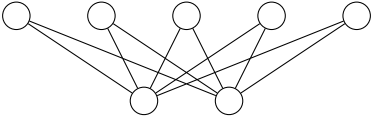


Counting independent sets in graphs with a given minimal degree

John Engbers* David Galvin

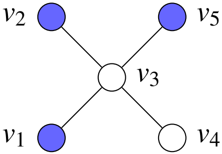
University of Notre Dame
Department of Mathematics

April 2012

An extremal question

Independent set (of vertices): A set of vertices which are pairwise non-adjacent.

$G:$



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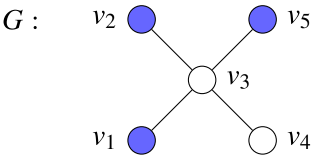

$i(G)$: Total number of independent sets in a graph G.

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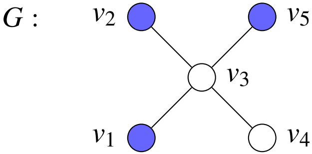

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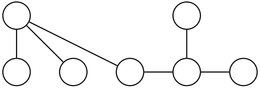
Question

Given a family of graphs $\mathcal{G}$, what is the maximum value of $i(G)$ and $i_t(G)$ as G ranges over $\mathcal{G}$?

Fixed order, trees

$\mathcal{G}(n)$: trees on n vertices



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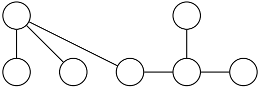

Theorem (Prodinger, Tichy 1982)

For $G \in \mathcal{G}(n)$,

- $i(G)$ *maximized by the star* $K_{1,n-1}$.

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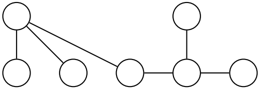

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Theorem (Wingard 1995)

For $G \in \mathcal{G}(n)$,

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Fixed order, fixed number of edges

$\mathcal{G}(n, m)$: graphs with n vertices, m edges

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Theorem (Cutler, Radcliffe 2011)

For $G \in \mathcal{G}(n, m)$,

- $i(G)$ *maximized by* Lex(n, m)
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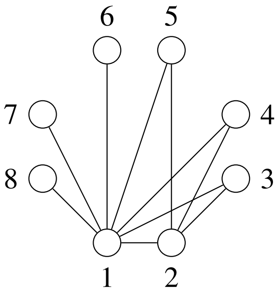
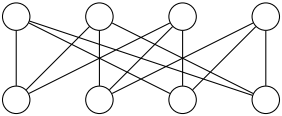


Lex$(8,11)$

Fixed order, regular of fixed degree

$\mathcal{G}(n, d)$: graphs with n vertices, d-regular



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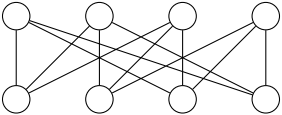

Theorem (Kahn 2001; Zhao 2011)

For $G \in \mathcal{G}(n,d)$,

- $i(G)$ *maximized by* $\frac{n}{2d}K_{d,d}$, *disjoint union of* $\frac{n}{2d}$ *copies of* $K_{d,d}$.

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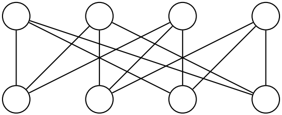

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- Asymptotic evidence for conjecture given by Carroll, G., Tetali (2009)

Today's family: fixed order, fixed minimum degree

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Intuition: fewer edges implies more independent sets.

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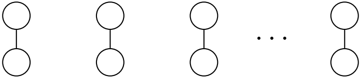

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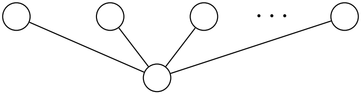

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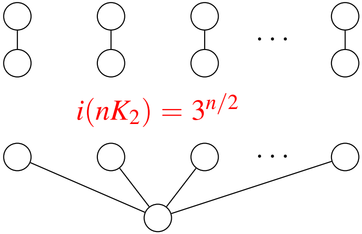

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New intuition: Maximize $\alpha(G)$

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For $n \geq 4\delta^2$ *and* $G \in \mathcal{G}(n, \delta)$,

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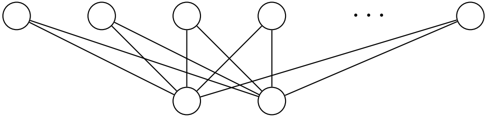

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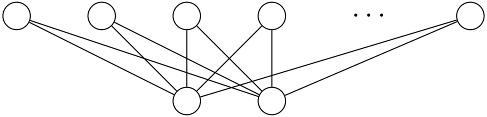

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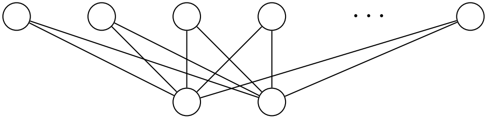

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Ordered $(t+1)$-set: choose first t, which rules out $t + \delta$ vertices, so:

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$$i_{t+1}(G) \leq \binom{n-\delta}{t+1} = i_{t+1}(K_{\delta,n-\delta}).$$

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Ordered independent t-sets starting with vertex of degree $> \delta$:

$$\#_{>\delta} \leq k(n - (\delta + 2))(n - (\delta + 3)) \cdots (n - (\delta + t))$$

where k is the number of vertices with degree $> \delta$.

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Missing term?

- Worst case situation: each new choice shares same δ neighbors

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Missing term?

- Worst case situation: each new choice shares same δ neighbors
- Can't happen $\delta + 1$ times (or we're in case 1.)
- $(\delta + 1)$st choice (at worst) removes a new vertex

Proof for $t \geq 2\delta + 1$

Have

$$\#_{>\delta} \leq k(n-(\delta+2))(n-(\delta+3))\cdots(n-(\delta+t))$$

and

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Worst case is $k=0$:

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Future improvements?

- Consider second/third/etc. choices more carefully
- Condition on the degrees of neighbors [linear programming]

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