Hamiltonian Cycle Problem

Definition: A Hamiltonian cycle is a cycle in a graph that visits each vertex exactly once.

To show Hamiltonian Cycle Problem is NP-complete, we first need to show that it actually belongs to the class NP, and then use a known NP-complete problem to Hamiltonian Cycle.

Does Hamiltonian Cycle Problem ∈ NP?

Given: $Graph\ G = (V, E)$

Certificate: List of vertices on Hamiltonian Cycle

To check if this list is actually a solution to the Hamiltonian cycle problem, one counts the vertices to make sure they are all there, then checks that each is connected to the next by an edge, and that the last is connected to the first.

It takes time proportional to n, because there are n vertices to count and n edges to check. n is a polynomial, so the check runs in polynomial time.

Therefore, Hamiltonian Cycle ∈ NP.

Prove Hamiltonian Cycle Problem ∈ NP-Complete

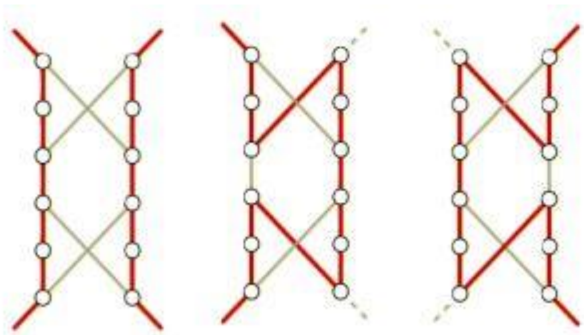
Reduction: Vertex Cover to Hamiltonian Cycle

Definition: Vertex cover is set of vertices that touches all edges in the graph.

Given a $graph\ G$ and integer k, construct a $graph\ G'$ such that G has a vertex cover of size k iff G' has a $Hamiltonian\ cycle$.

Idea: To construct widget for each edge in the graph.

$i.e$ $\quad \forall\ uv$ in the $Graph\ G$, create a widget shown below;



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As shown above, there are three ways to traverse a widget;

1. Enter from u, go somewhere else in the graph, and then come back through the other side i.e v
2. Enter and Exit through u
3. Enter and Exit through v

Construct G' for G (Vertex cover) of size $k = 2$

With the construction, any graph with a vertex cover, can be used to make a graph with a Hamiltonian Cycle graph. Since creating such a graph can be done under polynomial time, simply replace edges with widgets and make proper connections, we have a reduction from Vertex Cover to Hamiltonian Cycle.

This means that finding whether a graph has a Hamiltonian Cycle or not is NP Hard. As we have seen earlier it's also in NP, therefore, Hamiltonian Cycle is an NP Complete Problem