Mod n representations of complete multipartite graphs

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Mod r Representations

Let G be a (finite, simple) graph and r a positive integer.

Definition

A *representation* of G modulo r is an injective map

$$f : V(G) \to \{0, 1, \ldots, r-1\}$$

such that u, v are adjacent iff $\gcd(f(u) - f(v), r) = 1$.

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such that u, v are adjacent iff $\gcd(f(u) - f(v), r) = 1$.

Equivalently, we could define a representation modulo r as an injective map

$$f : V(G) \to \mathbb{Z}_r$$

such that u, v are adjacent iff $f(u) - f(v)$ is a *unit* in (the ring) $\mathbb{Z}_r$.

Unitary Cayley Graph

If we define Cay(r) to be the graph with vertex set $\{0, 1, \ldots, r-1\}$ where adjacency is defined by

$$i \leftrightarrow j \text{ iff } \gcd(i-j, r) = 1$$

then clearly Cay(r) is representable modulo r.

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Moreover, for any graph G:

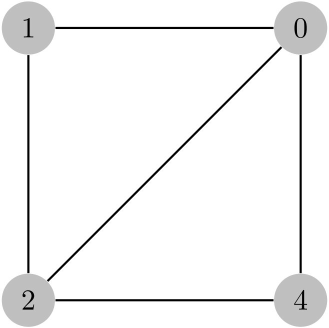
G is representable modulo r

iff

G is isomorphic to an induced subgraph of Cay(r)

Example

Figure: A representation modulo 9



Representation Number

The *representation number* of a graph G is defined by:

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Theorem

(Erdös and Evans 1989, Narayan 2003)

For every graph G, rep(G) exists. In particular, if $n = |V(G)|$ and $p_1, \ldots, p_{n-1}$ are the first $n-1$ primes $\geq n-1$, then

$$rep(G) \leq \prod_{i=1}^{n-1} p_i$$

and this bound is sharp.

Relationship to Product Dimension

If G is *reduced* (no two vertices have the same neighborhood), rep(G) is closely related to the product dimension of G:

The number of distinct prime divisors of rep(G) is *at least* the product dimension.

Simple example: edgeless graphs

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Proposition

$$\mathsf{rep}(\overline{K_n}) = 2n$$

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Upper bound:

Lower bound: if $k < 2n$, then any labeling modulo k must assign consecutive labels to some pair of vertices, contradicting the definition of representation.

General Bounds

- In general, rep(G) is **not at all** well-behaved with respect to standard graph operations (deleting a vertex, etc.)

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- Proofs are much more number-theoretic than combinatorial and typically involve results on the *distribution of primes*.

Proposition

Let G be a graph and p the smallest prime divisor of rep(G). Then

$$\omega(G) \leq p \leq \frac{\text{rep}(G)}{\alpha(G)}$$

Chinese Remainder Theorem

Given a prime factorization

$$r = p_1^{e_1} \dots p_s^{e_s}$$

we have a ring isomorphism:

$$\mathbb{Z}_r \cong \mathbb{Z}_{p_1^{e_1}} \times \dots \times \mathbb{Z}_{p_s^{e_s}}$$

so we can interpret a representation of G modulo r as a labeling of $V(G)$ by s-tuples as above.

This is particularly convenient because

$$\mathbb{Z}_r^* \cong \mathbb{Z}_{p_1^{e_1}}^* \times \dots \times \mathbb{Z}_{p_s^{e_s}}^*$$

Representation Numbers of Stars

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Theorem

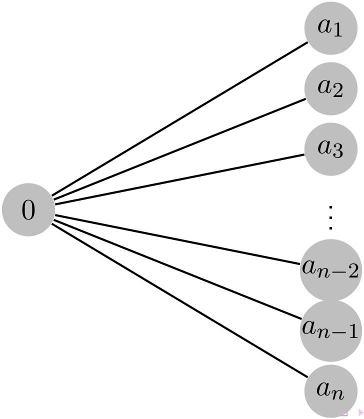
$$\mathit{rep}(K_{1,n}) = \min\{r : 2|r,\ \phi(r) \geq n\}$$

where

$$\phi(r) = |\{i : 1 \leq i \leq r-1, \gcd(i,r) = 1\}| = r\prod_{p|r}(1 - \frac{1}{p})$$

Upper bound

Given r even, $\phi(r) \geq n$,
Choose distinct (odd) integers $a_1, \ldots, a_n$ between 1 and $r-1$ such that $\gcd(a_i, r) = 1$.



Lower bound

If we pick an optimal labeling of $K_{1,n}$ modulo r, then by translating the labels, we may assume that the root is labeled 0. This forces the labels on all the leaves to be relatively prime to r; hence $\phi(\text{rep}(K_{1,n})) \geq n$.

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Thus for $n \geq 5$, $\text{rep}(K_{1,n}) < 3n$.
Finally, let q be the smallest prime divisor of $\text{rep}(K_{1,n})$. Then

$$q \leq \text{rep}(K_{1,n})/\alpha(K_{1,n}) < 3n/n = 3; \text{ so } q = 2.$$

Prime factorization of rep($K_{1,n}$)

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Based on calculations using MAGMA:

Conjecture

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Theorem

For n sufficiently large, rep$(K_{1,n})$ takes one the forms

$$2^a, \ 2^a p, \ 2^a pq$$

where $a \geq 1$ and p, q are odd primes.

Sketch of Proof

- The key ingredient is a result of Ingham (1937) that for sufficiently large n there is always a prime in $(n, n + n^{2/3})$.

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- The idea is to argue that if $r = \text{rep}(K_{1,n})$ has at least three odd prime divisors, then there is a prime $q \in (\phi(r), \frac{r}{2})$. Then $2q < r$, but $\phi(2q) = q - 1 \geq \phi(r) \geq n$, a contradiction.

  (Recall: $r = \min\{k : 2|k,\ \phi(k) \geq n\}$.)

Lingering questions

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  If one can prove that for sufficiently large n, there is a prime in $(n, n + n^{1/2})$, then we can eliminate this case – but there doesn't seem to be enough reason to believe this!

Complete Bipartite Graphs

Next, consider the complete bipartite graph $K_{m,n}$ with bipartition (A, B), $|A| = m$, $|B| = n$. Let $N = m + n$.

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Proposition

- *When $N \geq 640$, rep$(K_{m,n})$ is always divisible by 2 or 3.*
- *More precisely, rep$(K_{m,n})$ is either 2^a, 3^a or $2^a t$, where $a \geq 1$ and t is odd. In the last case, rep$(K_{m,n}) \geq 2N$.*

Idea of Proof: "Label Wastage"

If $r = \text{rep}(K_{m,n}) = 2^a t$, where t is odd, then in an optimal labeling by coordinate pairs in $\mathbb{Z}_{2^a} \times \mathbb{Z}_t$:

- All labels on vertices in A take the form $(odd, *)$
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Now if (x, y) is any label used on a vertex, $(x + 1, y)$ **cannot** be used as a label anywhere else.

Thus, the total number of "available" labels is at least $2N$, i.e. $r \geq 2N$.

Bounds for rep($K_{m,n}$)

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$$\psi(k) = \phi(k) + \frac{k}{\text{rad } k} = k[\prod_{p|k} \frac{1}{p} + \prod_{p|k}(1 - \frac{1}{p})]$$

Theorem

$$\min\{k : \psi(k) \geq N\} \leq \textit{rep}(K_{m,n}) \leq \min\{k : 2|k,\ \phi(k) \geq N\}$$

Both bounds are sharp.

The equipartite case

Proposition

$$rep(K_{n,n}) = \min\{r : r \geq 2n,\ r = 2^a \text{ or } r = 3^b\}$$

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For n sufficiently large, rep$(K_{1,n})$ takes one the forms

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- One can construct examples of each of the first three types, but not of the fourth.
- The proof is similar in spirit to that for stars (use Ingham, work with ψ instead of ϕ), but much more technical.

Complete Multipartite Graphs

Finally we consider the complete multipartite graph $G = K_{n_1,\ldots,n_t}$ with partite sets $A_1, \ldots, A_t$ of respective sizes $|A_i| = n_i$; assume $n_1 \leq \ldots \leq n_t$ and let $N = \sum_{i=1}^{t} n_i$.

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Additional complications:

- t may not be prime
- Even if t is prime, there is no guarantee that in a representation of G, all elements in a given partite set will be congruent to each other modulo the same prime divisor of rep(G).

Basic bounds

Proposition

Let ℓ be the smallest prime $\geq t$, p the smallest prime $\geq N$ and q the smallest prime divisor of rep(G). Then

$$\ell \leq q \leq \ell^2 \text{ and } qn_t \leq \text{rep}(G) \leq \ell p$$

Coherent labelings

Fortunately, some of the framework from the bipartite case may be salvaged:

Lemma

(Coherent labeling lemma) Let $f : V(G) \to \{0, 1, \ldots, r-1\}$ *be a representation of (a complete multipartite graph)* G *modulo* r*. Then there exists a* coherent *representation modulo* r*, i.e. a representation* $\tilde{f} : V(G) \to \{0, 1, \ldots, r-1\}$ *such that for each* i*,* $1 \leq i \leq t$*, there exists a prime divisor* p_i *of* r *such that*

$$\tilde{f}(u) \equiv \tilde{f}(v) (\textit{mod } p_i)$$

for all $u, v \in A_i$*.*

Prime Factorization of rep$(K_{n_1,\ldots,n_t})$

Theorem

Let G be a complete t-partite graph, where $t \geq 2$. When $|V(G)|$ is sufficiently large, $rep(G)$ takes one of the following forms:

$$p^a,\ p^a q^b,\ p^a q^b u$$

where p, q and u are primes with $p < q < u$ and $a, b \geq 1$.

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- The proof hinges on Ingham's result.
- One can't use nice functions like ϕ or ψ, so one needs to rely on "label wastage" arguments.

References

- R. Akhtar, A. B. Evans, and D. Pritikin. Representation Numbers of Stars. *Integers* **10** (2010), 733-745.
- R. Akhtar, A. B. Evans, and D. Pritikin. Representation Numbers of Complete Multipartite Graphs, *Discrete Mathematics* **3112** (2012), 1158-1165.

References

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Questions?