On TC^0 Lower Bounds for the Permanent

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Note: slides online at kinnejeff.com

Main Result, Definitions and Context

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Lower bound for Perm *on threshold circuits with depth* d*, advice* a*, size* s*, s.t.*

- $d = O(1)$, $a = \text{poly-log}(n)$, s s.t. $s^{(O(1))}(n) < 2^n$
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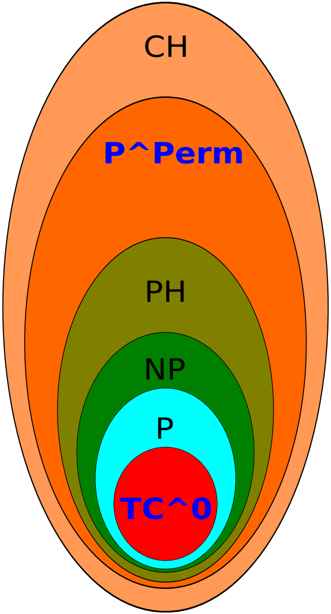
Main result also discovered by [CK12]


CH
P^Perm
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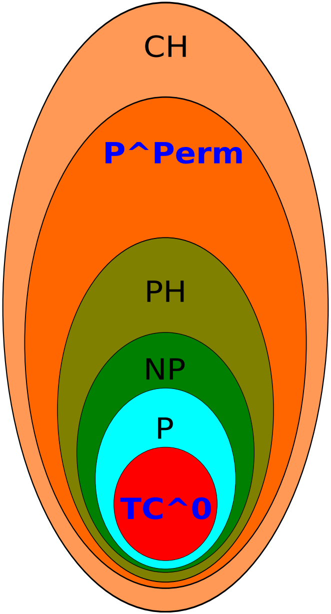

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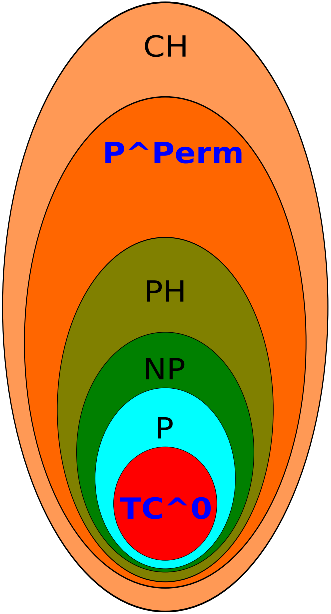

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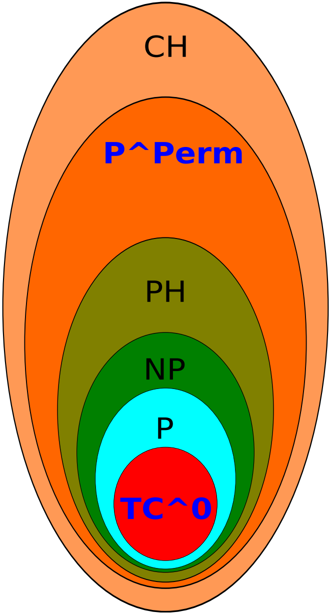

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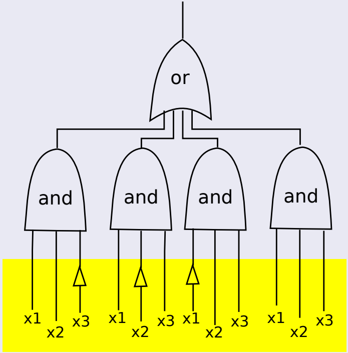

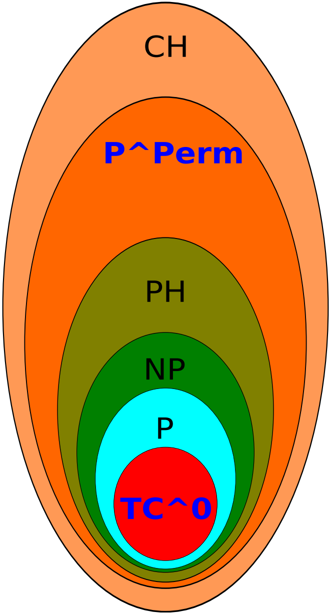

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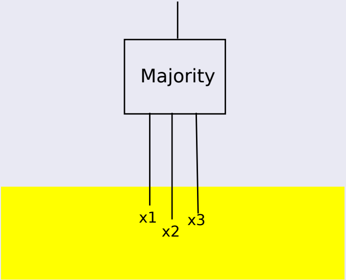

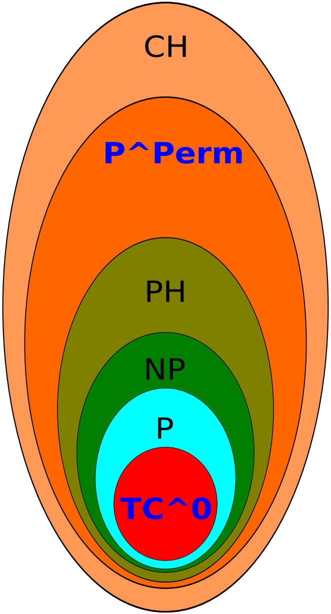

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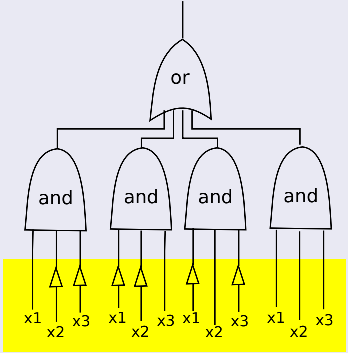

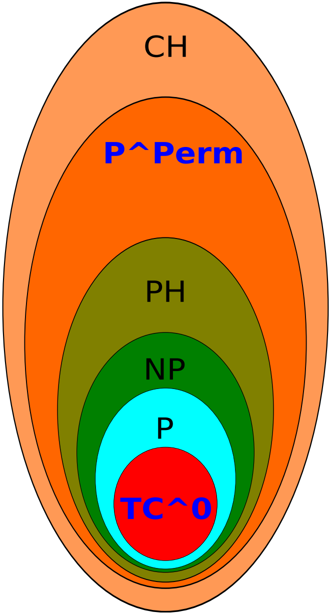

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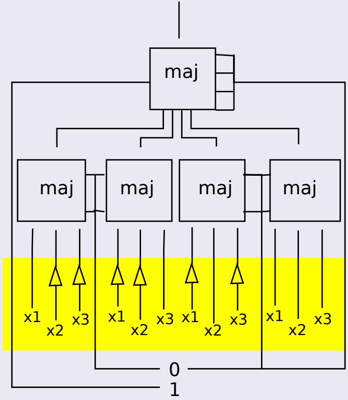

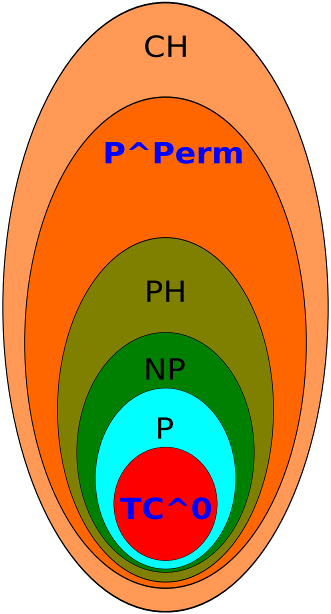

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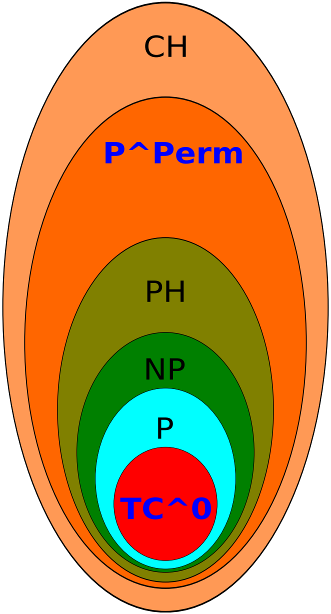

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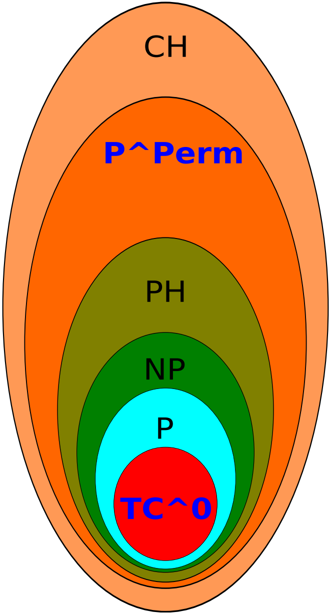

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- PRGs, crypto? [NRR02]

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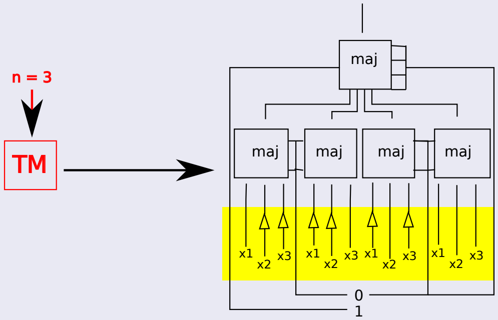

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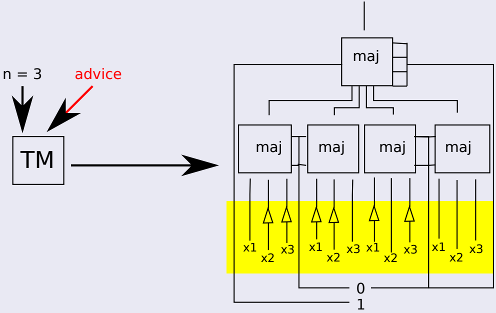

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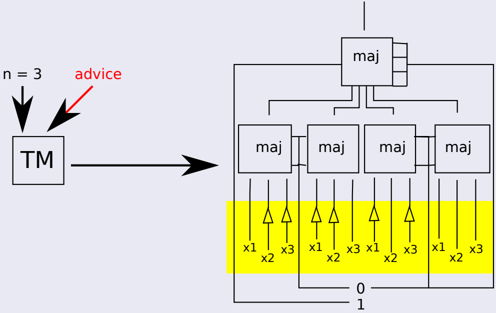

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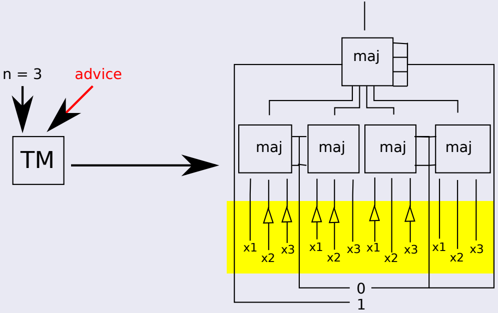

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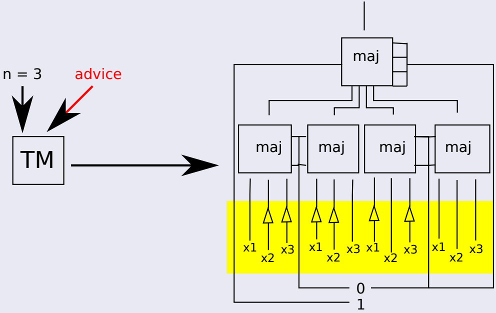

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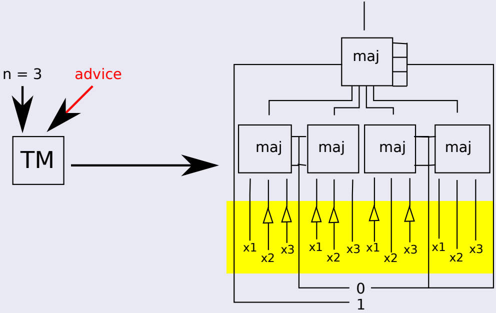

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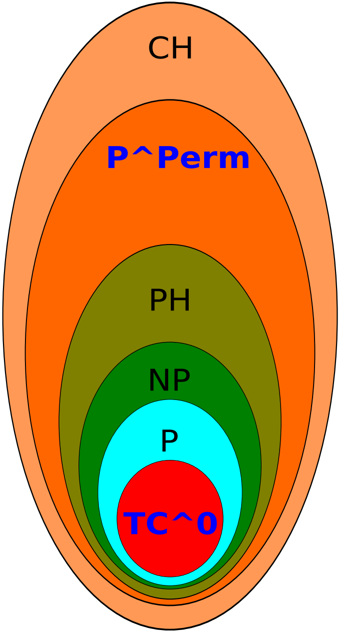
  derandomization $\Rightarrow$ non-uniform lower bounds [KI04, AM11]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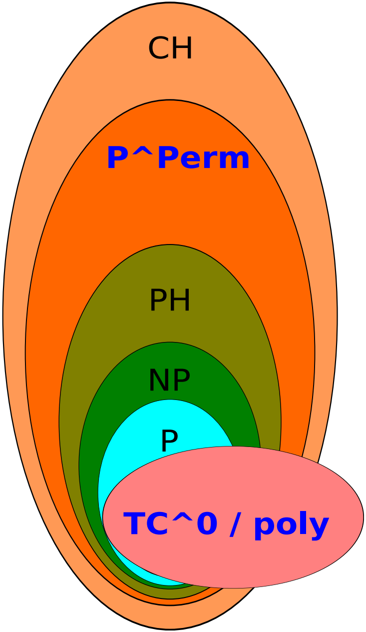
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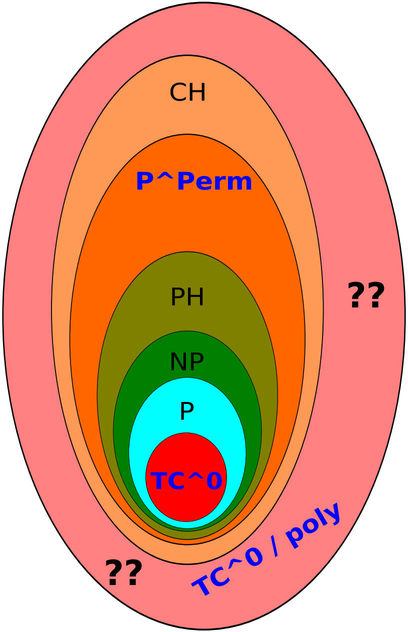
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- Conjecture: NP, Perm $\not\subseteq$ SIZE$(2^{n^{o(1)}})$

Proofs

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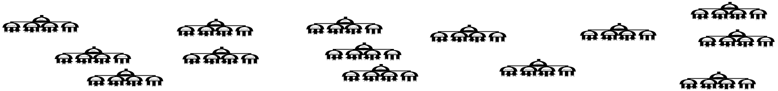

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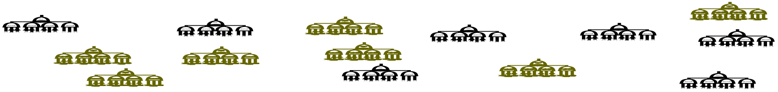

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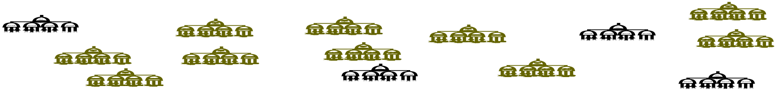

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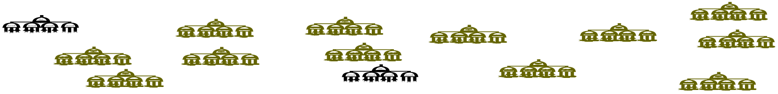

Theorem

Perm $\notin$ *TC circuits of depth* $O(1)$*, advice* $n^{o(1)}$*, and poly size.*

Proof

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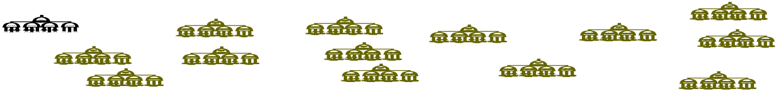

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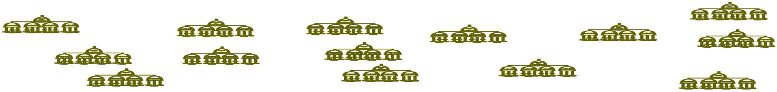

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$\mathrm{P}^{\mathrm{Perm}} \quad \subseteq \mathrm{P}^{\mathrm{TC}^0} + n^{o(1)}$ advice

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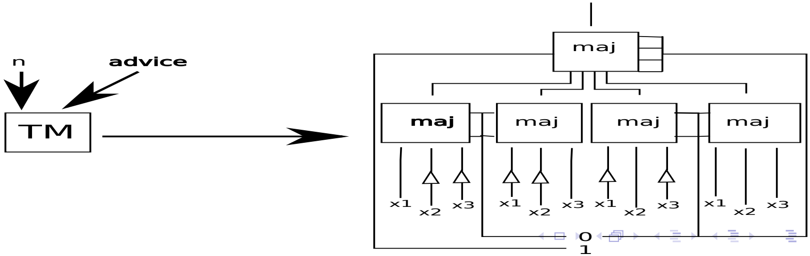

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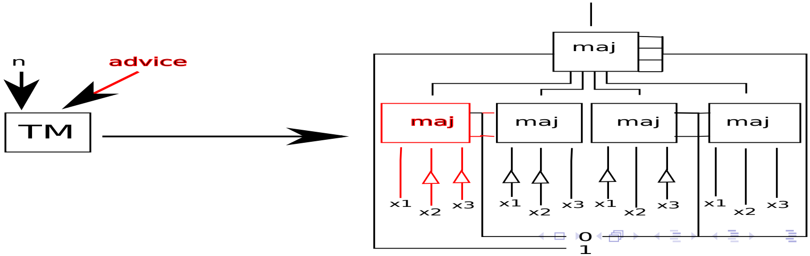

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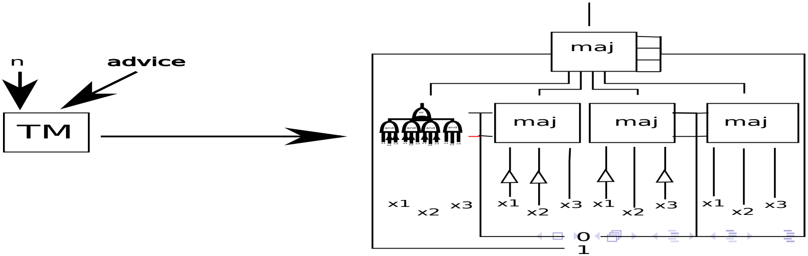

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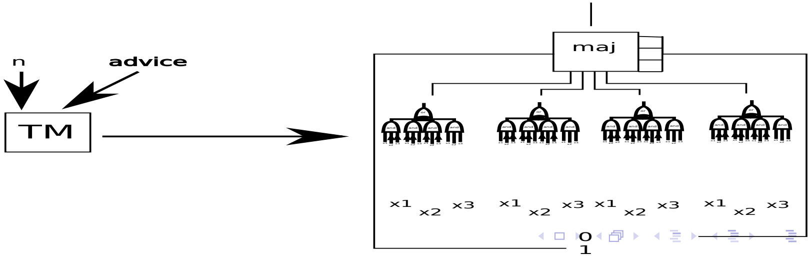

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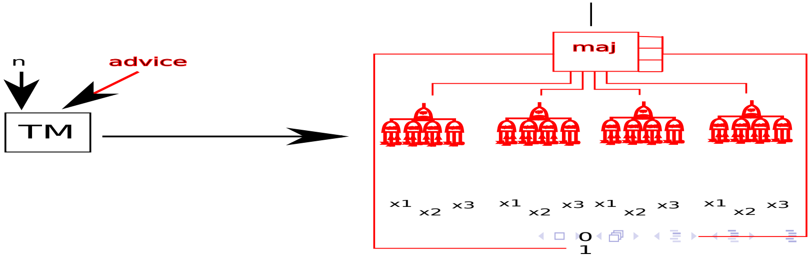

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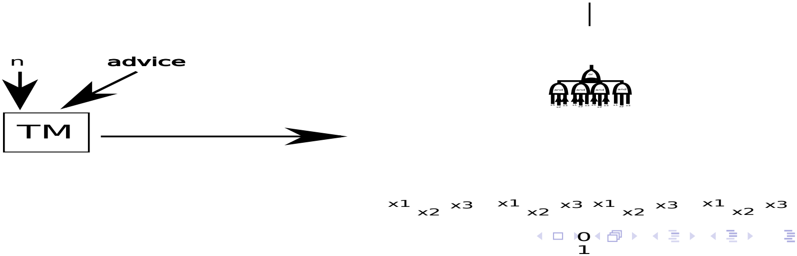

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$$\text{Level 1: PP question size } n^{o(1)} + \log(\mathrm{poly}(n))$$
$$\Rightarrow (n^{o(1)})^c = n^{o(1)} \text{ size circuits}$$

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Level 2: PP question size $n^{o(1)} + \log(\mathrm{poly}(n) \cdot n^{o(1)})$

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Level d: $\Rightarrow n^{o(1)}$ size circuit

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Level d: $\Rightarrow n^{o(1)}$ size circuit
(size $O(n)$ really)

Theorem

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Proof of [CK12]

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Proof of [CK12]

- TH-time-depth$(n^{k+1}, d) \not\subseteq$ TH-time-depth$(n^k, d) + o(n)$ advice

Theorem

$\mathrm{Perm} \notin$ *TC circuits of depth $O(1)$, advice $n^{o(1)}$, and poly size.*

Proof of [CK12]

- TH-time-depth$(n^{k+1}, d) \not\subseteq$ TH-time-depth$(n^k, d) + o(n)$ advice
- P has TC^0 size n^k, depth d, advice $n^{o(1)} \Rightarrow$ TH-time-depth$(n^{k+1}, d) \not\subseteq \mathrm{P} + n^{o(1)}$ advice

Theorem

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Proof of [CK12]

- TH-time-depth$(n^{k+1}, d) \not\subseteq$
  TH-time-depth$(n^k, d) + o(n)$ advice
- P has TC^0 size n^k, depth d, advice $n^{o(1)} \Rightarrow$
  TH-time-depth$(n^{k+1}, d) \not\subseteq$ P $+ n^{o(1)}$ advice
- theorem false: TH-time-depth$(n^{k+1}, d) \subseteq$ P $+ n^{o(1)}$ advice

What Next?

Extensions

- Poly advice

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- Poly advice (obstacles - natural pfs [RR97], relativization [AW09])

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- Make use nice properties of permanent

Extensions

- Poly advice (obstacles - natural pfs [RR97], relativization [AW09])
- Average-case, almost-everywhere hardness
- ...
- Make use nice properties of permanent
  - low-degree poly, downward/random self-reduction, ...

Thank you.

Slides online at http://www.kinnejeff.com

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