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Problem: The Traveling Salesman Problem (TSP)
Claim: TSP is NP complete
Problem description.

Consider a salesman who must visit n cities labeled v_1, v_2,v_3, …, v_N. He desires to start in his home city v_1 , visit all other cities exactly once and return home - (call this a tour). He also wants to minimize the total distance covered - (call this cost).

Problem formulation.

Given a set of distances on N cities and a bound D, is there a tour of length (cost) at most D?

Proof of NP completeness

We now proceed to show that TSP is NP complete.

This problem is a naturally occurring form of the Hamiltonian cycle problem (HCP). The difference is that of cost. Literally, We can solve TSP by by using HCP as a black box that will give us a permutation of the various cities that forms a hamiltonian cycle - in our case, a tour. This produces a certificate that a TSP certifier can then check its cost and assert whether it lies below the threshold D or not.

We must show that $HCP \leq p\ TSP$
Let $G = (V, E)$ be a graph.
Define TSP_0 s.t $\forall v_i \in G$, $v_i' \in TSP_0$, an instance of TSP.

Define :

$d(v_i', v_j') = 1$, if $(v_i, v_j) \in G$
$d(v_i', v_j') = 2$, if Otherwise.

Claim:

G has a hamiltonian cycle H_0 $\Leftrightarrow$ $\exists T_0$, a tour in TSP s.t $|T_0| \leq N$.
$(\Rightarrow)$ Suppose G has a hamiltonian cycle, H_0 . Then we let $T_0 = H_0$, and thus $|T_0| = |H_0| = N$.
$(\Leftarrow)$ Suppose $\exists T_0 \in TSP$ s.t $|T_0| \leq N$.
Let $T_0 = vi_1, vi_2, vi_3, \ldots, vi_N$. With $vi_1 = vi_N$.

$$\text{Then } |T_0| = \sum_j d(vi_j, vi_{j+1}) + d(vi_N, vi_1)$$
$$= N - 1 + 1 \quad = N.$$

Therefore, $\forall v_i, v_j \in G$ s.t $(v_i', v_j') \in T_0$, $(v_i, v_j) \in G$. This proves that the ordering of corresponding nodes in G must form a hamiltonian cycle. ◊

Sources:

Jon Kleinberg, Eva Tardos. Algorithm Design PEARSON