# Graphs and Combinatorics 

# On $\boldsymbol{n}$-partite Tournaments with Unique $\boldsymbol{n}$-cycle 

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#### Abstract

An $n$-partite tournament is an orientation of a complete $n$-partite graph. An $n$-partite tournament is a tournament, if it contains exactly one vertex in each partite set. Douglas, Proc. London Math. Soc. 21 (1970) 716-730, obtained a characterization of strongly connected tournaments with exactly one Hamilton cycle (i.e., $n$-cycle). For $n \geq 3$, we characterize strongly connected $n$-partite tournaments that are not tournaments with exactly one $n$-cycle. For $n \geq 5$, we enumerate such non-isomorphic $n$-partite tournaments.


## 1. Introduction

We use terminology and notation of [1]; all necessary notation and a large part of terminology used in this paper are provided in the next section.

The problems of characterizing strongly connected tournaments having a unique Hamilton cycle and finding the number of such tournaments have attracted several researches. R.J. Douglas [4] gave a structural characterization of tournaments having a unique Hamilton cycle. This result implies a formula for the number $s_{n}$ of non-isomorphic tournaments of order $n$ with a unique Hamilton cycle. This characterization as well as the formula are rather complicated. G.P. Egorychev [5] obtained a simplification by considering a generating function for the number of tournaments with unique Hamiltonian cycle. M.R. Garey [6] showed that $s_{n}$ could be expressed as a Fibonacci number ( $s_{n}=f_{2 n-6}$ ); his derivation was based on Douglas's characterization. J.W. Moon [13] obtained a direct proof of Garey's formula that is essentially independent of Douglas's characterization.

A classical result of J.A. Bondy [3] on short cycles in strong $n$-partite tournaments implies that every strong $n$-partite tournament contains an $n$-cycle.

In this paper, we provide a characterization of strongly connected $n$-partite tournaments ( $n \geq 3$ ) that are not tournaments with a unique $n$-cycle. We show that our characterization is polynomial time verifiable. The characterization allows us to find easily the number of such non-isomorphic $n$-partite tournaments for $n \geq 5$. Thus, our work is a natural continuation of the papers mentioned in the previous paragraph.

Our result is also related to a question of L. Volkmann [14]: let $n \geq 4$, is there a strongly connected $n$-partite tournament with at least two vertices in some partite
set and with exactly $n-m+1$ cycles of length $m$ for each fixed $m \in\{4,5, \ldots, n\}$ ? The first two authors [12] gave an affirmative answer to this question and raised a natural problem of characterizing such $n$-partite tournaments. Our paper solves the problem for $n=m$, which seems the most interesting case.

A recent paper [14] of L. Volkmann is the latest survey on cycles in multipartite tournaments. Cycles in multipartite tournaments were earlier overviewed in [2, 8, 10].

## 2. Terminology and Notation

A digraph obtained from an undirected graph $G$ by replacing every edge of $G$ with a directed edge (arc) with the same end-vertices is called an orientation of $G$. An oriented graph is an orientation of some undirected graph. A tournament is an orientation of a complete graph, and an n-partite tournament is an orientation of a complete $n$-partite graph. Partite sets of complete multipartite graphs become partite sets of $n$-partite tournaments.

The terms cycle and path mean simple directed cycle and path. A cycle of length $k$ is a $k$-cycle. For a cycle $C=v_{1} v_{2} \ldots v_{k} v_{1}, C\left[v_{i}, v_{j}\right]$ denotes the path $v_{i} v_{i+1} \ldots v_{j}$ which is part of $C$. A digraph $D$ is strong (or, strongly connected) if for every ordered pair $x, y$ of vertices in $D$ there exists a path from $x$ to $y$. For a set $X$ of vertices of a digraph $D, D\langle X\rangle$ denotes the subdigraph of $D$ induced by $X$.

For sets $T, S$ of vertices of a digraph $D=(V, A), T \rightarrow S$ means that for every vertex $t \in T$ and for every vertex $s \in S$, we have $t s \in A$, and $T \Rightarrow S$ means that for no pair $s \in S, t \in T$, we have $s t \in A$. While for oriented graphs $T \rightarrow S$ implies $T \Rightarrow S$, this is not always true for general digraphs. If $u \rightarrow v$ (i.e., $u v \in A$ ), we say that $u$ dominates $v$ and $v$ is dominated by $u$.

For a digraph $D=(V, A)$ and a vertex $x \in V, N^{+}(x)=\{y \in V: x y \in A\}$. Similarly, $N^{-}(x)=\{y \in V: y x \in A\}$. We denote, by $\mathcal{U} C_{n}$, the set of all strong $n$-partite tournaments, $n \geq 3$, which are not themselves tournaments, with exactly one cycle of length $n$.

We call $X=X_{1} \cup X_{2} \cup \ldots \cup X_{s}$ a partition of $X$ and denote it by $X=X_{1} \uplus X_{2} \uplus$ $\ldots \uplus X_{s}$, if $X_{i} \cap X_{j}=\emptyset$ for each $i \neq j$.

## 3. The Case of $n \geq 4$

We start from two known results.

Theorem 3.1. [9] Every partite set of a strong n-partite tournament, $n \geq 3$, contains a vertex which lies on an $m$-cycle for each $m \in\{3,4, \ldots, n\}$.

Theorem 3.2. [7] Every vertex in a strong n-partite tournament, $n \geq 3$, belongs to a cycle that contains vertices from exactly $q$ partite sets for each $q \in\{3,4, \ldots, n\}$.

Lemma 3.3. Let $D$ be a digraph with a cycle C. Iffor some vertices $x, y \in V(D)-V(C)$ we have $\left|N^{+}(x) \cap V(C)\right|+\left|N^{-}(y) \cap V(C)\right|>|V(C)|$, then there is an $(x, y)$-path of length $l$ in $D$, for all $l=2,3, \ldots,|V(C)|+1$.

Proof. Suppose that there was no $(x, y)$-path of length $l$ in $D$. Let $z \in N^{-}(y) \cap V(C)$ and let $u$ be the vertex on $C$ such that the path from $u$ to $z$ in $C$ has length $l-2$. Observe that $u$ does not dominate $x$. Thus, $\left|N^{+}(x) \cap V(C)\right| \leq|V(C)|-\left|N^{-}(y) \cap V(C)\right|$, a contradiction.

Lemma 3.4. If $n \geq 3, D \in \mathcal{U} C_{n}$ and $C$ is the unique $n$-cycle in $D$, then $C$ contains vertices from all partite sets. There is a vertex $y \in D-V(C)$ such that $D\langle V(C) \cup y\rangle$ is strong.

Proof. Let $D \in \mathcal{U} C_{n}$ and let $C$ be its unique $n$-cycle. If $C$ contains vertices from less than $n$ partite sets, then by Theorem 3.1 there exists another $n$-cycle, which is impossible. Therefore $C$ contains a vertex from every partite set of $D$. Let $V_{1}, V_{2}, \ldots, V_{n}$ be partite sets of $D$ and let $C=v_{1} v_{2} \ldots v_{n} v_{1}, v_{i} \in V_{i}, i=1,2, \ldots, n$.

Assume that there is no vertex $y \in D-V(C)$ for which $D\langle V(C) \cup y\rangle$ is strong. Then the following two sets $S$ and $T$ are non-empty: $S(T)$ is the set of vertices in $D-V(C)$ that do not dominate (are not dominated by) any vertex in $C$. Since $D$ is strong, there exist vertices $u \in S$ and $v \in T$ such that $u \rightarrow v$. By Lemma 3.3 we obtain an $n$-cycle containing $u$ and $v$, a contradiction.

Lemma 3.5. Let $n \geq 4$, let $D \in \mathcal{U} C_{n}$ and let $C$ be the unique $n$-cycle in $D$. We may order the partite sets in $D$ such that $C=v_{1} v_{2} \ldots v_{n} v_{1}$, where $v_{i} \in V_{i}$ and $v_{i} \rightarrow v_{j}$ for all $1<j+1<i \leq n$. If $y \in V(D)-V(C)$ such that $D\langle V(C) \cup y\rangle$ is strong, then one of the following statements hold.
(i): $y \in V_{2}, y \rightarrow\left(\left\{v_{1}\right\} \cup\left\{v_{3}, v_{4}, \ldots, v_{n-1}\right\}\right)$ and $v_{n} \rightarrow y$.
(ii): $y \in V_{n-1},\left(\left\{v_{n}\right\} \cup\left\{v_{n-2}, v_{n-3}, \ldots, v_{2}\right\}\right) \rightarrow y$ and $y \rightarrow v_{1}$.

Proof. By Lemma 3.4 $D\langle V(C)\rangle$ is a tournament and there exists a vertex $y \in V(D)-$ $V(C)$ such that $D\langle V(C) \cup y\rangle$ is strong. Let $C=w_{1} w_{2} \ldots w_{n} w_{1}$ and assume that $y$ belongs to the same partite set as $w_{i}$. By Theorem 3.2, $y$ lies in a cycle $C^{\prime}$ of $D\langle V(C) \cup y\rangle$ that contains vertices from exactly $n-1$ partite sets. If $C^{\prime}$ contains $w_{i}$, then the length of $C^{\prime}$ is $n$, a contradiction. Thus, $C^{\prime}$ does not contain $w_{i}$ and a vertex $w_{j}, j \neq i$. Let $T=D\left\langle V\left(C^{\prime}\right) \cup w_{j}\right\rangle ; T$ is a tournament. If $T$ is strong, then it has a Hamilton cycle, which is an $n$-cycle in $D$ distinct from $C$, a contradiction. Therefore we may assume that $T$ is not strong.

Thus, either $C^{\prime} \rightarrow w_{j}$ or $w_{j} \rightarrow C^{\prime}$. First assume that $C^{\prime} \rightarrow w_{j}$ (we will show that this implies (i) above). Since every vertex in $C$ dominates a vertex, $C^{\prime} \rightarrow w_{j}$ implies $w_{j}=w_{i-1}$. Let $C^{\prime}=z_{1} z_{2} \ldots z_{n-1} z_{1}$ where $z_{n-1}=y$. Note that if $w_{i} \rightarrow z_{r}$, for any $r>1$, then $w_{i} C^{\prime}\left[z_{r}, z_{r-2}\right] w_{i-1} w_{i}$ is an $n$-cycle containing $y=z_{n-1}$, and thus distinct from $C$, a contradiction. Therefore $z_{1}=w_{i+1}$ and $V\left(C^{\prime}\right)-\left\{w_{i+1}\right\} \Rightarrow w_{i}$.

We will now show that $z_{q} \rightarrow z_{k}$ for all $1<k+1<q \leq n-1$. To prove this claim, assume that $z_{k} \rightarrow z_{q}$ for some $1<k+1<q \leq n-1$. If $q=k+2$, then the
cycle $w_{i} C^{\prime}\left[z_{1}, z_{q-2}\right] C^{\prime}\left[z_{q}, z_{n-1}\right] w_{i-1} w_{i}$ gives us a contradiction. Assume that $q$ is maximum, and subject to that $k$ is maximum. Then the cycle $w_{i} C^{\prime}\left[z_{1}, z_{k}\right] C^{\prime}\left[z_{q}, z_{n-1}\right]$ $C^{\prime}\left[z_{k+2}, z_{q-1}\right] w_{i-1} w_{i}$ gives us a contradiction. Since $C=w_{i} C^{\prime}\left[z_{1}, z_{n-2}\right] w_{i-1} w_{i}$, we have proved the first part of the lemma by letting $v_{1}=w_{i-1}, v_{2}=w_{i}, \ldots, v_{n}=w_{i-2}$. In order to prove (i), we observe that $y \in V_{2}$ follows immediately from the fact that $y$ and $w_{i}=v_{2}$ lie in the same partite set. The facts that $y \rightarrow\left\{v_{1}, v_{3}, v_{4}, \ldots, v_{n-1}\right\}$ and $v_{n} \rightarrow y$ also follow from the above as $y=z_{n-1}$, which dominates $w_{i-1}$ and all $z_{k}$ with $1 \leq k \leq n-3$.

The case when $w_{j} \rightarrow C^{\prime}$ can be considered analogously to the above and, in particular, implies (ii).

In what follows, $D \in \mathcal{U} C_{n}$ and $C=v_{1} v_{2} \ldots v_{n} v_{1}$ is the unique $n$-cycle of $D$. Assume that the ordering of the partite sets satisfies the conditions of Lemma 3.5.

Let us partition the vertices of $V_{i}-v_{i}(1 \leq i \leq n)$ into four classes. We say that $x \in V_{i}-\left\{v_{i}\right\}$ is of type 1 if it satisfies condition (i) of Lemma 3.5, $x$ is of type 2 if it satisfies condition (ii) of Lemma 3.5, and $x$ is of type 3 if $V(C) \Rightarrow x$ and finally $x$ is of type 4 if $x \Rightarrow V(C)$. Let $T_{i}$ denote the set of vertices of type $i(1 \leq i \leq 4)$ in $D-V(C)$.

Remark. From the last lemma, we have that the vertices of type 1 are all in one partite set, $V_{2}$, the vertices of type 2 are all in the same partite set, $V_{n-1}$.

Lemma 3.6. If $n \geq 4$, then $\left(T_{1} \cup T_{4}\right) \Rightarrow\left(T_{2} \cup T_{3}\right)$.

Proof. If there is an arc $u w$ from $T_{2} \cup T_{3}$ to $T_{1} \cup T_{4}$, then we observe that $\mid N^{+}(w) \cap$ $V(C)\left|+\left|N^{-}(u) \cap V(C)\right| \geq(n-2)+(n-2) \geq n\right.$. We are now done by Lemma 3.3, unless equality holds, in which case $n=4, u \in T_{2}$ and $w \in T_{1}$. However in this case the 4 -cycle $u w v_{1} v_{2} u$, gives us the desired contradiction.

Remark. It is possible that $T_{1}$ or $T_{2}$ is empty, but by Lemmas 3.4 and 3.5 at least one of them is non-empty.

Lemma 3.7. Let $n \geq 4$. Let $R_{0}=T_{1}$ and $Q_{0}=T_{2}$, and define $R_{i}$ and $Q_{i}, i \geq 1$, as follows.
$R_{i}=\left\{r \in T_{4}:\right.$ the shortest path from $R_{0}$ to $r$ in $D-V(C)-T_{2}-T_{3}$ has length $\left.i\right\}$.
$Q_{i}=\left\{q \in T_{3}:\right.$ the shortest path from $q$ to $Q_{0}$ in $D-V(C)-T_{1}-T_{4}$ has length $\left.i\right\}$.
Note that $R_{j} \Rightarrow R_{i}$ and $Q_{i} \Rightarrow Q_{j}$ for all $j>i+1 \geq 1$, by definition. The following must also hold for $i \geq 0$.
(a) $R_{i} \subseteq V_{\left(i \bmod _{n-1)+2}\right.}$
(b) $Q_{i} \subseteq V_{(n-2-i} \bmod _{n-1)+1}$

Proof. We start of by proving (a) by induction on $i$. The claim (a) is clearly true for $i=0$, by Lemma 3.5, so we may assume that $i>0$. Also assume that $i<n-1$. If there exists a vertex $r_{i} \in R_{i}-V_{i+2}$, then let $P$ be a path of length $i$ from $r_{0} \in R_{0}$ to $r_{i}$ and observe that $P C\left[v_{i+2}, v_{n}\right] r_{0}$ is an $n$-cycle, a contradiction. Therefore $R_{i} \subseteq V_{i+2}$ when $i<n-1$.

If $i \geq n-1$, then $R_{i} \Rightarrow R_{i-(n-1)}$ by the definition of sets $R_{i}$. This would give us an $n$-cycle different from $C$, unless all vertices in $R_{i}$ belong to the same partite set as all vertices in $R_{i-(n-1)}$. We are now done by induction hypothesis as $\left.R_{i-(n-1)} \subseteq V_{((i-(n-1)} \bmod n-1\right)+2=V_{(i \bmod n-1)+2}$.
(b) can be proved analogously to (a).

Lemma 3.8. Consider $R_{i}$ and $Q_{i}$ as defined in Lemma 3.7. The following now holds.
(a) If $n \geq 5$ then $R_{i} \rightarrow R_{i+1}$ and $Q_{i+1} \rightarrow Q_{i}$ for all $i \geq 0$.
(b) If $n \geq 6$ and $\left|R_{i}\right|>0$ then $\left|R_{i-2}\right|=1$ for all $i \geq 2$.

Furthermore if $\left|Q_{i}\right|>0$ then $\left|Q_{i-2}\right|=1$ for all $i \geq 2$.

Proof. We first prove the first part of (a). Assume that it is not true and that there exists $r_{i} \in R_{i}$ and $r_{i+1} \in R_{i+1}$ with $r_{i+1} \rightarrow r_{i}$. Let $P=p_{0} p_{1} \ldots p_{i} r_{i+1}$ be a path from $R_{0}$ to $r_{i+1}$ in $D-V(C)-T_{2}-T_{3}$ of length $i+1$. Note that $r_{i} \notin P$. If $i+3 \geq n$, then we obtain the $n$-cycle $r_{i+1} r_{i} P\left[p_{i-n+3}, r_{i+1}\right]$, a contradiction (as $r_{i} \rightarrow p_{i-n+3}$, by Lemma 3.7 and the fact that $i-n+3 \leq i-2$ ). If $i+3<n$ then we obtain the $n$-cycle $r_{i+1} r_{i} C\left[v_{i+4}, v_{n}\right] P$, a contradiction. This proves the first part of (a).

The second part of (a) can be proved analogously.
We now prove the first part of (b). Assume that it is not true. Let $r_{i} \in R_{i}$ be arbitrary and let $P=p_{0} p_{1} \ldots p_{i-1} r_{i}$ be any path from $R_{0}$ to $r_{i}$ in $D-V(C)-T_{2}-T_{3}$ of length $i$. Let $r_{i-2} \in R_{i-2}-p_{i-2}$ be arbitrary. If $i+2 \geq n$, then we obtain the $n$-cycle $r_{i} r_{i-2} P\left[p_{i-n+2}, r_{i}\right]$, a contradiction (as $r_{i-2} \rightarrow p_{i-n+2}$, by Lemma 3.7 and the fact that $i-n+2 \leq i-4)$. If $i+2<n$ then we obtain the $n$-cycle $r_{i} r_{i-2} C\left[v_{i+3}, v_{n}\right] P$, a contradiction. This proves (b).

The second part of (b) is proved analogously.

Theorem 3.9. Let $n \geq 4$ and let $Z=\left\{v_{1}, v_{2}, v_{3}, \ldots, v_{n}\right\}$. Let $R_{0}, R_{1}, \ldots R_{a}$ and $Q_{0}, Q_{1}, \ldots, Q_{b}$ be disjoint non-empty sets (if $a=-1$ then there are no $R$-sets and if $b=-1$ then there are no $Q$-sets). Let $R=R_{0} \uplus R_{1} \uplus \ldots \uplus R_{a}$ and $Q=Q_{0} \uplus Q_{1} \uplus$ $\ldots \uplus Q_{b}$.

Let $D$ be a multipartite tournament with $V(D)=Z \uplus R \uplus Q$. Let $V_{1}, V_{2}, \ldots, V_{n}$ be the partite sets of $D$, where $v_{i} \in V_{i}, R_{i} \subseteq V_{\left(i \bmod _{n-1)+2}\right.}$ and $Q_{i} \subseteq V_{(n-2-i \bmod n-1)+1}$ for all $i$. Furthermore assume that the following statements hold.
(a) For all $i=1,2, \ldots, n-1$ we have $v_{i} \rightarrow v_{i+1}$.
(b) For all $1 \leq j<i-1<n$ we have $v_{i} \rightarrow v_{j}$.
(c) For all $0 \leq j<i-1<a$ we have $R_{i} \Rightarrow R_{j}$, and for all $0 \leq j<i-1<b$ we have $Q_{j} \Rightarrow Q_{i}$.
(d) We have $R \Rightarrow Z \Rightarrow Q$ and $R \Rightarrow Q$, except for $Q_{0} \rightarrow v_{1}$ and $v_{n} \rightarrow R_{0}$.
(e) If $n \geq 6$, then $\left|R_{0}\right|=\left|R_{1}\right|=\ldots=\left|R_{a-2}\right|=1$ and $\left|Q_{0}\right|=\left|Q_{1}\right|=\ldots=$ $\left|Q_{b-2}\right|=1$.
(f) If $n \geq 5$, then $R_{i} \rightarrow R_{i+1}$ for all $i=0,1,2, \ldots, a-1$, and $Q_{i+1} \rightarrow Q_{i}$ for all $i=0,1,2, \ldots, b-1$.
(g) If $n=4$, then the following holds. Every vertex in $R_{i+1}$ has an arc into it from $R_{i}$, for all $i=0,1, \ldots, a-1$. Every vertex in $Q_{i+1}$ has an arc into $Q_{i}$, for all $i=0,1, \ldots, b-1$. Furthermore there is no 4 -cycle in $D\langle R\rangle$ or $D\langle Q\rangle$.

Then $D$ contains a unique n-cycle. Any n-partite tournament, which is not a tournament, with a unique n-cycle, can be constructed as above.

Proof. We note that if $D$ is a $n$-partite tournament, which is not a tournament, containing a unique $n$-cycle, then all of the above must hold by the previous lemmas (note that (g) follows trivially).

Now we prove that if $D$ satisfies the above conditions then $C=v_{1} v_{2} v_{3} \ldots v_{n} v_{1}$ is a unique $n$-cycle in $D$. Assume that $C^{\prime}$ is another $n$-cycle. By (a) and (b), $C^{\prime}$ is not completely in $D\langle Z\rangle$. We claim that $D\langle R\rangle$ and $D\langle Q\rangle$ do not contain an $n$-cycle. This follows from (g) for $n=4$, from (c), (e), (f) and the fact that two $R$-sets or $Q$-sets with indices $n$ apart are in the same partite set for $n \geq 5$ (for $n \geq 6$ (e) is important to prevent an $n$-cycle visiting the same $R$-set or $Q$-set twice). As $R \Rightarrow Q$, this implies that there is no $n$-cycle in $D\langle R \cup Q\rangle$. This implies that $C^{\prime}$ must contain at least one vertex from $Z$. We now consider the following cases.

Case 1. Assume that $V\left(C^{\prime}\right) \cap R \neq \emptyset$ and $V\left(C^{\prime}\right) \cap Q \neq \emptyset$. By (d), $C^{\prime}$ must contain $v_{n}$ followed by vertices in $R$, then by vertices in $Q$ and then the vertex $v_{1}$. However then we must have a path from $v_{1}$ to $v_{n}$ in $D\langle Z\rangle$, and the only such path has length $n$, a contradiction to the length of $C^{\prime}$.

Case 2. Assume that $V\left(C^{\prime}\right) \cap R \neq \emptyset$ and $V\left(C^{\prime}\right) \cap Q=\emptyset$. As before $C^{\prime}$ must contain $v_{n}$ followed by vertices in $R$, then by vertices in $Z$, ending in $v_{n}$.

Let $r=\left|R \cap V\left(C^{\prime}\right)\right|$. If $n=4$ we get a contradiction by considering the cases $r=1,2,3$ separately. In particular, for $r=3$, the path $C^{\prime} \cap R$ has both initial and terminal vertices in $R_{0}$ or the initial vertex in $R_{0}$ and the terminal one in $R_{2}$. However, $v_{4} \rightarrow R_{0}$, and $v_{4}$ and $R_{2}$ are in the same partite set.

If $n=5$ and $r=4$, then the predecessor of $v_{n}$ on $C^{\prime}$ must lie in either $R_{0}$ or $R_{3}$, a contradiction. If $n \geq 6$, or $r \leq 3$ and $n=5$, then the only path with $r$ vertices in $D\langle R\rangle$ starting at $R_{0}$, has the form $s_{0} s_{1} s_{2} \ldots s_{r-1}$, where $s_{i} \in R_{i}$. Since the only path into $v_{n}$ with $n-r$ vertices in $D\langle Z\rangle$ is $v_{r+1} v_{r+2} \ldots v_{n}$, we get a contradiction ( $s_{r-1}$ and $v_{r+1}$ lie in the same partite set).

Case 3. Assume that $V\left(C^{\prime}\right) \cap R=\emptyset$ and $V\left(C^{\prime}\right) \cap Q \neq \emptyset$. We can obtain a contradiction analogously to Case 2.

Corollary 3.10. We can check whether a strong n-partite tournament, $n \geq 4$, belongs to $\mathcal{U} C_{n}$ in polynomial time.

Proof. Let $D$ be a strong $n$-partite tournament, $n \geq 4$. The above mentioned result of Bondy [3] implies that $D$ has an $n$-cycle. The constructive proof in [3] implies a polynomial algorithm to construct such a cycle $C$. If $C$ has two or more vertices from the same partite set, then by Theorem 3.1 $D \notin \mathcal{U} C_{n}$. Otherwise, we have to check the conditions of the above theorem that can be done in polynomial time. $\square$

Corollary 3.11. The number of non-isomorphic strong n-partite tournaments, $n \geq 6$, with $n+q$ vertices belonging to $\mathcal{U} C_{n}$ is

$$
t(n, q)=2+(q-1) q+\sum_{j=1}^{q-1}(1+j(j-1) / 2)(1+(q-j)(q-j-1) / 2)
$$

The number of non-isomorphic strong 5-partite tournaments, with $q+5$ vertices belonging to $\mathcal{U} C_{5}$ is $t(5, q)=(q+3) 2^{q-2}$.

Proof. It follows from Theorem 3.9 that $t(n, q)$, for $n \geq 5$, equals the number of possible partitions of sets $R$ and $Q$ (or of the non-empty one, if one of them is empty).

Let $n \geq 6$. If $R=\emptyset$, then $q$ vertices of $Q$ may all be in $Q_{0}$ or exactly one vertex in each of $Q_{0}, Q_{1}, \ldots, Q_{b-2}$ and $q-b+1$ vertices are in $Q_{b-1} \cup Q_{b}$. Since there are exactly $q-b$ ways to distribute $q-b+1$ vertices among non-empty $Q_{b-1}$ and $Q_{b}$ and $1 \leq b \leq q-1$, the overall number of possible partitions of $Q$ equals $1+(q-1) q / 2$.

To compute $t(n, q)$ for $n \geq 6$, it suffice to observe that the number of possible partitions of $R$ also equals $1+(q-1) q / 2$, if $Q=\emptyset$, and if both $R$ and $Q$ are non-empty, then the number of possible partitions of $R$ and $Q$ equals $(1+j(j-$ 1) $/ 2)(1+(q-j)(q-j-1) / 2)$ provided $|R|=j$.

Let $n=5$. If $R=\emptyset$, then $q$ vertices of $Q$ may be distributed among the $Q$-sets as follows: we "open" $Q_{0}$ and put the first vertex in $Q_{0}$, the second vertex goes into $Q_{0}$ or into $Q_{1}$, but if it goes into $Q_{1}$, we "close" $Q_{0}$ and "open" $Q_{1}$ (similarly to the first fit heuristic in bin packing), etc. The $k$ th vertex goes into the open $Q_{j}$ or we "close" $Q_{j}$ and "open" $Q_{j+1}$ and put the vertex there. This algorithm implies that the number of possible partitions of $Q$ equals $2^{q-1}$.

Similarly to the case of $n \geq 6$, we see that $t(5, q)$ equals $2 \times 2^{q-1}+\sum_{j=1}^{q-1}$ $2^{j-1} 2^{q-j-1}$ 。

Remark. It is interesting to observe that while $t(n, q)$ is polynomial for $n \geq 6$, the number is exponential, in $q$, for $n=5$. Another interesting observation is that we have been able to find the exact number of certain non-isomorphic n-partite tournaments. For non-isomorphic acyclic and close to acyclic families of $n$-partite tournaments, an exact enumeration was given in [11].

## 4. The Case of $\boldsymbol{n}=\mathbf{3}$

Theorem 4.1. A strong 3-partite tournament $D$ with partite sets $V_{1}, V_{2}$ and $V_{3}$ contains a unique 3-cycle if and only if it can be constructed in one of the following two ways.
(i) A partite set, say $V_{1}$, has only one vertex $x$ and there is exactly one arc from $N^{+}(x)$ to $N^{-}(x)$.
(ii) Each partite set has at least two vertices. Let $v_{1} \in V_{1}, v_{2} \in V_{2}$ and $V_{3}=\left\{v_{3}\right\} \uplus W_{1} \uplus$ $W_{2}$ ( $W_{1}$ or $W_{2}$ may be empty). Let $D$ contain arcs such that $\left(V_{1} \cup W_{1}\right) \Rightarrow\left(V_{2} \cup W_{2}\right)$
and $\left(\left(V_{1} \cup v_{2}\right)-v_{1}\right) \rightarrow v_{3} \rightarrow\left(\left(V_{2} \cup v_{1}\right)-v_{2}\right)$. The arcs between $W_{1}$ and $V_{1}$ as well as between $W_{2}$ and $V_{2}$ are arbitrary.

Proof. It is not difficult to check that a digraph constructed as in (i) above has a unique 3-cycle. Now let $D$ be a digraph constructed as in (ii) above, and let $S_{1}=$ $W_{1} \cup V_{1}$ and $S_{2}=W_{2} \cup V_{2}$. Observe that $D\left\langle S_{i} \cup\left\{v_{3}\right\}\right\rangle$ is bipartite for $i=1,2$. Therefore any 3-cycle must contain vertices from both $S_{1}$ and $S_{2}$. However, the only path from $S_{2}$ to $S_{1}$ is $v_{2} v_{3} v_{1}$, which gives us the unique 3-cycle $v_{1} v_{2} v_{3} v_{1}$.

Now assume that $D$ is a strong 3-partite tournament containing a unique 3-cycle. Let $V_{1}, V_{2}$ and $V_{3}$ be partite sets of $D$. We first consider the case when some partite set is of size 1 . Without loss of generality assume that $V_{1}=\{x\}$. Clearly, every arc from $N^{+}(x)$ to $N^{-}(x)$ results in a 3-cycle, and there must be at least one such arc as $D$ is strong. This implies that $D$ can be constructed as in (i) above. So now consider the case when all partite sets contain at least two vertices.

Let $C=v_{1} v_{2} v_{3} v_{1}$ be the unique 3-cycle in $D$, with $v_{i}$ belonging to the partite set $V_{i}$. Note that if $X \subseteq V(C)$ is non-empty then $D-X$ is not strong, by Theorem 3.1. Let $Q_{1}, Q_{2}, \ldots Q_{s}$ be the strong components of $D-V(C)$, such that $Q_{i} \Rightarrow Q_{j}$ for all $1 \leq i<j \leq s$. Observe that each $Q_{i}$ has vertices from at most two partite sets. Let $Q^{*}$ contain all vertices in $D-V(C)$ which have a path into it from $Q_{1}$ and a path from it to $Q_{s}$. The set $Q^{*} \neq \emptyset$ since $Q_{1} \cup Q_{s}$ are in $Q^{*}$.

Observe that $Q^{*}$ contains vertices from all three partite sets. As $D$ is strong, there is an arc from $C$ to $Q_{1}$. Without loss of generality assume that $v_{1} q_{1}$ is such an arc. There is also an arc from $Q_{s}$ to $C$. Note that this arc must enter $v_{2}$, as if it enters $v_{1}$ then $D\left\langle Q^{*} \cup\left\{v_{1}\right\}\right\rangle$ is strong and if it enters $v_{3}$ then $D\left\langle Q^{*} \cup\left\{v_{1}, v_{3}\right\}\right\rangle$ is strong. Let $q_{s} v_{2}$ be such an arc. By an analogous argument $q_{1} \Rightarrow V(C)-v_{1}$ and $V(C)-v_{2} \Rightarrow q_{s}$. This implies that $q_{1}$ and $q_{s}$ belong to the partite set $V_{3}$, as we otherwise would obtain the 3 -cycles $v_{1} q_{1} v_{3} v_{1}$ and/or $q_{s} v_{2} v_{3} q_{s}$.

Now assume that there exists an arc $r_{2} r_{1}$ in $D-V(C)$, where $r_{2} \in V_{2}$ and $r_{1} \in V_{1}$. If $r_{2} \rightarrow v_{3}$, then let $P$ be any path from $q_{1}$ to $r_{2}$ in $D-V(C)$ (which exists as $q_{1} \in Q_{1} \cap V_{3}$ ). Thus, $D\left\langle V(P) \cup\left\{v_{3}, v_{1}\right\}\right\rangle$ is strong and contains vertices from three partite sets. By Theorem 3.1, it contains a 3-cycle, which does not include $v_{2}$, a contradiction. Therefore $v_{3} \rightarrow r_{2}$. We can analogously prove that $r_{1} \rightarrow v_{3}$. However, this gives us the 3-cycle $v_{3} r_{2} r_{1} v_{3}$, a contradiction. Therefore $\left(V_{1}-v_{1}\right) \rightarrow\left(V_{2}-v_{2}\right)$.

This means, in particular, that there exists an index $i \geq 1$, such that $V_{1}-v_{1} \subseteq$ $Q_{1} \cup Q_{2} \cup \ldots \cup Q_{i}$ and $V_{2}-v_{2} \subseteq Q_{i+1} \cup Q_{i+2} \cup \ldots \cup Q_{s}$ (as all $Q^{\prime}$ s can contain vertices from at most two partite sets). Furthermore there is no $r_{2} \in V_{2}$, such that $r_{2} \rightarrow v_{1}$ as we could then obtain the 3-cycle $v_{1} q_{1} r_{2} v_{1}$ not including $v_{2}$ or $v_{3}$. Similarly there is no $r_{1} \in V_{1}$, such that $v_{2} \rightarrow r_{1}$. Setting $W_{1}=\left(Q_{1} \cup Q_{2} \cup \ldots \cup Q_{i}\right) \cap V_{3}$ and $W_{2}=\left(Q_{i+1} \cup Q_{i+2} \cup \ldots \cup Q_{s}\right) \cap V_{3}$ and using the conclusions above, we obtain $\left(V_{1} \cup W_{1}\right) \Rightarrow\left(V_{2} \cup W_{2}\right)$.

Suppose there exists $r_{1} \in V_{1}-v_{1}$ such that $v_{3} \rightarrow r_{1}$. Let $P$ be a path in $D-V(C)$ from $r_{1}$ to $q_{s}$. Then $D\left\langle V(P) \cup\left\{v_{2}, v_{3}\right\}\right\rangle$ is strong and has vertices from three partite sets. Thus, by Theorem 3.1, it has a 3-cycle. The 3-cycle does not include $v_{1}$, a contradiction. Hence, $\left(V_{1}-v_{1}\right) \rightarrow v_{3}$. Similarly, we can prove that $v_{3} \rightarrow\left(V_{2}-v_{2}\right)$. This completes the proof that $D$ satisfies the conditions in (ii).

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Received: April 8, 2004
Final version received: August 4, 2005

