Constructing universal graphs

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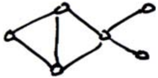
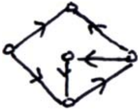
MIGHTY LII
28 April 2012

Types of graphs.

Simple graphs: edges are unordered pairs of vertices (no repetitions)



Directed graphs: edges are ordered pairs of vertices

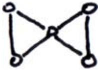


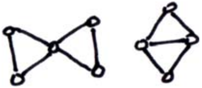
Multi-graphs: edges are unordered "pairs" of vertices (repetition allowed)

Universal graphs

Let $\mathcal{F}$ be a collection of graphs. Then we say that a graph U is a universal graph for $\mathcal{F}$ if every graph in $\mathcal{F}$ is a subgraph of U.

Example:

$\mathcal{F} = \{$  ,  $\}$

$U_1 =$ 

$U_2 =$ 

Families for universal graphs

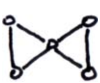
Families for which universal graphs have been studied:

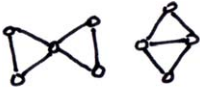
- Trees on n vertices [Bhatt et al. '89; Chung et al. '81; Friedman and Pipenger '87; Gol'dberg and Livšic '68; Nebeský '75; Yang '92]
- Planar graphs with bounded degree [Capalbo '02]
- Caterpillars [Chung and Graham '81]
- Cycles [Bondy '71]
- Sparse graphs [Babai et al. '82; Rodl '81]
- Graphs with bounded degree [Alon et al. and Capalbo et al. '99 '00 '01 '02 '07]

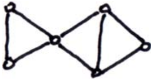
Induced universal graphs

Let $\mathcal{F}$ be a collection of graphs. Then we say that a graph U is an induced universal graph for $\mathcal{F}$ if every graph in $\mathcal{F}$ is an induced subgraph of U.

Example:

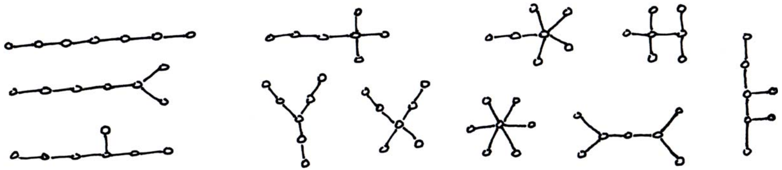
$\mathcal{F} = \{$  ,  $\}$

$U_1 =$ 

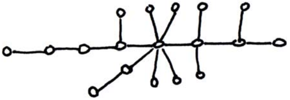
$U_2 =$ 

An example with trees

Let $\mathcal{F}$ be the set of trees on 7 vertices.



Then the following is an induced universal graph for $\mathcal{F}$.



Families for induced universal

Families for which induced universal graphs have been studied include

- All graphs on n vertices [Moon '65]
- Tournaments [Moon '68]
- Trees on n vertices [Chung et al. '81]
- Planar graphs [Chung '90]
- Graphs with bounded arboricity [Chung '90]
- Graphs with bounded degree [Butler]

Main result

Let $\mathcal{F}$ be all graphs on n vertices with maximum degree at most r. Then there is an induced universal graph U such that

$$|V(U)| \leq Cn^{\lceil r/2 \rceil} \qquad \text{and} \qquad |E(U)| \leq Dn^{2\lceil r/2 \rceil - 1}$$

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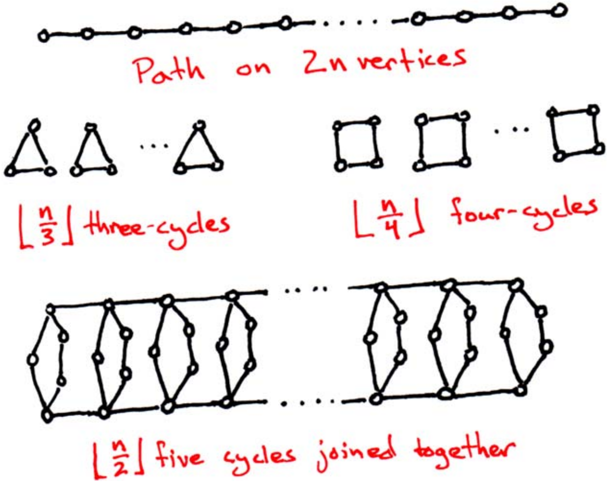
The proof consists of three major parts:

1. We can decompose our graphs into $\lceil r/2 \rceil$ graphs each one of which has degrees at most 2. (Petersen)
2. Find induced universal graph for the case $r = 2$.
3. Small induced universal graphs can be combined to form large induced universal graphs.

Induced universal graph for r=2



Note: $|V(U)| \leq 6.5n$ and $|E(U)| \leq 7.5n$.

Claim

The previous graph is an induced universal graph for the family of graphs on n vertices with maximum degree at most 2.

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Sketch of proof

If G has maximum degree 2 then it is composed of paths and cycles.

- Embed the paths of G in the long path of U.
- Embed the three-cycles of G in the three-cycles of U.
- Embed the four-cycles of G in the four-cycles of U.

How to embed longer cycles

Example:

To insert a cycle of length b we need to use $\lfloor b/2 \rfloor - 1$ five cycles. Since we also need to add a “buffer” five cycle between consecutive embedded cycles; it follows we need at most $\lfloor n/2 \rfloor$ five cycles strung together in order to embed all the cycles.

Theorem (Chung '90)

Let $\mathcal{F}$ be a family of graphs and U a corresponding induced universal graph for $\mathcal{F}$. If $\mathcal{H}$ is a family of graphs where each graph can be broken into k subgraphs each belonging in $\mathcal{F}$, then there is an induced universal graph W for $\mathcal{H}$ such that

$$|V(W)| = |V(U)|^k \qquad \text{and} \qquad |E(W)| \leq k|V(U)|^{2k-2}|E(U)|.$$

Note: in general do not need to assume that all the $\mathcal{F}$ are the same, i.e., can have k different families of graphs and k different corresponding universal graphs.

Construction of Chung

Given the graph U we form W as follows:

- Vertices of W are k-tuples of vertices of U, i.e., $(u_1, u_2, \ldots, u_k)$.
    - This gives W exactly $|V(U)|^k$ vertices.

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- Vertices of W are k-tuples of vertices of U, i.e., $(u_1, u_2, \ldots, u_k)$.
  - This gives W exactly $|V(U)|^k$ vertices.
- $(u_1, u_2, \ldots, u_k)$ is adjacenct to $(u_1', u_2', \ldots, u_k')$ in W if and only if for some i the vertex u_i is adjacent to u_i' in U.
  - Any edge $\{u, u'\}$ in U can form at most $k|V(U)|^{2k-2}$ edges in W. Namely pick an i from 1 to k and then fix the two entries u_i and u_i', finally all the remaining $2k-2$ entries can vary. So there are at most $k|V(U)|^{2k-2}|E(U)|$ edges.

How good is the result?

We certainly have that the number of induced subgraphs of our induced universal graph is as least as large as the number of graphs in the family. So for n large:

$$\frac{|V(U)|^n}{n!} \geq \binom{|V(U)|}{n} \geq |\mathcal{F}| \geq e^{-(r^2-1)/4}\left(\frac{r^{r/2}}{e^{r/2}r!}\right)^n n^{rn/2}/n!$$

So $|V(U)| \geq cn^{r/2}$ for a constant c depending only on r.

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So for r even within a constant multiple of smallest number of vertices. For r odd off by a factor of $n^{1/2}$. Noga Alon has closed this gap for odd n.

Generalizations

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Multigraph result

Let $\mathcal{F}$ be all multi-graphs on n vertices with maximum degree at most r. Then there is an induced universal multi-graph U such that

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Generalizations

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Directed graph result

Let $\mathcal{F}$ be all directed graphs on n vertices with maximum in-degree and out-degree at most r. Then there is an induced universal directed graph U such that

$$|V(U)| \leq Cn^{r} \qquad \text{and} \qquad |E(U)| \leq Dn^{2r-1}.$$

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