Vertex-transitive graphs

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Basic Definitions

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Intuitively, a graph is vertex-transitive if there is no structural (i.e. non-labeling) way to distinguish vertices of the graph.

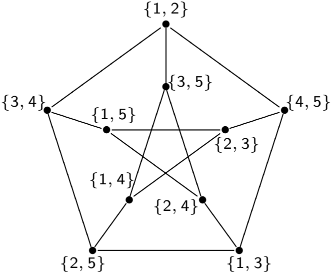
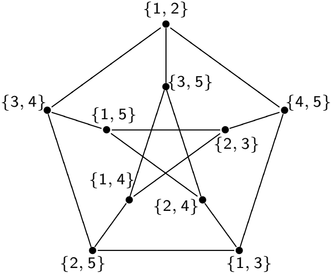


Figure : The 2-subset labeling of the Petersen graph

Here the vertices of the Petersen graph P are labeled with 2-element subsets of $\{1, 2, 3, 4, 5\}$ and two vertices are adjacent if and only if their intersection is empty.



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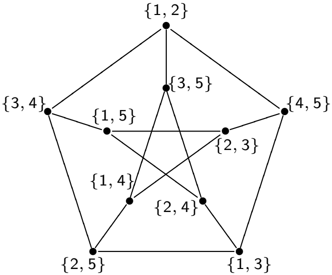

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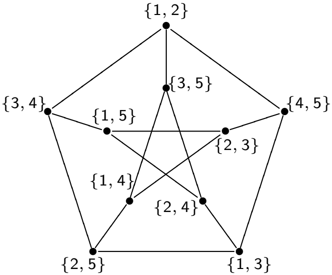

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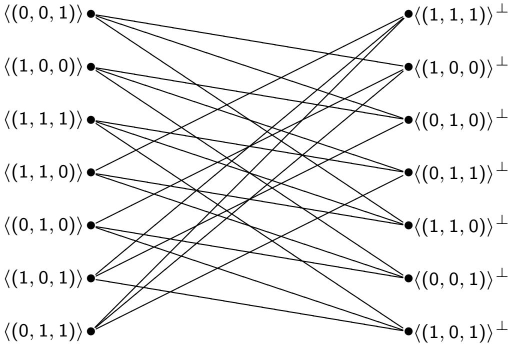
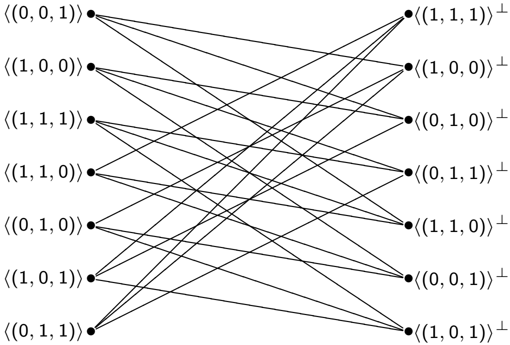


Figure : The Heawood graph labeled with the lines and hyperplanes of $\mathbb{F}_2^3$

Form a bipartite graph with bipartition sets the lines of $\mathbb{F}_2^3$ and the hyperplanes of $\mathbb{F}_2^3$.



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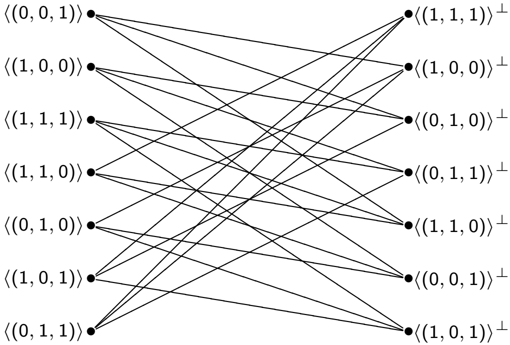

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A group $G \leq S_X$ is doubly-transitive *if whenever $(x_1, y_1), (x_2, y_2) \in X \times X$ such that $x_1 \neq y_1$ and $x_2 \neq y_2$, then there exists $g \in G$ such that $g(x_1, y_1) = (x_2, y_2)$.*

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As in a 3-dimensional vector space there is a linear transformation which maps any two different one-dimensional subspaces to any other two different one-dimensional subspaces, there is a subgroup of $\mathrm{Aut}(H)$ which is doubly-transitive on lines (and hyperplanes).

Cayley graphs

Definition

Let G be a group and $S \subset G$ such that $1 \notin S$ and $S = S^{-1}$. Define a Cayley digraph of G, denoted $\mathrm{Cay}(G, S)$, to be the graph with $V(\mathrm{Cay}(G, S)) = G$ and $E(\mathrm{Cay}(G, S)) = \{(g, gs) : g \in G, s \in S\}$. We call S the connection set of $\mathrm{Cay}(G, S)$.

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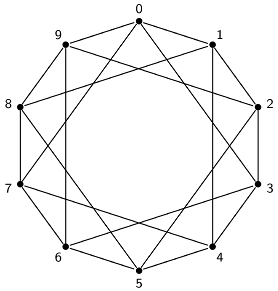

Figure : The Cayley graph $\mathrm{Cay}(\mathbb{Z}_{10}, \{1, 3, 7, 9\})$.

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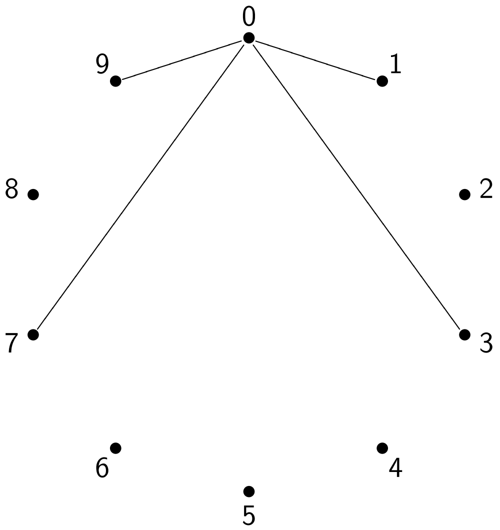

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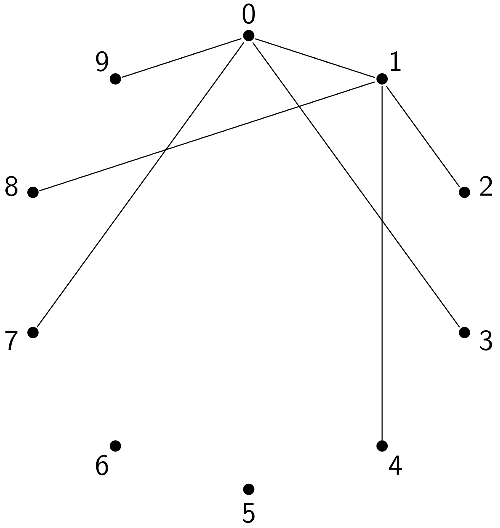

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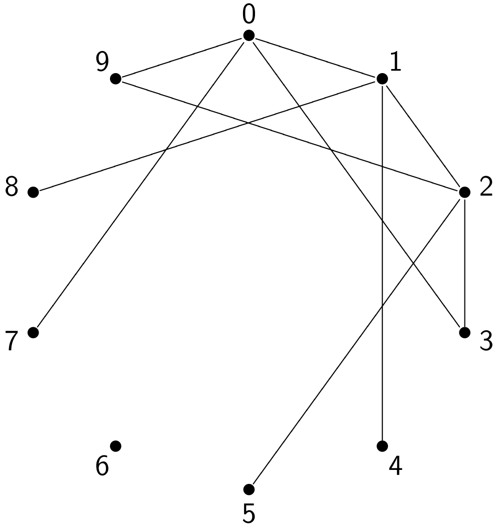

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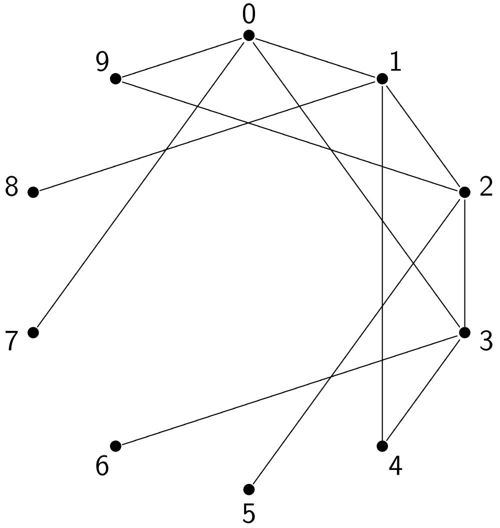

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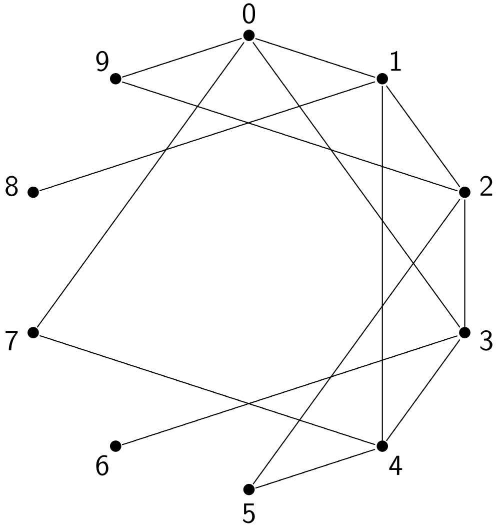

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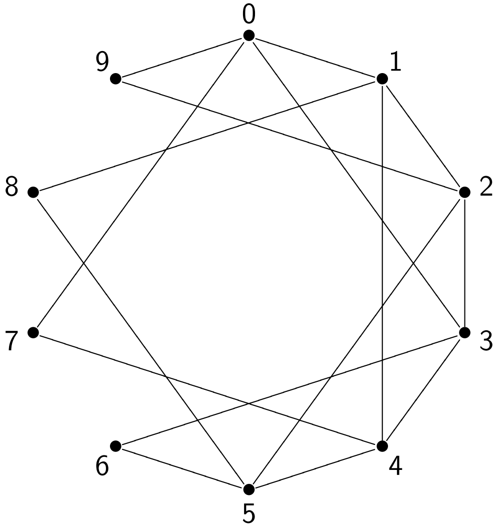

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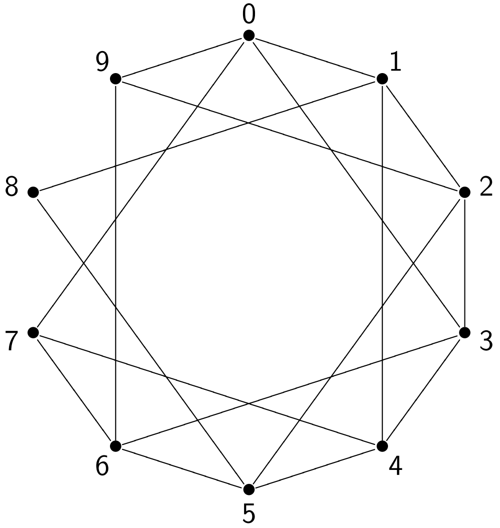

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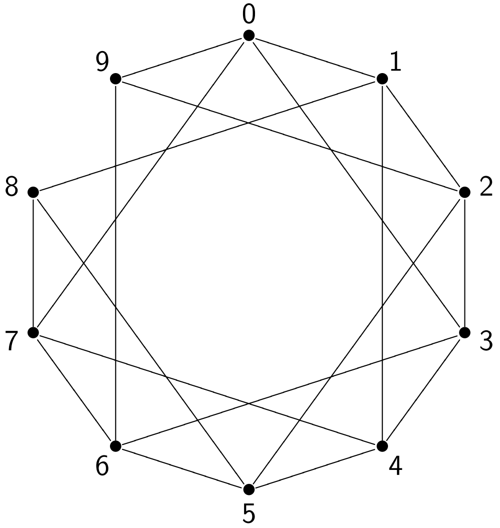

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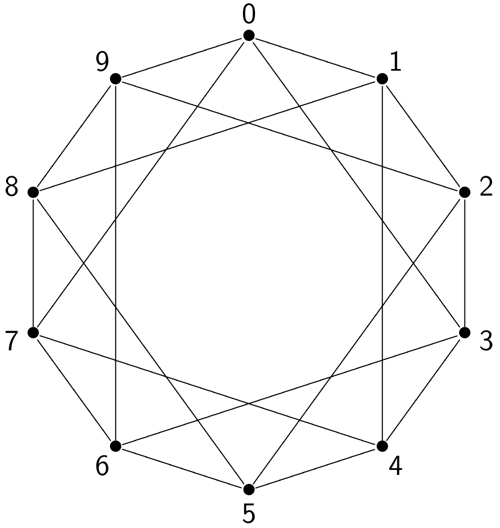

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It has also been conjectured that every connected Cayley graph on at least 3 vertices contains a Hamilton cycle, as the only 4 such graphs known are non-Cayley (the Petersen graph, the Coxeter graph, and graphs obtained from these by replacing a vertex with a triangle).

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- Cayley graphs of groups whose commutator subgroup is a cyclic p-group (Keating and Witte (1985))

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- Cayley graphs of groups of order less than 120 except some groups of order 72, 96, and 108 (Kutnar, Marušič, Witte Morris, Morris and Sparl (2012))

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Theorem (Burnside, 1901)

Let G be a transitive group of prime degree p that contains $(\mathbb{Z}_p)_L$. Then either $G \leq \mathrm{AGL}(1,p) = \{x \to ax + b : a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$ or G is doubly-transitive.

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Corollary

Let Γ be a Cayley graph of Z_p, p a prime. Then $\mathrm{Aut}(\Gamma) \leq \mathrm{AGL}(1,p)$ or $\mathrm{Aut}(\Gamma) = S_p$.

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Theorem (D., C.H. Li, P. Spiga, 2012?)

Let G be a transitive group of degree n such that contains the left-regular representation of some abelian group H. If H is a Hall π-subgroup of G, then either H is normal in G or G is doubly-transitive. Here π is the set of divisors of n.

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So to test isomorphism between two Cayley graphs of a group G, we must check whether group automorphisms of G are graph isomorphisms.

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There are more general versions of this lemma for when G is not a CI-group with respect to graphs, and to when a graph is not a Cayley graph. All versions essentially say that the isomorphism problem boils down to the conjugacy classes of G_L (or some other appropriate group if the graph is not Cayley).

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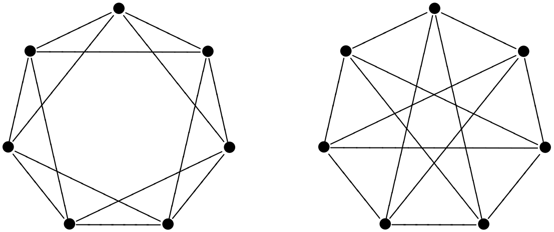
Show that the following two graphs are isomorphic.



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For example, the subgroup of the automorphism group of the Heawood graph that permutes the lines amongst themselves is doubly-transitive but not a symmetric group.

THANKS!