CONVEX HULL

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Abstract

In mathematics, the convex hull or convex envelope of a set X of points in the Euclidean plane or Euclidean space is the smallest convex set that contains X. For instance, when X is a bounded subset of the plane, the convex hull may be visualized as the shape formed by a rubber band stretched around X.Formally, the convex hull may be defined as the intersection of all convex sets containing X or as the set of all convex combinations of points in X. With the latter definition, convex hulls may be extended from Euclidean spaces to arbitrary real vector spaces. The algorithmic problem of finding the convex hull of a finite set of points in the plane or in lowdimensional Euclidean spaces is one of the fundamental problems of computationalgeometry.

1 Introduction

The convex hull is one of the most fundamental structures in computational geometry and plays a central role in pure mathematics. No wonder, the convex hull of a set of points is one of the most studied geometric problems both in algorithms and in pure mathematics. Indeed, computing convex hull is a fundamental operation in computational geometry. The in depth study of the hulls is useful in its own right and as a tool for constructing other structures in a variety of circumstances in the design of algorithms. The convex hull represents something of a success story in the area of algorithmic geometry.

There has been an amazing variety of research on hulls which ultimately leading to optimal algorithm known as Graham's scan. To trace the history of Graham's scan algorithm is worth it especially, in the context of algorithm design and analysis. The importance of the topic demands an intuitive appreciation. So intuitively, if we think of each point in given set as being a nail sticking out from a board, the convex hull is the shape formed by a tight rubber band that surrounds all the nails.

There are many algorithms which are used to find the convex hull for a set of points. We use Graham Scan Algorithm to find the convex hull of set of points. This is an efficient algorithm that has a time complexity of O(n log n) where as all the algorithms to find convex hull has a time Complexity of O(n2) during 1970. Apart from time complexity of its implementation, Convex hulls find its applications in many fields of scientific development and analysis.

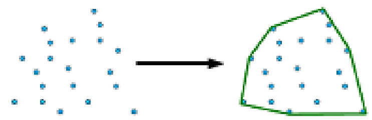

Figure 1: This convex hull was uploaded to writeLaTeX via the project menu.

2 Definitions

Definition 1. *The convex hull Q is the set of all convex combinations of points in the given set Q.*

Definition 2. *The convex hull of a set Q of points is the smallest convex polygon P for which each point in set Q is ether on the boundary of P or in its interior*

Definition 3. *The convex hull of a set points Q is the intersection of all convex sets that contains Q.*

Definition 4. *The convex hull in d-dimensions is the set of all convex combinations of d + 1 (or fewer points) of points in the given set Q*

2.1 Other Definitions

The convex hull is the smallest convex polygon containing the points.

- Polygon - A region of the plane bounded by a cycle of line segments, called edges, joined end-to-end in a cycle. Points where two successive edges meet are called vertices.
- Convex- For any two points p, q inside the polygon, the line segment pq is completely inside the polygon. Formally, a set S is convex if p is in S and q is in S implies that the segment pq is a subset of S. See the figure below. Note that this can be taken as the primary definition of convexity.
- Smallest - Any convex proper subset of the convex hull excludes at least one point in P.This implies that every vertex of the convex hull is a point in P.
- Equivalently, we can also define the convex hull as the largest convex polygon whose vertices are all points in P, or the unique convex polygon that contains P and whose vertices are all points in P. Notice that P might have interior points that are not vertices of the convex hull.

2.1.1 Problem

Let us consider an example
Given a set Q = p1, p2, ..., pn of points in the plane, compute a list that contains
those points from Q that are the vertices of CH(Q), listed in counterclockwise order.
Input = Set of points: p1, p2, p3, p4, p5, p6, p7, p8, p9.
Output = Representation of the convex hull: p8, p5, p4, p9, p2.

3 History

- **Gift Wrapping Algorithm**
  One of the simplest (although not the most time efficient in the worst Case) planar algorithms. Discovered independently by Chand and Kapur in 1970 and R. A. Jarvis in 1973. The simplest (although not the most time efficient in the worst case) algorithm in the plane was proposed by R.A. Jarvis in 1973. It is also called gift wrapping algorithm. It has O(nh) time complexity, where n is the number of points in the set, and h is the number of points in the hull. In the worst cases the complexity is O(n2).

- **Graham Scan Algorithm**
  The first paper published in the field of computational geometry was on the construction of convex hull on the plane. In the late 1960, the best algorithm for convex hull was O(n2). In 1972, R.L.Graham developed his simple and efficient algorithm in response to this need. The algorithm is efficient in the sense that no matter how many points are on the boundary of the convex,it runs efficiently. Historically, this is the first publication that showed convex hull computation in O(n log n) time in the worst case.

- **Divide and Conquer Algorithm**
  Another O(n log n) solution is the divide and conquer algorithm for the convex hull, published in 1977 by Preparata and Hong. This algorithm is also applicable to the three dimensional case.

- **QuickHull Algorithm**
  Discovered independently in 1977 by W. Eddy and in 1978 by A. Bykat. Just like the Quicksort algorithm, it has the expected time complexity of O(n log n), but may degenerate to O(nh) = O(n2) in the worst case.
  QuickHull begins with the generation of a quadrilateral that connects the extreme points along the compass directions. Only those points that are outside of the quadrilateral can be on the hull and will be considered in further computation. Each of the four regions outside of this quadrilateral will be considered in turn and each will be recursively processed using the Quick Hull function.

- **Chan Algorithm**
  A simpler optimal output-sensitive algorithm discovered by chan in 1996 to compute the convex hull of a set P of n points, in 2 or 3 dimensional space. The algorithm takes O(n log h) time, where h is the number of points in the output (the convex hull). In the planar case, the algorithm combines an O(n log n) algorithm with Jarvis march, in order to obtain an optimal O(n log h) time. Chan algorithm is not able because it is much simpler than the ultimate planar convex hull algorithm, and it naturally extends to 3-dimensional space.

4 The Graham Scan Algorithm

Graham scan is a method of computing the convex hull of a finite set of points in the plane with time complexity O (n log n). It is named after Ronald Graham, who published the original algorithm in 1972. The algorithm finds all vertices of the convex hull ordered along its boundary. The first paper published in the field of computational geometry was on the construction of convex hull on the plane. In the late 1960, the best algorithm for convex hull was O (n2). At Bell Laboratories, they required the convex hull for about 10,000 points and they found out this O (n2) was too slow. The algorithm is efficient in the sense that no matter how many points are on the boundary of the convex, it runs efficiently.

In 1981, A.C.Yao, proved that it is optimal in the worst-case sense. The problem with Graham algorithm is that it has no obvious extension to three dimensions. The reason is that the Graham scan depends on angular sorting, which has no direct counterpart in three dimensions.
textbf Working Graham scan Algorithm begins with finding the base point which is the point with smallest Y- coordinate. Then by using the base point, we sort all the points based on angle made by them with base point. After sorting, we choose three points in sequence and check for point that leads to clockwise rotation. We remove the points that lead to a clockwise rotation and these are the point that does not form convex hull boundaries.

Let points [0...n-1] be the input array.
1. Find the bottom-most point by comparing y coordinates of all points. If there are two points with same y value, then the point with smaller x coordinate value is considered. Put the bottom-most point at first position.
2. Consider the remaining (n-1) points and sort them by polar angle in counterclockwise order around points [0]. If polar angle of two points is same, then put the nearest point first.
3. Create an empty stack S and push points[0], points[1] and points[2] to S.
4. Process remaining (n-3) points one by one. Do following for every point points[i].
4.1. Keep removing points from stack while orientation of following 3 points is not counterclockwise or they do not make a left turn.
a) Point next to top in stack
b) Point at the top of stack
c) points[i]
4.2 Push points[i] to S
5. Print contents of S

5 Pseudocode

Pseudocode GRAHAM SCAN (Q)
1.Find p0 in Q with minimum y-coordinate (and minimum x-coordinate if there are ties)
2.Sorted the remaining points of Q (that is, Q minus p0) by polar angle in counterclockwise order with respect to p0.
3.TOP [S] = 0; Lines 3-6 initialize the stack to contain, from bottom to top, first three points.
4.PUSH (p0, S)
5.PUSH (p1, S)
6.PUSH (p2, S)
7.For i = 3 to m; Perform test for each point p3... pm.
8.do while the angle between NEXT TO TOP[S], TOP[S], and pi makes a non left turn
9.do POP(S)
10.PUSH (S, pi)
11.return S

6 Time Complexity

Running Time Computation:
Line 1 requires O (n) time to find the anchor point po (i.e., point with minimum y-coordinate). Since sorting based on polar angle takes O (n log n) time, Line 2 requires O (n log n) time (using merge or heap sort to sort polar angles).
Moreover removal of n minus m points with duplicate angles takes O(n) time. Lines 3, 4, 5, and 6 require constant time, O(1), to initialize the stack to contain the first three points. For-loop in Line 7 is executed m minus 2 times, so it takes O (n) time. Now the while-loop in Line 8.

This is a ”problem”. It may iterate as many as O (n) time, which leads to an over-estimate of O (n2). Note that each pass through the while statement, POP is executed.
We use the aggregate method of amortized analysis to analyze the while-loop. As in analysis of MULTIPOP, we observe there is at most one pop for each push operation. In case of GRAHMS SCAN, at least three points (p0, p1 and pm) are never popped from the stack. So in fact, at most m minus 2 pop operations are performed. Therefore, the amortized cost of each Iteration of while loop is O (n)/n = O (1). It implies that the overall worst-case for for-loop is O (n). The worst case running time of GRAHAM SCAN is
T(n) = O(n) + O(n log n) + O(1) + O(n) = O(n log n),where n = —Q—.

7 Applications

The computation of the convex hull of a finite set of points has found applications in diverse areas. They are:

- pattern recognition
- Geographical Information Systems (GIS)
- image processing
- robotics
- Stock cutting and allocation
- Collision avoidance
- Fitting ranges with a line
- Shape analysis

- **Geographical Information Systems (GIS)**:
  A geographic information system (GIS), also known as a geospatial information system, is a system for capturing, storing, analyzing and managing data and associated attributes which are spatially referenced to the Earth. GIS In the strictest sense, it is an information system capable of integrating, storing, editing, analyzing, sharing, and displaying geographically referenced information. In a more generic sense, GIS is a tool that allows users to create interactive queries (user created searches), analyze the spatial information, edit data, maps, and present the results of all these operations.
  Geographic information science is the science underlying the geographic concepts, applications and systems, taught in degree and GIS Certificate programs at many universities. Geographic information system technology can be used for scientific investigations, resource management, asset management, Environmental Impact Assessment, Urban planning, cartography, criminology, history, sales, marketing, and logistics. For example, GIS might allow emergency planners to easily calculate emergency response times in the event of a natural disaster, GIS might be used to find wetlands that need protection from pollution, or GIS can be used by a company to site a new business to take advantage of a previously Undeserved market.

- **Visual Pattern Matching** - Detecting Car License Plates
  Pattern recognition aims to classify data (patterns) based on either a priori knowledge or on statistical information extracted from the patterns. The patterns to be classified are usually groups of measurements or observations, defining points in an appropriate multidimensional space. This is in contrast to pattern matching, where the pattern is rigidly specified within medical science pattern recognition creates the basis for CAD

Systems (Computer Aided Diagnosis).
CAD describes a procedure that supports the doctor interpretations and findings. A complete pattern recognition system consists of a sensor that gathers the observations to be classified or described, a feature extraction mechanism that computes numeric or symbolic information from the observations and a classification or description scheme that does the actual job of classifying or describing Observations relying on the extracted features.

- **Collision avoidance** : Suppose our problem is to plan the path for a robot from point a to point b in the presence of obstacles. If the convex hull of a robot avoids collision with obstacles, then so does the robot. So, the convex hull structure often used to plan paths.
- **Fitting ranges with a line** : Suppose our problem is to find a straight line that fits between collections of data ranges. Our solution requires finding a straight line that fits between a collection of data ranges maps to finding the convex region common to a collection of half planes.
- **Smallest box** : Suppose our problem is to place an object (convex polygon) in a smallest box. The smallest area rectangle that encloses a polygon has at least one side flush with the convex hull of the polygon, and so hull is computed at the first step of minimum rectangle algorithms. Similarly, finding the smallest three-dimensional box surrounding an object in space depends on the convex hull of the object.
- **Shape analysis** : Shapes may be classified for the purpose of matching by their "convex deficiency trees," structures that depend for their computation on convex hulls.

8 References

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