# Vector Space Secret Sharing Scheme

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# Introduction

1. Security in cryptography is based on the secret key K.

2. In private-key cryptography, some time it is not secure to give secret key to an individual(participant).

3. Therefore secret sharing scheme was introduced to share secret key $K$ among authorized group of participants.

# Secret Sharing Scheme

Secret sharing scheme works as follows: Let $\mathcal{P} = \{P_1, P_2, ..., P_n\}$ be set of all participants.

STEP 1: Determine authorized group

STEP 2: Secure and public information are given to all participants for secret key $K$.

STEP 3: When authorized group of participants pool their share, then they will recover the secret key $K$.

STEP 4: If one or more participants are missing from the group, then remaining members of the authorized group cannot determine the secret key K.

**Example:** Time magazine(May 4, 1992)

Russian nuclear ignition key

$\mathcal{P} = \{$Boris Yeltsin, Yevgeni Shaposhnikov, Defence Ministry$\}$

Authorized group $B \subset \mathcal{P}$ such that $|B| = 2$.

# Basic Secret Sharing Schemes

Some of the well-known secret sharing schemes are:

1) The Shamir Threshold Scheme (also Blakley)

2) The Monotone Circuit Construction

3) Brickell Vector Space Construction

# Brickel Vector Space Construction

Let $\mathcal{P} = \{P_i, P_2, ..., P_n\}$ be set of participants and $\Gamma = \{B_1, B_2, ..., B_k\}$ be an access structure on $\mathcal{P}$.

Let $p$ be large enough prime number and $d \geq 2$ be an integer number.

Suppose there exist a function $\phi : \mathcal{P} \longrightarrow (\mathcal{Z}_p)^d$ with the following property:

$$(1, 0, ..., 0) =< \phi(P_i) : P_i \in B > \Leftrightarrow B \in \Gamma = \{B_1, ..., B_k\}. \qquad \textbf{(1)}$$

# Brickel Vector Space Construction

**Algorithm I: Vector Space Sharing Scheme**
(Due to Brickell)
**Input:** access structure $\Gamma$ and $\phi$ function satisfying (**1**)
**Initial Phase:**
1) for $1 \leq i \leq n$
2) $\quad D$ gives public share $\phi(P_i) \in (\mathcal{Z}_p)^d$ to $P_i$
**Share Computation:**
3) $D$ chooses secret key $K \in \mathcal{Z}_p$
4) $D$ secretly chooses $a_2, a_3, ..., a_d \in \mathcal{Z}_p$ and forms vector $\mathbf{a} = (K, a_2, a_3, ..., a_d)$
5) for $i = 1$ to $n$
6) $\quad D$ computes $y_i = \mathbf{a}.\phi(P_i)$
7) $\quad D$ gives secret share $y_i$ to $P_i$

# Brickel Vector Space Construction

**Example:** Let $\mathcal{P} = \{P_1, P_2, P_3, P_4\}$ be set of participants and $\Gamma = \{B_1, B_2\} = \{\{P_1, P_2, P_3\}, \{P_1, P_4\}\}$ be access structure. By trial and error we can find the following $\phi$ function, where $d = 3, p \geq 3$:

$$\phi(P_1) = (0, 1, 0)$$
$$\phi(P_2) = (1, 0, 1)$$
$$\phi(P_3) = (0, 1, -1)$$
$$\phi(P_4) = (1, 1, 0)$$

$(1, 0, 0) = \phi(P_2) - \phi(P_1) + \phi(P_3)$, where $B_1 = \{P_1, P_2, P_3\} \in \Gamma$

$(1, 0, 0) = \phi(P_4) - \phi(P_1)$, where $B_2 = \{P_1, P_4\} \in \Gamma$

No other subset of $\mathcal{P}$ which does not contain $B_1$ or $B_2$ cannot create $(1, 0, 0)$

# Brickel Vector Space Construction

We will represent $\phi$ as a mmatrix

$\phi =$

| | | |
|---|---|---|
| 0 | 1 | 0 |
| 1 | 0 | 1 |
| 0 | 1 | -1 |
| 1 | 1 | 0 |

**Algorithm I** is very efficient algorithm but requirement of existence of function $\phi$ is the only drawback

There is no known efficient algorithm to construct such function $\phi$ for any given access structure $\Gamma$

Stinson indicated in his book that trail and error(brute force search) is the only way to find it

For large parameters $n, p, d$ exhausted search is time consuming

# $\phi$ Functions for Special Access Structures

Even if construction of such function $\phi$ is not very easy for every access structure

There is very elegant algorithm to construct a $\phi$ function for one particular access structure.

Let $G = (V, E)$ be a complete multipartite graph

Then define participant set $\mathcal{P} = V$ and access structure $\Gamma = E$

Construction of $\phi$ function for the vector space secret sharing is very easy(based on theorem in Stinson)

# $\phi$ Functions for Special Access Structures

**Example:** Complete bipartite graph $G = (V, E)$
$V = \{P_1, P_2, P_3, P_4, P_5\}$ and
$E = \{\{P_1, P_3\}, \{P_1, P_4\}, \{P_1, P_5\}, \{P_2, P_3\}, \{P_2, P_4\}, \{P_2, P_5\}\}$
$\mathcal{P} = V$, $\Gamma = E$, and $V(G) = V_1 \cup V_2 = \{P_1, P_2\} \cup \{P_3, P_4, P_5\}$.

Pick two $x_1 = 1, x_2 = 2$, of $(\mathcal{Z}_p)^2$, where $p \geq 2$ and function as follows:

$$\phi = \begin{array}{|c|c|} \hline x_1 & 1 \\ \hline x_1 & 1 \\ \hline x_2 & 1 \\ \hline x_2 & 1 \\ \hline x_2 & 1 \\ \hline \end{array} = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 1 & 1 \\ \hline 2 & 1 \\ \hline 2 & 1 \\ \hline 2 & 1 \\ \hline \end{array}$$

# $\phi$ Functions for Special Access Structures

**Algorithm II: Construction of $\phi$ for multipartite graph**

**Input:** Complete multipartite graph $G = (\mathcal{P}, \Gamma)$

1) determine disjoint partitions of $V(G) = \cup_{i=1}^{k} V_i$
2) choose distinct $x_i \in \mathcal{Z}_p$ for $i = 1, 2, ..., k$, where $p \geq k$
3) for $j = 1$ to $|\mathcal{P}|$
4) if $P_j \in V_i$, for some $i$
5) define $\phi(P_j) = (x_i, 1)$
6) return $\phi$

# Special Access Structure I

Let $G = (V, E)$ a multipartite graph but not complete

$\mathcal{P} = V$ and $\Gamma = E$ such that

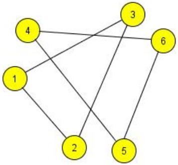
$\Gamma = \{B_1, B_2, ..., B_m\}$ has the following properties:

1) $B_i \cap B_j = \emptyset$ for all $i \neq j$

2) $|B_i| = k$ for $i = 1, 2, ..., m$

# Special Access Structure I

**Example:** $G = (V, E)$ with $V = \{1,4\} \cup \{2,5\} \cup \{3,6\}$ and $E = \{(1,2),(1,3),(2,3),(4,5),(4,6),(5,6)\}$




$\mathcal{P} = V = \{1,2,3,4,5,6\}$ and $\Gamma = \{B_1, B_2\} = \{\{1,2,3\},\{4,5,6\}\}$

$|B_i| = k = 3$ so $d = 2k - 1 = 6 - 1 = 5$, and let us take $p = 5$
First construct $A_1$ and $A_2$ for $B_1 = \{1, 2, 3\}$ and $B_2 = \{4, 5, 6\}$, respectively

$A_1 =$

| | | | | |
|---|---|---|---|---|
| 1 | 1 | 0 | 2 | 0 |
| 0 | 1 | 1 | 2 | 2 |
| 0 | 0 | 1 | 0 | 2 |

$A_2 =$

| | | | | |
|---|---|---|---|---|
| 1 | 1 | 0 | 3 | 0 |
| 0 | 1 | 1 | 3 | 3 |
| 0 | 0 | 1 | 0 | 3 |

Then $\phi$ is

$\phi =$

| |
|---|
| $A_1$ |
| $A_2$ |

$=$

| | | | | |
|---|---|---|---|---|
| 1 | 1 | 0 | 2 | 0 |
| 0 | 1 | 1 | 2 | 2 |
| 0 | 0 | 1 | 0 | 2 |
| 1 | 1 | 0 | 3 | 0 |
| 0 | 1 | 1 | 3 | 3 |
| 0 | 0 | 1 | 0 | 3 |

**Algorithm III: Construction of $\phi$**

**Input:** $\mathcal{P} = \{P_1, P_2, ..., P_n\}$, $\Gamma = \{B_1, B_2, ..., B_m\}$,
where $B_i \cap B_j = \emptyset$ for all $i \neq j$ and $|B_i| = k$

1) pick $x_i \in \mathcal{Z}_p$ such that $1 < x_1 < x_2 < ... < x_m$
2) for $s = 1$ to $m$
3) construct $A_s = (a_{ij})_{k \times 2k-1}$ with all 0 entries
4) for $i = 1$ to $k$
5) $a_{ii} = 1$
6) for $i = 1$ to $k - 1$
7) $a_{i(i+1)} = 1$
8) for $i = 1$ to $k - 1$
9) $a_{i(k+i)} = x_s$
10) for $i = 2$ to $k$
11) $a_{i(k+i-1)} = x_s$
12) return $\phi = \begin{array}{|c|} \hline A_1 \\ \hline A_2 \\ \hline ... \\ \hline A_m \\ \hline \end{array}$

Matrix $A_i$ constructed by **Algorithm III** will be like

| 1 | 2 | 3 | 4 | .. | k-1 | k | k+1 | k+2 | k+3 | .. | 2k-1 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | .. | 0 | 0 | $x_i$ | 0 | 0 | .. | 0 |
| 0 | 1 | 1 | 0 | .. | 0 | 0 | $x_i$ | $x_i$ | 0 | .. | 0 |
| 0 | 0 | 1 | 1 | .. | 0 | 0 | 0 | $x_i$ | $x_i$ | .. | 0 |
| .. | .. | .. | .. | .. | .. | .. | .. | .. | .. | .. | .. |
| 0 | 0 | 0 | 0 | .. | 1 | 1 | 0 | 0 | .. | $x_i$ | $x_i$ |
| 0 | 0 | 0 | 0 | .. | 0 | 1 | 0 | 0 | .. | 0 | $x_i$ |

Properties of block $A_i$:

1. The first column has unique 1.
2. Columns 2 through $k$ have exactly two 1's.
3. Columns $k+1$ through $2k-1$ have exactly two $x_i$'s.

**Lemma**

*Let $B_i = \{P_{i_1}, P_{i_2}, ..., P_{i_k}\}$ be an authorized set. Assume $A_i$ is created by* **Algorithm III** *for $B_i$. Then $(1, 0, 0, ..., 0)$ can be written as linear combination of shares, i.e. rows of $A_i$, of $B_i$ but if one or more rows of $A_i$ is missing, then $(1, 0, 0, ..., 0)$ cannot be written as linear combination of remaining rows of $A_i$.*

## Proof.

Let $a_j$ be $j - th$ row of $A_i$. Then
$(1,0,0,...,0) = (a_1 + a_3 + ...) - (a_2 + a_4 + ...)$ by properties of $A_i$
Now let $C = \{P_{i_{j_1}}, P_{i_{j_2}}, ..., P_{i_{j_l}}\} \subset B_i$. Without loose of generality we can assume that $i_{j_1} < i_{j_2} < ... < i_{j_l}$.
If $i_{j_i} \neq 1$, then it is obvious that $(1,0,0,...,0)$ cannot be linear combination of these rows. Hence $P_{i_{j_1}} = P_1$.
Since $C$ is unauthorized, there is at least one participant $P_{i_{j_s}}$ which is not in $C$. Let $s$ be the smallest index such that $P_{i_{j_s}} \notin C$
Let $a_1, a_2, ..., a_l \in \mathcal{Z}_p$
Suppose:
$(1,0,0,...,0) = a_1(1,1,...,x_i,0,...,0) + \sum_{r=2}^{l} a_r \phi(P_{i_{j_r}}) \Leftrightarrow$
$a_1 = 1, a_1 + a_2 = 0, ...., a_{s-2} + a_{s-1} = 0, a_{s-1} = 0, ...$ where $s \geq 2$.
Since $a_1 = 1$, then $a_2 = -1(p-1$ in $\mathcal{Z}_p)$ so on, hence we get $a_{s-1} = 1$ (or $-1$ based on even or odd $s$ value) contradiction with $a_{s-1} = 0$. □

**Theorem**

*Let $\mathcal{P} = \{P_1, P_2, ..., P_n\}$ be set of participants. Access structure $\Gamma = \{B_1, B_2, ...,$
$B_m\}$ is given where $B_i \cap B_j = \emptyset$ for all $i \neq j$ and $|B_i| = k$ for $i = 1, 2, ..., m$. Then the function $\phi$, which is constructed by* **Algorithm III**, *satisfies* **(1)**.

**Proof.**

Let $C = \{P_{j_1}, P_{j_2}, ..., P_{j_l}\} \subset \mathcal{P}$. If $C$ is an authorized set, then $B_i \subset C$ for some $i$. Hence by previous lemma we are done.

If $C$ is not authorized set, then we have the following cases:

Case 1: If $|C| = l < k$

Case 2: If $|C| = l = k$

Case 3: If $|C| = l > k$ $\square$