Theorem: *Prove that Post Correspondence Problem (PCP) is undecidable*

Post Correspondence Problem

This problem is introduced by Emil Leon Post in 1946 [1]. Problem asks to find a match from the given collection of two faced dominos where each face of dominos contains strings. A two faced domino will look like: $[\frac{a}{ab}]$

So the collection of two faced dominos can be represented in a set as:

$\{[\frac{a}{ab}],[\frac{b}{ca}],[\frac{ca}{a}],[\frac{a}{ab}], [\frac{abc}{c}]\}$

If we choose a list of dominos from the dominos (repetition is allowed while choosing) set in such a way that concatenated of strings in one face must be same as concatenated string of other face. This list is called match. This problem asks us to find a match from any given set of two faced dominos.

In the above collection of two faced dominos, there exists a match because the string "abcaaabc" can be obtained by concatenating strings of each face of the dominos separately.

Problem Definition

An instance of this problem can be expressed in mathematically as:

Let D be the collection of n dominos.

$D = \{\left[\frac{u1}{v1}\right], \left[\frac{u2}{v2}\right], \left[\frac{u3}{v3}\right], \ldots, [\frac{un}{vn}]\}$

And there exists match sequence k_1, k_1, k_2, k_3… k_l where

$u1u2u_3 \ldots u_l = v_1v_2v_3 \ldots v_l$ (Concatenation of strings of one face is u and other face v) problem is to find the existence of match in D. Let

PCP = {<D>| D is the instance of the Post Correspondence Problem with a match}

Proof:

Consider a Turing machine M to simulate this PCP's input string w can be represented as <M, w>.

$$M = (Q,\Sigma,\Gamma,\delta,q_0,q_{accept},q_{reject})$$

If there is a match in the input string w, then Turing machine M halts in an accepting state. This halting state of Turing machine is an acceptance problem A_{TM}. We know that the acceptance problem A_{TM} is undecidable from proof mentioned in Sipser's book [1]. Therefore PCP problem is also undecidable.

If a Turing Machine M accepts input w, then the computation history of M gives us the accepting configuration, from which we can determine the instance D of the PCP problem has a match. How can construct the instance D so that there exists a match and Turing Machine M halts in accept state on input w? By choosing the dominos in such a way a concatenation of strings of each faces are same at each step of the selection. We can force the simulation of M to accept the w.

To force the simulation of M, we make 2 modifications to Turing Machine M and a one change to our PCP problem.

1. M on input w can never attempt to move tape head beyond the left end of the input tape.
2. If input is empty string € we use ␣
3. PCP problem starts the match with first domino $[\frac{u1}{v1}]$ This is called Modified PCP problem

   MPCP = {<D>| D is an instance of PCP starts with first domino}

Construction Steps:

1. Put $[\frac{\#}{\#q0w1w2..wn\#}]$ into $D^{|}$ as the first domino, where is instance of $D^{|}$ is MPCP. A partial match is obtained in first domino is # at one face is same #symbol in other face.
2. Transition functions for Turing Machine M can have moves Left L, Right R

   For every x,y z in Γ tape alphabets and q,r in Q where q is not equal to q_{reject}.

   If δ(q,x) = (r,y,R) put the domino $[\frac{qx}{by}]$ into $D^{|}$ and δ(q,x) = (r,y,L) put the domino $[\frac{zqx}{rzy}]$ into $D^{|}$
3. For every tape alphabets x in Γ put $[\frac{x}{x}]$ into $D^{|}$

   To mark the separation of each configurations put $[\frac{\#}{\#}]$ and $[\frac{\#}{␣\#}]$ into $D^{|}$
4. To read input alphabets x in Γ even after Turing Machine is accepting state put $[\frac{xqa}{qa}]$ and $[\frac{qax}{qa}]$ and $[\frac{qa\#}{\#}]$ into $D^{|}$

These steps concludes the construction of $D^{|}$ Since this a instance of MPCP, we need to convert this to PCP. So to convert to D, we consider the below dominos and strings matchings.

Converting MPCP To PCP

Let u = $u_1, u_2, \ldots, u_n$ be any string of input length n and modify these strings as

$u = *$u_1$*$u_2$*$u_3$* …*$u_n$

u$ = u_1* u_2* u_3* … u_n*

u = * u_1* u_2* u_3* … u_n*

Let D be the set of two faced Dominos

$D^{|} = \{\left[\frac{u1}{v1}\right], \left[\frac{u2}{v2}\right], \left[\frac{u3}{v3}\right], \ldots, [\frac{un}{vn}]\}$ and $\{\left[\frac{\$u1}{\$v1\$}\right], \left[\frac{\$u2}{\$v2\$}\right], , [\frac{*\sqcup}{\sqcup}]\}$

From above dominos collection, we could see only domino has partial match starts with $\left[\frac{\$u1}{\$v1\$}\right]$ and to place marker the end of inputs$[\frac{*\sqcup}{\sqcup}]$ There by we can avoid stating explicit requirement of domino should start with first domino.

If number of configurations of the Turing machine does not lie within the value of qng^n [Lemma specified in Sipsers Book], then Turing machine is looping state. It does not halt.

References

[1] Michael Sipser (2005). "A Simple Undecidable Problem". Introduction to the Theory of Computation (2nd ed.). Thomson Course Technology. pp. 199–205. ISBN 0-534-95097-3.

[2] E. L. Post (1946). "A variant of a recursively unsolvable problem" (PDF). Bull. Amer. Math. Soc 52.