple yields two similar directed walks $P(x; y), P(x; z)$ from (x, y) and (x, z) to $(s(x), b(x))$ in H^+.

Recall that an undirected asteroidal triple u, v, w is defined by the property that for any vertex x from u, v, w, there exists a walk joining the other two vertices which is avoided by the neighbours of x. Our definition of a permutable triple already sounds vaguely reminiscent of this. However, we shall need another technical condition.

A *digraph asteroidal triple* (DAT) is a permutable triple in which each of the three pairs $(s(u), b(u))$, $(s(v), b(v))$, and $(s(w), b(w))$ is invertible. This turns out to imply that the entire permutable triple "is inside" one strong component of H^+, in the following sense.

THEOREM 3.2. *If u, v, w is a DAT, then there exist, for each permutation x, y, z of u, v, w, walks $P(x, s(x))$, $P(y, b(x))$, and $P(z, b(x))$, as in the definition above, such that all the associated walks $P(x; y), P(x; z)$ lie entirely inside one fixed strong component C of H^+.*

In particular, all six pairs $(u, v), (v, u)$, (u, w), $(w, u), (v, w)$, (w, v), and all three pairs $(s(u), b(u))$, $(s(v), b(v))$, $(s(w), b(w))$ are invertible, and belong to C.

Proof. Indeed, consider in H the three vertices u, v, w of a DAT, the three invertible pairs $s(u), b(u)$, and $s(v), b(v)$, and $s(w), b(w)$, and the nine walks $P(u, s(u))$, $P(v, b(u))$, $P(w, b(u))$, $P(v, s(v))$, $P(u, b(v))$, $P(w, b(v))$, and $P(w, s(w))$, $P(u, b(w))$, $P(v, b(w))$, from the definition. Consider now the walks $P(u; v), P(u; w)$, and $P(v; u)$, $P(v; w)$, and $P(w; u), P(w, v)$, joining $(u, v), (u, w)$ to $(s(u), b(u))$, and $(v, u), (v, w)$ to $(s(v), b(v))$, and $(w, u), (w, v)$ to $(s(w), b(w))$, respectively, in H^+.

Suppose x is any vertex from u, v, w. Since $s(x), b(x)$ is an invertible pair, we also have in H^+ a directed walk $Q(x)$ from$(s(x), b(x))$ to $(b(x), s(x))$. Consider now the following directed walk from (u, v) to (v, u): concatenate $P(u; v)$ from (u, v) to $(s(u), b(u))$, with $Q(u)$ from $(s(u), b(u))$ to $(b(u), s(u))$, and then concatenated with the skew-symmetric walk to $P(u; v)$ (taking us from $(b(u), s(u))$ to (v, u)). Replacing the last segment by the skew-symmetric walk to $P(u; w)$ yields a directed walk from (u, v) to (w, u). By similar concatenations we see that all the pairs $(u, v), (v, u)$, (u, w), (w, u), (v, w), (w, v), as well as all vertices on the walks $P(u; v), P(u; w)$, $P(v; u)$, $P(v; w)$, $P(w; u), P(w, v)$, including $(s(u), b(u))$, $(s(v), b(v))$, $(s(w), b(w))$ are in the same component of H^+.

We have already observed that a strong component C of H^+ either has no invertible pairs, or all pairs are invertible (observation 3, above). It is easy to check that we also have the following property:

4. if u, v, w is a permutable triple in H, and (x, y) is in the same component of H^+ as $(s(u), b(u))$, then there exists a walk $P(u, x)$ (from u to x) and two walks $P(v, y)$ (from v to y), and $P(w, y)$ (from w to y), both congruent to $P(u, x)$, such that $P(u, x)$ avoids both $P(v, y)$ and $P(w, y)$.

This allows us to define a permutable triple u, v, w as having just one common pair $(s, b) = (s(u), b(u)) = (s(u), b(u)) = (s(u), b(u))$. We may call a pair s, b a *base pair* if there is a permutable triple u, v, w with $(s, b) = (s(u), b(u)) = (s(u), b(u)) = (s(u), b(u))$. Property 4 says that a strong component C of H^+ either has no base pairs, or all pairs are base pairs. Thus H has a DAT if and only if some component of H^+ contains both invertible pairs and base pairs.

We note, for future reference, that in the proof of Theorem 3.2 we have only used walks from $(b(u), s(u))$ to $(s(u), b(u))$, from $(b(v), s(v))$ to $(s(v), b(v))$, and from $(b(w), s(w))$ to $(s(w), b(w))$. (Although invertibility of these pairs also ensures directed walks from $(b(u), s(u))$ to $(s(u), b(u))$ and so on, we did not use these walks in the proof.)

We are now ready to state our main result.

THEOREM 3.3. *Let H be a digraph.*

If H contains a DAT, the problem LHOM(H) is NP-complete.

If H is DAT-free, the problem LHOM(H) is polynomial time solvable.

Deciding whether or not a given digraph H contains a DAT is easily seen to be polynomial in the size of $V(H)$. One only needs to check for connectivity properties in suitable auxiliary digraph defined on the triples of vertices of H. Specifically, let H^{++} be the digraph with the vertex set $V(H)^3$ and an arc from (u, v, w) to (u', v', w') just when H has arcs uu', vv', ww' but not uv' and not uw', or H has an arcs $u'u, v'v, w'w$ but not $v'u$ and not $w'u$. Then u, v, w is a DAT if and only if for every permutation x, y, z of u, v, w, the digraph H^{++} contains an invertible pair s, b such that (s, b, b) is reachable from (x, y, z).

We conclude this section with the following observation.

THEOREM 3.4. *Let H be a reflexive digraph and U the underlying graph of H.*

If U contains an asteroidal triple then H contains a DAT.

Proof. Suppose u, v, w is an asteroidal triple in U, with $P(u, v)$ (respectively $P(u, w)$, respectively $P(v, w)$) a path in U not containing any neighbours w (respectively of v, respectively of u). We first note that each of the pairs u, v and u, w and v, w is invertible in H: for instance to obtain a walk from u to v avoiding a walk from v to u, it suffices to follow the path in H corresponding to $P(u, w)$ while taking the corresponding number of loops at v, then taking loops at w while following the path in H corresponding to the reverse of $P(u, v)$, and finally taking the path in H corresponding to the reverse of $P(v, w)$, while taking loops at u. Now we observe that u, v, w is a DAT: for instance there is a walk from u to u, consisting of loops at u, which avoids both the path corresponding to $P(v, w)$ and the congruent walk consisting of loops at w.

4 Conservative Polymorphisms.

It is easy to see that if a digraph (or other relational structure) H admits a conservative polymorphism defined by identities, such as the semi-lattice, majority, or Maltsev polymorphisms, then so does any induced subgraph (substructure) of H. Thus one may want to characterize such digraphs by a forbidden substructure characterization. For example, for reflexive undirected graphs H [5, 14], a conservative majority exists if and only if H is an interval graph, i.e., does not admit an asteroidal triple or a chordless cycle of length greater than three [34].

The following results illustrate our approach, and introduce a technique used in the proof of Theorem 5.2.

THEOREM 4.1. *A digraph H admits a conservative majority polymorphism if and only if it has no permutable triple.*

Proof. Let f be a conservative polymorphism on H, and suppose that u, v, w is a permutable triple on H. Assume that $f(u, v, w) = u$. There exist mutually congruent walks from u, v, w to $s(u), b(u), b(u)$ respectively, where the first walk avoids the second two walks. Let u_1, v_1, w_1 be the first vertices on these walks (respectively), just after u, v, w. We must have $f(u_1, v_1, w_1) = u_1$, since u lacks the right kind of arc to v_1 and w_1. Similarly, for the i-th vertices of these walks, we must have $f(u_i, v_i, w_i) = u_i$. This would imply that $f(s(u), b(u), b(u)) = s(u)$, which is impossible if f is majority on the pair $s(u), b(u)$. Symmetric arguments handle the cases when $f(u, v, w) = v$ and $f(u, v, w) = w$. Thus there is no conservative majority function.

Next, assume that there is no permutable triple in H. We proceed to define a conservative majority function f on H as follows. Consider three vertices u, v, w . Let x be one vertex of u, v, w, and y, z the other two vertices. We say that x is a *distinguisher* for x, y, z, if for any three mutually congruent walks from x, y, z to any s, b, b respectively, the first walk does not avoid one of the other two walks. Since there is no permutable triple, at least one of u, v, w must be a distinguisher for u, v, w. This definition can be applied even if the vertices u, v, w are not distinct. Note that if, say, $u = v$ then no walk starting in u can avoid a walk starting in v, whence u is a distinguisher, and similarly for v, w.

We define the values of f as follows: for a triple (u, v, w), we set $f(u, v, w)$ to be the first vertex from u, v, w, in this order, that is a distinguisher for u, v, w. Note that the last remark ensures that if $u = v$ or $u = w$, then $f(u,v,w) = u$. On the other hand, if $v = w$, it is possible that $u, u \neq v$ is a distinguisher of u, v, w, and we make an exception and define $f(u,v,w) = v$.

It remains to show f is a polymorphism. Thus, for contradiction, suppose $uu' \in E(H)$, $vv' \in E(H)$, $ww' \in E(H)$, and $f(u,v,w)f(u',v',w') \notin E(H)$. (A symmetric proof applies if $u'u \in E(H)$, $v'v \in E(H)$, $w'w \in E(H)$, and $f(u',v',w')f(u,v,w) \notin E(H)$.) This clearly implies that at least one of the triples u, v, w or u', v', w' are distinct vertices. Suppose one of the triples, say u, v, w has a repetition. If $u = v$, then $f(u,v,w) = u$, and $f(u',v',w') = w'$ (since $f(u,v,w)f(u',v',w') \notin E(H)$). This contradicts the fact that w' is a distinguisher, since the walk $w'w$ avoids both paths $u'u, v'w$. A similar proof applies if $u = w$. If $v = w$, we have defined $f(u,v,w) = v$ regardless of whether u is a distingiusher, so we have the same proof as well. It remains to consider the case when both triples x, y, z and x', y', z' consist of distinct vertices. Assume first that $f(x,y,z) = x$, $f(x',y',z') = y'$. Note that $xz' \notin E(H)$, else y' would not be a distinguisher of x', y', z', because of the paths $y'y, x'x, z'x$. The fact that $f(x',y',z') = y'$ means that x' is not a distinguisher for x', y', z', thus there exist congruent walks X', Y', Z' from x', y', z' respectively, such that Y', Z' end at the same vertex, and X' avoids Y' and Z'. Then xx' followed by X', together with yy' followed by Y' and zz' followed by Z' are also congruent walks, and the walk xx', X' avoids the other two walks, contradicting the fact that x is a distinguisher for x, y, z.

Because of the symmetry between x, y, z and x', y', z', the only other cases to consider are $f(x,y,z) = x, f(x',y',z') = z'$ and $f(x,y,z) = y, f(x',y',z') = z'$; in both situations a similar proof applies.

The proof implies the following fact.

Corollary 4.1. *A conservative ternary polymorphism on H cannot be majority on all three pairs $(s(u), b(u))$, $(s(v), b(v))$, and $(s(w), b(w))$, of any permutable triple u, v, w.*

We define N to be any digraph with four vertices a, a', b, b' which contains the arcs aa', ab', bb' but not the arc ba'. We define *an end* in H to be four vertices a, a', b, b', and three congruent walks, $P(a,a')$ from a to a', $P(a,b')$ from a to b', and $P(b,b')$ from b to b', such that $P(b,b')$ avoids $P(a,a')$. Note that an end with paths of length one is a copy of N.

We now define a *strong end* to be an end with vertices a, a', b, b' and walks, $P(a,a'), P(a,b'), P(b,b')$, such that additionally $P(b,b')$ avoids $P(a,b')$ (except at the last vertex). Note that each N is also a strong end. We claim that any end in H contains a strong end. Indeed, if i is the smallest subscript such that the i-th vertex of $P(b,b')$ has an arc to the $(i+1)$-st vertex x (where $x \neq b'$) of $P(a,b')$, forward or backward according to $P(b,b')$, then there is a smaller end over a, b, x and the $(i+1)$-st vertex of $P(a,a')$, which is strong.

Theorem 4.2. *A digraph H admits a conservative Maltsev polymorphism if and only if it has no end.*

Proof. Let f be a conservative polymorphism on H. We may assume that a, a', b, b' is a strong end in H. Thus there exist mutually congruent walks $P(a,a'), P(a,b'), P(b,b')$ from a, a, b to a', b', b' respectively, where the third walk avoids the first two walks. If f is Maltsev on a, b, then $f(a,a,b) = b$, and since $P(b,b')$ avoids $P(a,a')$ and $P(a,b')$, we derive $f(a',b',b') = b'$, which is impossible if f is also Maltsev on a', b'.

For the converse, assume that there is no end in H. We proceed to define a conservative Maltsev polymorphism f on H as follows. Consider three vertices u, v, w. If there do not exist vertices s and b and congruent walks $P(u,b)$ from u to b, $P(v,b)$ from v to b, and $P(w,s)$ from w to s, such that both $P(u,b)$ and $P(v,b)$ avoid $P(w,s)$, then we define $f(u,v,w) = u$. If such vertices s, b, and walks $P(u,b), P(v,b), P(w,s)$ do exist, then we claim there cannot exist vertices s', b' and congruent walks $P(u,s'), P(v,b'), P(w,b')$, such that both $P(v,b')$ and $P(w,b')$ avoid $P(u,s')$. (Otherwise the concatenation of $P^{-1}(u,s')$ with $P(u,b)$, of $P^{-1}(v,b')$ with $P(v,b)$, and of $P^{-1}(w,b')$ with $P(w,s)$ would form an end.) In this case we define $f(u,v,w) = w$. Note that in the degenerate cases when the paths have length zero, we obtain the Maltsev equations. It can be checked that f is a polymorphism. For instance, suppose that ux, vy, wz are arcs of H, and $f(u,v,w) = u$, $f(x,y,z) = z$, but uz is not an arc. This means there exist congruent walks $P(x,b'), P(y,b'), P(z,s')$, such that both $P(x,b')$ and $P(y,b')$ avoid $P(z,s')$. Extending these walks by the arcs ux, vy, wz yields congruent walks from u to b', from v to b', and w to s'. Since $f(u,v,w) = u$, there must be an arc from v to z, else the two former walks would avoid the latter walk. This means that we obtain an end formed by the walks from u to b', from v to b' and from v to s', a contradiction.

Corollary 4.2. *A conservative ternary polymorphism on H cannot be Maltsev on both pairs a, b and a', b' of any strong end in H.*

We have similar results on digraphs which admit conservative semi-lattice polymorphisms. In particular [17], a reflexive digraph admits a conservative semi-lattice (i.e., a min ordering) if and only if it has no invertible pairs. A similar situation occurs for bipartite digraphs. However, we do not at this time have a characterization of general digraphs with a conservative semi- lattice (min ordering). For structures with two binary relations, the existence of conservative semi-lattice is NP-complete, [1].

Finally, we observe the following fact.

PROPOSITION 4.1. *A conservative binary polymorphism on H cannot be semi-lattice on any invertible pair in H.*

Proof. Suppose the conservative binary polymorphism f is semi-lattice on the invertible pair u, v and assume $f(u, v) = u$. Consider the congruent walks P from u to v and Q from v to u, such that P avoids Q. It follows that $f(v, u) = v$, contradicting the fact that f is commutative on u, v.

5 The Dichotomy.

We first prove the following fact.

THEOREM 5.1. *If H contains a DAT, then LHOM(H) is NP-complete.*

In the interest of brevity, we give here a short proof using the results of the previous section, together with Theorem 2.1. We have given a direct combinatorial proof in [25].

Proof. Suppose u, v, w is a DAT in H. Then by Theorems 3.1 and 3.2, if there is a conservative semi-lattice, majority, or Maltsev polymorphism for one of the invertible pairs $(s(u), b(u))$, $(s(v), b(v))$, $(s(w), b(w))$, then the same conservative polymorphism is semi-lattice, majority, or Maltsev for all three pairs. We know such a polymorphism cannot be semi-lattice on the pairs, by Proposition 4.1. Similarly, Corollary 4.1 implies that it cannot be majority on the pairs. We now prove it cannot be Maltsev on the pairs, and thus LHOM(H) is NP-complete by Theorem 2.1.

The claim is proved by showing that a DAT contains a strong end joining two pairs in C and applying Corollary 4.2. Consider the walks $P(u, s(u))$, $P(v, b(u))$, $P(w, b(u))$ and $P(u, b(w))$, $P(v, b(w))$, $P(w, s(w))$ from the definition of DAT. Since $P(w, s(w))$ avoids $P(u, b(w))$ and $P(v, b(w))$ by definition, we conclude that conversely, also $P(u, b(w)$ and $P(v, b(w))$ avoid $P(w, s(w))$. Otherwise there would be a copy of N joining two consecutive pairs of C (cf. Theorem 3.2). This means that $P^{-1}(w, s(w))$ avoids both $P^{-1}(v, b(w))$ and $P^{-1}(u, b(w))$. Similarly, $P(w, b(u))$ and $P(v, b(u))$ avoid $P(u, s(u))$. We obtain an end consisting of the pairwise concatenations of $P^{-1}(u, b(w))$ with $P(u, s(u))$, of $P^{-1}(v, b(w))$ with $P(v, b(u))$, and of $P^{-1}(w, s(w))$ with $P(w, b(u))$. Recall the proof that each end contains a strong end. It can be checked that for this particular end, the resulting smaller end (which is strong), connects two pairs from C (cf. Theorem 3.2), contradicting Corollary 4.2. (Consider the position of the vertex x from that proof: if x is in $P(v, b(u))$, Theorem 3.2 applies; and x cannot be in $P^{-1}(v, b(w))$ because we would obtain an N from the fact that $P(w, s(w))$ avoids $P(v, b(w))$.)

On the other hand, we now proceed to show that a DAT-free digraph H has a tractable LHOM(H). We will again use Theorem 2.1, cf. also [4].

Specifically, we shall show that a DAT-free digraph H admits two special conservative polymorphisms - a binary polymorphism f and a ternary polymorphism g - such that for all pairs u, v of vertices of H, the restriction $f|_{u,v}$ is a semi-lattice, or the restriction $g|_{u,v}$ is a majority. It will then follow from Theorem 2.1 that for DAT-free digraphs H, the problem LHOM(H) is polynomial time solvable. It will also follow, using also Theorem 5.1, that we may omit the mention of Maltsev polymorphisms from the statement of Theorem 2.1, to obtain Corollary 2.1.

THEOREM 5.2. *Suppose H is a DAT-free digraph.*

Then H admits a binary polymorphism f and a ternary polymorphism g such that

- *if u, v is not invertible then $f|_{u,v}$ is semi-lattice, and*
- *if u, v is invertible then $g|_{u,v}$ is majority.*

COROLLARY 5.1. *If H is DAT-free, then LHOM(H) is polynomial time solvable.*

Proof. The rest of this section is devoted to the proof of Theorem 5.2. Thus we shall assume for the remainder of the section that H is a DAT-free digraph. We will first define the binary polymorphism f. To start, we define $f(x, x) = x$ for all vertices x. It remains to define f on pairs (x, y) that are vertices of H^+.

We say that the pair (x, y) is a *special* vertex of H^+ if there is a directed walk in H^+ from (x, y) to (y, x). Note that skew-symmetry of H^+ implies that if (x, y) has a directed walk to a special vertex, then (x, y) is also special. In particular, a strong component either has all its vertices special (in which case we say it is a *special component*), or none of its vertices are special. For each component C of H^+ we define the *coupled* component $C' = \{(u, v) : (v, u) \in C\}$. We say that C is *co-special* if C' is special. It is possible that $C = C'$, in which case we say that C is *self-coupled.* Note that a strong component C is self-coupled if and only if it is both special and co-special; this happens if and only if all pairs in C are invertible.

The *condensation* of a digraph H is obtained from H by identifying each strong component of H to a vertex and placing an arc between the shrunk vertices just if there was an arc between the corresponding strong components. The condensation of any digraph H is acyclic. The condensation of H^+ has a particular structure, arising from the properties of special and co-special strong components. Recall that if C_1 is special and reachable from C_2, then C_2 is also special; thus by skew-symmetry, if C_1 is co-special and can reach C_2, then C_2 is also co-special. This means that following any directed path in the condensation of H^+ we first encounter some (possibly none) special strong components, followed by at most one self-coupled strong component, and then by the co-special strong components corresponding to the initial special strong components, in the reverse order.

Recall that (cf. the proof of Theorem 3.1) if f is any binary polymorphism of H and (x, y) has an arc to (x', y') in H^+, then $f(x, y) = x$ implies $f(x', y') = x'$. This suggests the following definition. If (x, y) lies in a co-special strong component which is not self-coupled, we set $f(x, y) = x$. If (x, y) lies in a special strong component which is not self-coupled, we set $f(x, y) = y$. For all pairs (x, y) in self-coupled strong components, we set $f(x, y) = y$. It remains to define $f(x, y)$ for pairs (x, y) in strong components that are neither special nor co-special.

Consider now the subgraph X of the condensation of H^+ induced by vertices corresponding to strong components that are neither special not co-special. It is easy to see that this subgraph has a topological sort, i.e., a linear ordering of its vertices (strong components of H^+), $C_1, C_2, \ldots C_k, C_{k+1}, \ldots, C_{2k}$ such that any arc of H^+ goes from some C_i to some C_j with $i < j$, and such that each C_i, $i \leq k$, is coupled with the corresponding C_{2k+1-i}. (To see this, set C_{2k} be any vertex of out-degree zero in X, and let C_1 be the strong component of H^+ coupled to C_{2k}. Then remove C_1 and C_{2k} from X and repeat.) Now we set $f(x, y) = y$ for all strong components $C_1, C_2, \ldots C_k$ and $f(x, y) = x$ for all (x, y) in strong components $C_{k+1}, \ldots C_{2k}$, and $f(x, y) = y$ for all (x, y) in strong components $C_1, C_2, \ldots C_k$.

This defines a mapping of $V(H)^2$ to $V(H)$. The definition ensures that if $f(x, y) = x$, then $f(x', y') = x'$ for any (x', y') with an arc from (x, y).

It is easy to see that f is a polymorphism; indeed, suppose if xx', yy' are arcs of H and $f(x, y)f(x', y')$ is not an arc of H. Asume $f(x, y) = x, f(x', y') = y'$. (The other case $f(x, y) = y, f(x', y') = x'$ is similar.) Then there is an arc in H^+ from (x, y) to (x', y'), contradicting the property of f mentioned just above.

It also follows from the definition that $f(x, y) = f(y, x)$ for all pairs (x, y) in special but not co-special strong components, co-special but not special strong components, and all strong components that are neither special not co-special. In other words, $f(x, y) = f(y, x)$ holds unless (x, y) is in a self-coupled component, i.e., unless x, y is an invertible pair.

Thus f is a conservative polymorphism of H which is commutative on pairs that are not invertible. It follows that it is semi-lattice on those pairs. (For conservative polymorphisms, associativity for pairs follows easily from commutativity.)

Lemma 5.1. *The mapping f is a polymorphism of H, and is semi-lattice on all pairs (x, y) that are not invertible.*

We now proceed to define the ternary conservative polymorphism g. It also depends on the location of the pairs of arguments in the condensation of H^+, but it will be more convenient to define it in terms of the polymorphism f (which itself was defined in terms of the condensation of H^+).

The proof has some similarity to that of Theorem 4.1. Consider three vertices u, v, w . Let x be one vertex of u, v, w, and y, z the other two vertices. We say that x is a *weak distinguisher* for x, y, z, if for any three mutually congruent walks from x, y, z to $s(x), b(x), b(x)$ respectively, *such that $s(x), b(x)$ is an invertible pair*, the first walk does not avoid one of the other two walks. Since there is no DAT, at least one of u, v, w must be a weak distinguisher for u, v, w.

Given a triple (x, y, z) of vertices of H, we consider the six-tuple of values

$$G(x, y, z) = [f(x, y), f(y, x), f(y, z), f(z, y), f(x, z), f(z, x)]$$

each value being a vertex x, y, or z. Note that each value can occur at most four times, for instance z can not occur in the first or second coordinate of $G(x, y, z)$. Moreover, our definition of f ensures that if $f(x, y) \neq f(y, x)$, then $f(x, y) = y, f(y, x) = x$. We set $g(x, y, z)$ to be the value which occurs most frequently in the six-tuple $G(x, y, z)$. If there is a tie, we choose as $g(x, y, z)$ the first vertex amongst the tied vertices, in the order of preference, first x, then y, then z, that is a weak distinguisher for x, y, z. (The detailed consideration of cases below shows such a weak distinguisher always exists.)

Note that if x fails to be a weak distinguisher of x, y, z, then the pairs (x, y) and (x, z) have similar directed walks in H^+ to some $(s(x), b(x))$ that lies in a self-coupled strong component. It follows from our definition of f that this implies that $f(x, y) = y, f(x, z) = z$. This is helpful in deciding whether or not x, y, or z can fail to be a weak distinguisher. In particular, it implies that the value $g(x, y, z)$ chosen is always a weak distinguisher of x, y, z.

LEMMA 5.2. *The mapping g is a polymorphism of H, and is majority on all pairs (x, y) that are invertible.*

Proof. For ease of reference, we list here all the possible six-tuples $G(x, y, z)$ and the resulting $g(x, y, z)$ (written after the / sign). In the first case, we grouped several possibilities together by using "?" as a wild card.

1. $[x, x, ?, ?, x, x]/x, [y, y, y, y, ?, ?]/y, [?, ?, z, z, z, z]/z$
2. $[x, x, z, z, z, x]/x, [y, x, y, y, x, x]/x, [y, y, z, y, z, z]/y$
3. $[x, x, y, y, z, x]/x, [x, x, z, y, z, z]/z, [y, y, z, y, x, x]/y$
   $[y, y, z, z, z, x]/z, [y, x, y, y, z, z]/y, [y, x, z, z, x, x]/x$
4. $[y, x, z, y, x, x]/x, [y, x, z, y, z, z]/z, [x, x, z, y, z, x]/x$
   $[y, y, z, y, z, x]/y, [y, x, y, y, z, x]/y, [y, x, z, z, z, x]/z$
5. $[x, x, y, y, z, z]/x, [y, y, z, z, x, x]/x$
6. $[y, x, z, y, z, x]$: in this case any of x, y, z could fail to be a weak distinguisher; since each occurs twice in the six-tuple, $g(x, y, z)$ is just the first weak distinguisher of x, y, z, in the order x, y, z.

The claim about being majority on invertible pairs follows directly from the definition of g. In fact, the six-tuples $G(x, x, y)$, $G(x, y, x)$, and $G(y, x, x)$ each contain two values $f(x, x) = x$, and in case of invertible pairs x, y the other four values are evenly divided between x and y. Therefore $g(x, x, y) = g(x, y, x) = g(y, x, x) = x$ if x, y is an invertible pair. (We observe in passing, although we do not need it, that in fact $g(x, x, y) = g(x, y, x) = g(y, x, x) = x$ on all pairs x, y except those where $f(x, y) = f(y, x) = y$.)

We proceed to prove that g is a polymorphism of H. Thus we consider two triples $(x, y, z), (x', y', z')$ where xx', yy', zz' are arcs of H, and show that $g(x, y, z)g(x', y', z')$ is also an arc of H. If instead $x'x, y'y, z'z$ are arcs of H, then $g(x', y', z')g(x, y, z)$ is also an arc of H, by a proof that is literally the same, except it reverses all arcs listed here as going from the unprimed to the primed vertices. (In other words, the arcs we consider here as forward arcs are viewed as backward arcs.) We proceed by contradiction, and assume that $g(x, y, z)g(x', y', z')$ is not an arc of H. The proof is technical, but the arguments in most cases are very similar. Hence we will focus here only on the case when $g(x, y, z) = x, g(x', y', z') = y'$. (By symmetry, this covers also the case when $g(x, y, z) = y, g(x', y', z') = x'$. The other cases, checked by very similar arguments, are $g(x, y, z) = x, g(x', y', z') = z'$, and $g(x, y, z) = y, g(x', y', z') = z'$.)

Thus we shall assume throughout this proof that the pair (x, y) has an arc to (x', y') in H^+.

If four of the values in $G(x, y, z)$ are x, and four of the values in $G(x', y', z')$ are y', i.e., if both six-tuples are in case (1), then in particular $f(x, y) = x, f(x', y') = y'$. Since (x, y) has an arc to (x', y') in H^+, this contradicts the fact that f is a polymorphism.

This argument still applies if four of the values in $G(x, y, z)$ are x and three of the values in $G(x', y', z')$ are y', i.e., if the six-tuple of primed vertices is in cases (2, 3, 4), because we must either have $f(x, y) = x$, $f(x', y') = y'$ or $f(y, x) = x$, $f(y', x') = y'$. The latter case similarly contradicts the fact that f is a polymorphism, since skew-symmetry implies that (y', x') has an arc to (y, x) in H^+.

In the case three of the values in $G(x, y, z)$ are x and three of the values in $G(x', y', z')$ are y', i.e., both six-tuples are in the cases (2, 3, 4), similar arguments handle all situations except when $f(y, x) = x, f(x, y) = y$, and $f(x', y') = y', f(y', x') = x'$. In this situation, we would have $f(x, z) = f(z, x) = x$, and $f(y', z') = f(z', y') = y'$. We must not have the arc xz' in H, else (y', z') would dominate (y, x) while $f(y', z') = y', f(y, x) = x$. Now (x, z) dominates (x', z') and hence $f(x', z') = f(z', x') = x'$. This means that $G(x', y', z')$ contains three x' and three y', and since both are weak distinguishers, we should have had $g(x', y', z') = x'$.

This leaves us to consider the cases where at least one of the triples x, y, z or x', y', z', say the first one, has two occurrences of each x, y, z. These are the situations in cases (5, 6), i.e., $[x, x, y, y, z, z]$, $[y, y, z, z, x, x]$, and $[y, x, z, y, z, x]$.

We continue to assume that $g(x, y, z) = x$ and $g(x', y', z') = y'$ and xy' is not an arc of H; recall that the pair (x, y) dominates the pair (x', y') in H^+.

In case $G(x, y, z) = [x, x, y, y, z, z]$, from $f(x, y) = x$ we obtain $f(x', y') = x'$ which further implies that $f(y', x') = x'$. Now there are at most two values y' in the six-tuple $G(x', y', z')$, and at least two values x'. Since $f(x', y') = x'$, the vertex x' is a weak distinguisher of x', y', z', and this contradicts the definition of $g(x', y', z')$.

In case $G(x, y, z) = [y, y, z, z, x, x]$, we first suppose that zy' is not an arc. Then (z, y) dominates (z', y'), and we have $f(y', z') = f(z', y') = z'$ (as $f(y, z) = f(z, y) = z$). On the other hand since (x, y) dominates (x', y') and $f(x, y) = f(y, x) = x$, we have $f(x', y') = f(y', x') = x'$. This contradicts the definition of $g(x', y', z') = y'$, since

x' occurs at least as frequently as y'. If zy' is an arc, then (x, z) dominates (x', y') and since $f(x, z) = f(z, x) = z$, we have $f(x', y') = f(y', x') = x'$. It follows that either z' occurs more frequently than y', or x' is a weak distinguisher and occurs at least as frequently as y', contradicting the definition of $g(x', y', z') = y'$.

Finally, in case of $[y, x, z, y, z, x]$, as (y', x') dominates (y, x) in H^+, we have $f(y, x) = x$ imply $f(y', x') = x'$. It is readily checked from (1-6) that now only the following cases from (3, 4, 6) lead to $g(x', y', z') = y'$:

- $[y', x', y', y', z', z']$
- $[y', x', y', y', z', x']$
- $[y', x', z', y', z', x']$

In the first case, $[y', x', y', y', z', z']$, we have $f(z', x') = z'$ and $f(z, x) = x$, thus xz' must be an arc of H. Now there are congruent walks (with one arc) in H from y', x', z' to y, x, x respectively, such that the first avoids the other two and x, y is an invertible pair (f is non-commutative only on invertible pairs); this means that y' is not a weak distinguisher of x', y', z' and contradicts the choice of $g(x', y', z') = y'$.

In the second case, $[y', x', y', y', z', x']$, similarly, we must have the arc zy' in H, and so we again have congruent walks (with one arc) in H from x, y, z to x', y', y' respectively, such that the first avoids the other two and x', y' is an invertible pair, implying that x is not a weak distinguisher of x, y, z, and contradicting the assumption that $g(x, y, z) = x$.

In the last case, we have all pairs in x, y, z and in x', y', z' invertible. Since x' is not a weak distinguisher of x', y', z', but x is a weak distinguisher of x, y, z, we must have in H the arc xz'; this arc yields congruent (one-arc) walks in H from y', x', z' to y, x, x, and x, y is an invertible pair. This again contradicts the choice of $g(x', y', z') = y'$.

Theorem 5.2 now follows from Lemmas 5.1 and 5.2.

6 Conclusions and Future Directions.

We have several applications of Theorem 3.3, classifying the complexity of problems LHOM(H) for digraphs H restricted to some natural classes of digraphs. In these cases, it turns out that either a conservative semi-lattice or a conservative majority polymorphism suffices to cover all the tractable cases. In this note we shall only state some results without proof.

THEOREM 6.1. *Let H be a digraph whose underlying graph is a tree.*

If H has a min-ordering, then $LHOM(H)$ is polynomial time solvable.

Otherwise H contains a DAT and LHOM(H) is NP-complete.

THEOREM 6.2. *Let H be a digraph whose underlying graph is a cycle.*

If H has a conservative majority function, then LHOM(H) is polynomial time solvable.

Otherwise H contains a DAT and LHOM(H) is NP-complete.

Feder [13] studied the complexity of the (non-list) homomorphism problem for digraphs H whose underlying graph is a cycle; the result was quite complex and no concrete description was given. (Larose and Zadori [33] did classify the cases in terms of types [36].) The situation for list homomorphisms turned out to be significantly simpler; we will provide a concrete description of the tractable cases.

For reflexive digraphs, Conjecture 5.5 in [24] (cf. [21]) states that conservative semi-lattice polymorphisms suffice to cover the tractable cases. A plausible algebraic approach to this is in progress [10].

We close with a few open problems.

1. Investigate the class of DAT-free digraphs.
2. Find a more efficient algorithm to recognize if a digraph is DAT-free.
3. Find an algorithm for LHOM(H) when H is a DAT-free digraph without using [4, 8].

Regarding 1, 2, we point out that in the undirected case the class of AT-free graphs is quite popular, as it unifies several known graph classes, has interesting structural properties, and allows efficient algorithms for computational problems intractable in general [6, 11, 12]. The recognition problem for AT-free graphs is easily seen to be polynomial, but the search for really efficient recognition of AT-free graphs appears to be continuing [30]. For 3, one could perhaps show directly that DAT-free graphs admit a list homomorphism algorithm of width $(2, 3)$.

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