Vertex-transitive graphs

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A subgroup \( G \) of the symmetric group \( S_X \) on the set \( X \) is \textit{transitive} if whenever \( x, y \in X \), then there exists \( g \in G \) such that \( g(x) = y \).
A subgroup $G$ of the symmetric group $S_X$ on the set $X$ is **transitive** if whenever $x, y \in X$, then there exists $g \in G$ such that $g(x) = y$. A graph $\Gamma$ is **vertex-transitive** if its automorphism group $\text{Aut}(\Gamma)$ is transitive on $V(\Gamma)$, the vertex set of $\Gamma$. 
Basic Definitions

Definition

A subgroup $G$ of the symmetric group $S_{X}$ on the set $X$ is transitive if whenever $x, y \in X$, then there exists $g \in G$ such that $g(x) = y$. A graph $\Gamma$ is vertex-transitive if its automorphism group $\text{Aut}(\Gamma)$ is transitive on $V(\Gamma)$, the vertex set of $\Gamma$.

Intuitively, a graph is vertex-transitive if there is no structural (i.e. non-labeling) way to distinguish vertices of the graph.
Here the vertices of the Petersen graph $P$ are labeled with 2-element subsets of $\{1, 2, 3, 4, 5\}$ and two vertices are adjacent if and only if their intersection is empty.
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Form a bipartite graph with bipartition sets the lines of $\mathbb{F}_2^3$ and the hyperplanes of $\mathbb{F}_2^3$.

Figure: The Heawood graph labeled with the lines and hyperplanes of $\mathbb{F}_2^3$.
Form a bipartite graph with bipartition sets the lines of $\mathbb{F}_2^3$ and the hyperplanes of $\mathbb{F}_2^3$. A line is adjacent to a hyperplane if and only if the hyperplane contains the line.

**Figure**: The Heawood graph labeled with the lines and hyperplanes of $\mathbb{F}_2^3$.
Form a bipartite graph with bipartition sets the lines of $\mathbb{F}_2^3$ and the hyperplanes of $\mathbb{F}_2^3$. A line is adjacent to a hyperplane if and only if the hyperplane contains the line. The graph is the Heawood graph.
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Consider all linear transformations of $\mathbb{F}_2^3$ to $\mathbb{F}_2^3$ (or matrices if you like) permuting lines and hyperplanes of $\mathbb{F}_2^3$. Such a linear transformation will take a line contained in a hyperplane to a line contained in a hyperplane, and so induces an automorphism of the Heawood graph $H$. Some linear algebra will also show that the function which maps a subspace to its orthogonal complement is also an automorphism of $H$. Thus $\text{Aut}(H)$ is vertex-transitive. These are all of the automorphisms of $H$, and in group theory language $\text{Aut}(H) = \mathbb{Z}_2 \rtimes \text{PΓL}(3,2)$.
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Definition

A group $G \leq S_X$ is **doubly-transitive** if whenever $(x_1, y_1), (x_2, y_2) \in X \times X$ such that $x_1 \neq y_1$ and $x_2 \neq y_2$, then there exists $g \in G$ such that $g(x_1, y_1) = (x_2, y_2)$. 

Note that if $\Gamma$ is a graph with doubly-transitive automorphism group, then it is complete or has no edges and so its automorphism group is the symmetric group.

As in a 3-dimensional vector space there is a linear transformation which maps any two different one-dimensional subspaces to any other two different one-dimensional subspaces, there is a subgroup of $\text{Aut}(H)$ which is doubly-transitive on lines (and hyperplanes).
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Cayley graphs

Definition

Let $G$ be a group and $S \subset G$ such that $1 \notin S$ and $S = S^{-1}$. Define a Cayley digraph of $G$, denoted $\text{Cay}(G, S)$, to be the graph with $V(\text{Cay}(G, S)) = G$ and $E(\text{Cay}(G, S)) = \{(g, gs) : g \in G, s \in S\}$. We call $S$ the connection set of $\text{Cay}(G, S)$.
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Figure: The Cayley graph $\text{Cay}(\mathbb{Z}_{10}, \{1, 3, 7, 9\})$. 
For $h \in G$, define $h_L : G \to G$ by $h_L(x) = hx$. Then $h_L(g, gs) = (hg, hgs)$, and so $h_L$ is an automorphism of a Cayley graph. We set $G_L = \{h_L : h \in G\}$. $G_L$ is the left regular representation of $G$. So $G_L \leq Aut(Cay(G, S))$. If $h, g \in G$, then $(gh^{-1})_L(h) = gh^{-1}h = g$. Thus Cayley graphs are vertex-transitive graphs. Think of a Cayley graph $Cay(G, S)$ as being constructed in the following way. First, the neighbors of a vertex, the identity in $G$, are specified via $S$. The rest of the edges of $Cay(G, S)$ are then obtained by translating the neighbors of 1 using elements of $G_L$. 
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In 1969, Lovász proposed the following problem, usually attributed as a conjecture:

Problem
Let us construct a finite, connected, undirected graph which is symmetric and has no simple path containing all elements. A graph is called symmetric, if for any two vertices $x$, $y$ it has an automorphism mapping $x$ into $y$.

It has also been conjectured that every connected Cayley graph on at least 3 vertices contains a Hamilton cycle, as the only 4 such graphs known are non-Cayley (the Petersen graph, the Coxeter graph, and graphs obtained from these by replacing a vertex with a triangle).
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There are many results on this conjecture, and we list some of the most well-known:

- Every connected Cayley digraph of a $p$-group, $p$ a prime, contains a directed Hamiltonian cycle (Witte, 1986)
- Every vertex-transitive graph of order $pq$ whose automorphism group does not contain a normal transitive simple group is Hamiltonian with the exception of the Petersen graph (Marušić (1983), Alspach and Parsons, 1982)
- Cayley graphs of groups whose commutator subgroup is a cyclic $p$-group (Keating and Witte, 1985)
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Some recent results

Cayley graphs on nilpotent groups with cyclic commutator subgroup are hamiltonian (Ghaderpour and Witte Morris (2012?))

Odd-order Cayley graphs with commutator subgroup of order \( pq \) are hamiltonian, \( p \) and \( q \) distinct primes (Witte Morris (2013?))

Cayley graphs of groups of order less than 120 except some groups of order 72, 96, and 108 (Kutnar, Marušič, Witte Morris, Morris and Sparl (2012))
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Theorem (Burnside, 1901)

Let $G$ be a transitive group of prime degree $p$ that contains $(\mathbb{Z}_p)^L$. Then either $G \leq AGL(1,p) = \{x \rightarrow ax + b: a \in \mathbb{Z}^*_p, b \in \mathbb{Z}_p\}$ or $G$ is doubly-transitive.

$AGL(1,p)$ is the normalizer of $(\mathbb{Z}_p)^L$ in $S_p$.

Recall that a graph with doubly-transitive automorphism group is necessarily complete or has no edges with automorphism group a symmetric group.

We then have

Corollary

Let $\Gamma$ be a Cayley graph of $\mathbb{Z}_p$, $p$ a prime. Then $\text{Aut}(\Gamma) \leq AGL(1,p)$ or $\text{Aut}(\Gamma) = S_p$. 

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**Theorem (D., 2005)**

Let $G \leq S_{p^k}$ be such that every minimal transitive subgroup of $G$ is cyclic of order $p^k$. Then either $G$ has a normal Sylow $p$-subgroup or $G$ is doubly-transitive.

**Theorem (D., C.H. Li, P. Spiga, 2012?)**

Let $G$ be a transitive group of degree $n$ such that contains the left-regular representation of some abelian group $H$. If $H$ is a Hall $\pi$-subgroup of $G$, then either $H$ is normal in $G$ or $G$ is doubly-transitive. Here $\pi$ is the set of divisors of $n$. 

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Theorem (Muzychuk, 1997)

The values of $n$ for which any two ciculant graphs of order $n$ are isomorphic if and only if they are isomorphic by an automorphism of $\mathbb{Z}_n$ are $n = m$ and $4m$, where $m$ is square-free, or $n = 8, 9, 18$. 

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It is not hard to show that the image of a Cayley graph \( \text{Cay}(G, S) \) under a group automorphism of \( G \) is the Cayley graph \( \text{Cay}(G, \alpha(S)) \).
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**Theorem (Muzychuk, 1997)**

The values of $n$ for which any two ciculant graphs of order $n$ are isomorphic if and only if they are isomorphic by an automorphism of $\mathbb{Z}_n$ are $n = m$ and $4m$, where $m$ is square-free, or $n = 8, 9, 18$. 
Ádám’s conjecture was generalized into the following question:

Problem
For which groups $G$ is it true that any two Cayley graphs of $G$ are isomorphic if and only if they are isomorphic by a group automorphism of $G$?

A group $G$ for which the answer to the preceding question is ‘Yes’ is called a CI-group with respect to graphs.

We say "with respect to graphs" as the same question can be asked of other "combinatorial objects" (and has been - even in the late 1920's and early 1930's for designs). Many papers have been written on this topic!
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Essentially every result on the isomorphism problem makes use of the following result of Babai published in 1977.

**Lemma**

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then $G$ and $\delta^{-1}G$ are conjugate in $\text{Aut}(\text{Cay}(G, S))$.

There are more general versions of this lemma for when $G$ is not a CI-group with respect to graphs, and to when a graph is not a Cayley graph. All versions essentially say that the isomorphism problem boils down to the conjugacy classes of $G$ (or some other appropriate group if the graph is not Cayley).
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Ted Dobson
Vertex-transitive graphs
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\( \mathbb{Z}_p \) is a CI-group

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Let $\delta \in S_p$ such that $\delta^{-1}(\mathbb{Z}_p)_L\delta \leq \text{Aut}(\text{Cay}(\mathbb{Z}_p, S))$. Note that $(\mathbb{Z}_p)_L$ has order $p$, and that a Sylow $p$-subgroup of $S_p$ has order $p$ as $|S_p| = p!$. 
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Hence \( \delta^{-1}(\mathbb{Z}_p)_L\delta \) and \( (\mathbb{Z}_p)_L \) are Sylow \( p \)-subgroups of \( \text{Aut}(\text{Cay}(G, S)) \) and so are conjugate by a Sylow Theorem.

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Show that the following two graphs are isomorphic.
Imprimitive groups

Definition

A subset $B \subset X$ is called a block of a transitive permutation group $G \leq S_X$ if $g(B) = B$ or $g(B) \cap B = \emptyset$ for all $g \in G$.

Singleton sets are always blocks as is $X$ itself—these are trivial blocks. If $B$ is a block of $G$, then $g(B)$ is also a block of $G$, and $\{g(B) : g \in G\}$ is a complete block system of $G$.

A permutation group with a nontrivial block is an imprimitive group, and if $G$ is primitive if it has no nontrivial blocks.

The automorphism group of the Petersen graph is primitive, while the automorphism group of the Heawood graph is imprimitive, with the lines and hyperplanes of $F_3^2$ forming a complete block system with 2 blocks of size 7.
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The direct product of all minimal normal subgroups of a primitive group is a direct product of isomorphic simple groups.
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The Big Problem

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For example, the subgroup of the automorphism group of the Heawood graph that permutes the lines amongst themselves is doubly-transitive but not a symmetric group.
THANKS!