Magic rectangle sets and ordered distance antimagic graphs

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Definition: A *magic rectangle* MR(a,b) is an $a \times b$ array with $a,b > 1$ in which the first ab positive integers are placed so that the sum over each column of MR(a,b) is $\sigma(a,b) = a(ab + 1)/2$ and the sum over each row is $\tau(a,b) = b(ab + 1)/2$.

Magic rectangle

1	7	6	4	
8	2	3	5	

MR(2,4)

Magic rectangle

1	7	6	4	**18**
8	2	3	5	**18**
9	**9**	**9**	**9**	

MR(2,4)

Magic rectangle

15	2	14	4	5	
8	10	7	9	6	
1	12	3	11	13	

MR(3,5)

Magic rectangle

15	2	14	4	5	**40**
8	10	7	9	6	**40**
1	12	3	11	13	**40**
24	**24**	**24**	**24**	**24**	

MR(3,5)

Incomplete round robin tournaments

We want to play a tournament with 8 teams such that each team plays 5 games rather than 7.

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What games to remove?

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What games to remove?

If my team avoids two best teams while your team avoids two worst teams, I have an unfair advantage.

Incomplete round robin tournaments

We want to play a tournament with 8 teams such that each team plays 5 games rather than 7.

What games to remove?

If my team avoids two best teams while your team avoids two worst teams, I have an unfair advantage.

So we may want each team play opponents with the same "strength" or maybe "mimic" the complete tournament.

Complete round robin

Team	Opponents' rankings			
1	35			
2	34			
3	33			
4	32			
5	31			
6	30			
7	29			
8	28			

Complete round robin

Team	Opponents' rankings			
1	35	= 2 + 3 + … + 8		
2	34			
3	33			
4	32			
5	31			
6	30			
7	29			
8	28			

Complete round robin

Team	Opponents' rankings			
1	35	= 2 + 3 + … + 8		
2	34	= 1 + 3 + … + 8		
3	33			
4	32			
5	31			
6	30			
7	29			
8	28			

Complete round robin

Team	Opponents' rankings			
1	35	= 2 + 3 + … + 8		
2	34	= 1 + 3 + … + 8		
3	33	= 1 + 2 + … + 8		
4	32			
5	31			
6	30			
7	29			
8	28	= 1 + 2 + … + 7		

Incomplete round robin

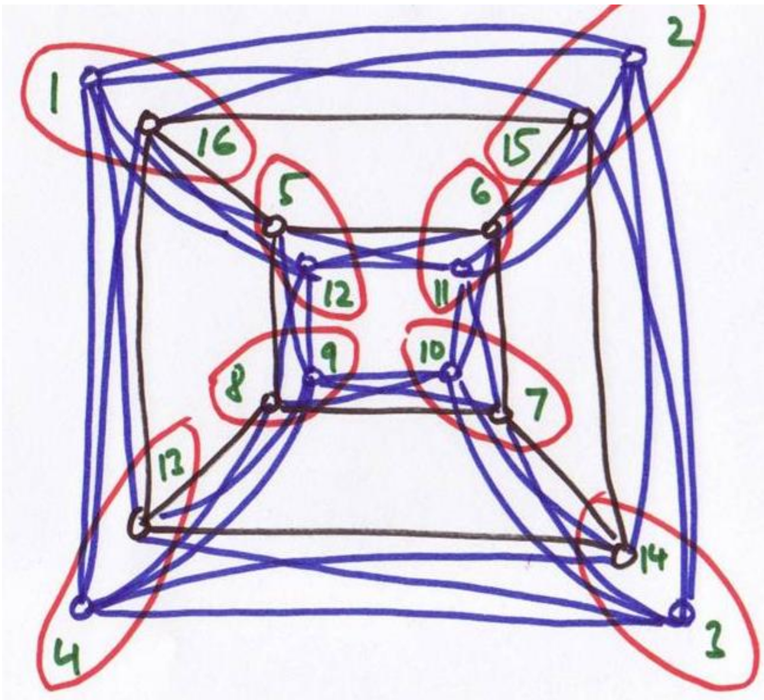
Team	Opponents' rankings		Team	Opponents' rankings
1	35		1	35 – ***m***
2	34		2	34 – ***m***
3	33		3	33 – ***m***
4	32		4	32 – ***m***
5	31		5	31 – ***m***
6	30		6	30 – ***m***
7	29		7	29 – ***m***
8	28		8	28 – ***m***

Distance magic labeling

Distance magic vertex labeling of a graph G with n vertices:

A bijection μ from the vertex set of G to $\{1, 2, \ldots, n\}$ such that sum $w(x)$ of labels of the neighbors of each vertex (called the *weight* of x) is equal to the same constant m.

Distance magic labeling



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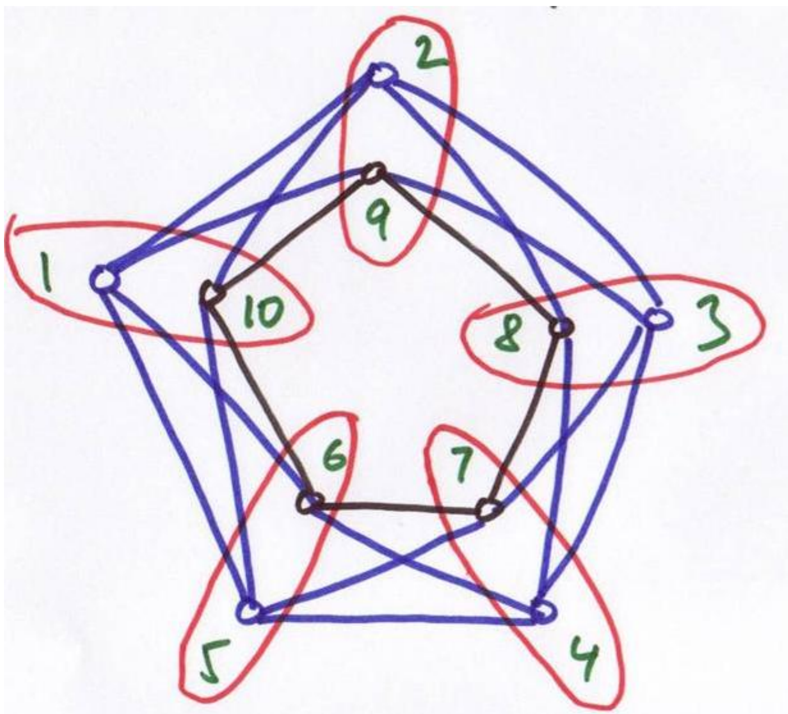

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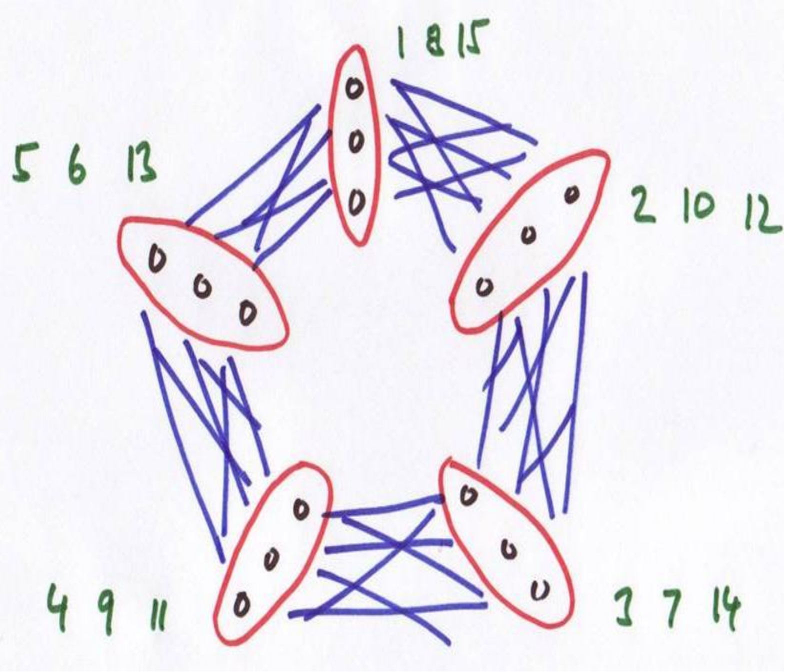
Distance magic labeling



Distance magic vertex labeling of a graph G with n vertices:

A bijection μ from the vertex set of G to $\{1, 2, \ldots, n\}$ such that sum $w(x)$ of labels of the neighbors of each vertex (called the *weight* of x) is equal to the same constant m.

Theorem 1: There is no r-regular DM graph for r odd.

Theorem 2: There is no r-regular DM graph with n vertices for $r \equiv n \equiv 2 \bmod 4$.

Fair incomplete round robin

Team	Opponents' rankings		Team	Opponents' rankings
1	35		1	35 – m
2	34		2	34 – m
3	33		3	33 – m
4	32		4	32 – m
5	31		5	31 – m
6	30		6	30 – m
7	29		7	29 – m
8	28		8	28 – m

Equal strength incomplete round robin

Team	Opponents' rankings		Team	Opponents' rankings
1	35		1	***m***
2	34		2	***m***
3	33		3	***m***
4	32		4	***m***
5	31		5	***m***
6	30		6	***m***
7	29		7	***m***
8	28		8	***m***

Equal chance? incomplete round robin

Team	Opponents' rankings		Team	Opponents' rankings
1	35		1	*m*
2	34		2	*m*
3	33		3	*m*
4	32		4	*m*
5	31		5	*m*
6	30		6	*m*
7	29		7	*m*
8	28		8	*m*

Handicap incomplete round robin

Team	Opponents' rankings		Team	Opponents' rankings
1	35		1	***k*+1**
2	34		2	***k*+2**
3	33		3	***k*+3**
4	32		4	***k*+4**
5	31		5	***k*+5**
6	30		6	***k*+6**
7	29		7	***k*+7**
8	28		8	***k*+8**

Tournament comparison

Team\Opps ranking	Complete RR	Incomplete RR FAIR	Incomplete RR EQUAL STRENGTH	Incomplete RR HANDICAP
1	35	35 – ***m***	***m***	***k*+1**
2	34	34 – ***m***	***m***	***k*+2**
3	33	33 – ***m***	***m***	***k*+3**
4	32	32 – ***m***	***m***	***k*+4**
5	31	31 – ***m***	***m***	***k*+5**
6	30	30 – ***m***	***m***	***k*+6**
7	29	29 – ***m***	***m***	***k*+7**
8	28	28 – ***m***	***m***	***k*+8**

What games to play?

We want to find an *r*-factor such that the sum of rankings of the neighbors of team ***i*** (i.e., the games which **will** be played) will be equal to $k+i$ for same constant k.

What games to play?

We want to find an *r*-factor such that the sum of rankings of the neighbors of team ***i*** (i.e., the games which **will** be played) will be equal to $k+i$ for same constant k.

Team\Opps ranking	Incomplete RR HANDICAP
1	***k*+1**
2	***k*+2**
3	***k*+3**
4	***k*+4**
5	***k*+5**
6	***k*+6**
7	***k*+7**
8	***k*+8**

What games to play?

We want to find an r-factor such that the sum of rankings of the neighbors of team ***i*** (i.e., the games which **will** be played) will be equal to $k+i$ for same constant k.

Distance-antimagic vertex labeling of a graph G:

A bijection μ from the vertex set of G to $\{1, 2, \ldots, n\}$ such that weights of all vertices form the set

$\{k+1, k+2, \ldots, k+n\}$

for same constant k.

What games to play?

So in fact a fair incomplete round robin tournament is a distance-antimagic graph

Distance-antimagic vertex labeling of a graph *G:*

A bijection μ from the vertex set of G to $\{1, 2, \ldots, n\}$ such that weights of all vertices form the set

$\{k+1, k+2, \ldots, k+n\}$

for same constant k.

What games to play?

We want to find an r-factor such that the sum of rankings of the neighbors of team ***i*** (i.e., the games which will be played) will be equal to $k+i$ for same constant k.

Ordered *distance-antimagic vertex labeling* of a graph G:

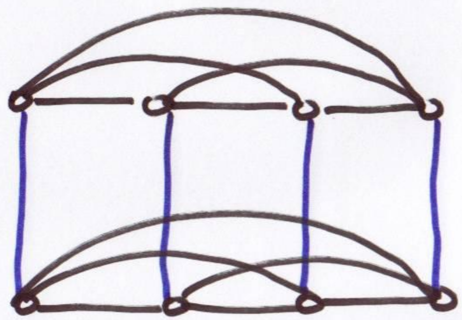
A bijection μ from the vertex set of G to $\{1, 2, \ldots, n\}$ such that weight of vertex ***i*** is equal to $k+i$ for same constant k.

What games to play?

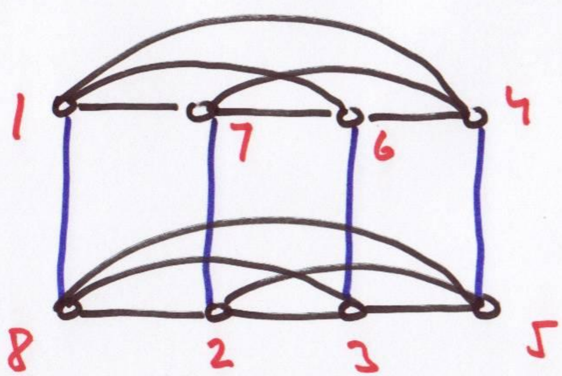
We want to find an *r*-factor such that the sum of rankings of the neighbors of team ***i*** (i.e., the games which will be played) will be equal to *k*+*i* for same constant *k*.

We want to find an *r*-factor *F* which has an *ordered distance-antimagic vertex labeling.*

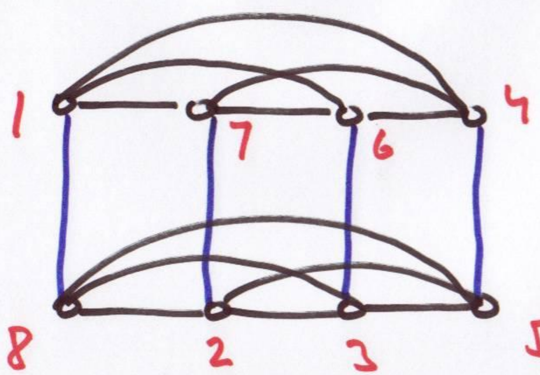
$K_2 \times K_4$



$K_2 \times K_4$



$K_2 \times K_4$



18

18

9 9 9

Remove an r-factor F with

a distance-antimagic vertex labeling with diff = 2

1	7	6	4	**18**
8	2	3	5	**18**
9	**9**	**9**	**9**	

$$K_2 \times K_4$$

Remove an *r*-factor *F* with *a distance-antimagic vertex labeling with diff* = 2

1	7	6	4	**18**
8	2	3	5	**18**
9	**9**	**9**	**9**	

Team 1 misses opponents with total rankings

18 + 9

Remove an r-factor F with

a distance-antimagic vertex labeling with diff $= 2$

1	7	6	4	**18**
8	2	3	5	**18**
9	**9**	**9**	**9**	

Team 1 misses opponents with total rankings

$$18 + 9 - 1 - 1 = 25$$

Remove an *r*-factor *F* with

a distance-antimagic vertex labeling with diff = 2

1	7	6	4	**18**
8	2	3	5	**18**
9	**9**	**9**	**9**	

Team 1 misses opponents with total rankings

$$18 + 9 - 1 - 1 = 25$$

therefore plays opponents with total rankings

$$35 - 25 = 10$$

Remove an r-factor F with

a distance-antimagic vertex labeling with diff $= 2$

1	7	6	4	**18**
8	**2**	3	5	**18**
9	**9**	**9**	**9**	

Team 2 misses opponents with total rankings

$$18 + 9 - 2 - 2 = 23$$

Remove an r-factor F with

a distance-antimagic vertex labeling with diff $= 2$

1	7	6	4	**18**
8	**2**	3	5	**18**
9	**9**	**9**	**9**	

Team 2 misses opponents with total rankings

$$18 + 9 - 2 - 2 = 23$$

therefore plays opponents with total rankings

$$34 - 23 = 11$$

Remove an *r*-factor *F* with *a distance-antimagic vertex labeling with diff* = 2

1	7	6	4	**18**
8	2	**3**	5	**18**
9	**9**	**9**	**9**	

Team 3 misses opponents with total rankings

$$18 + 9 - 3 - 3 = 21$$

Remove an *r*-factor *F* with *a distance-antimagic vertex labeling with diff* = 2

1	7	6	4	**18**
8	2	**3**	5	**18**
9	**9**	**9**	**9**	

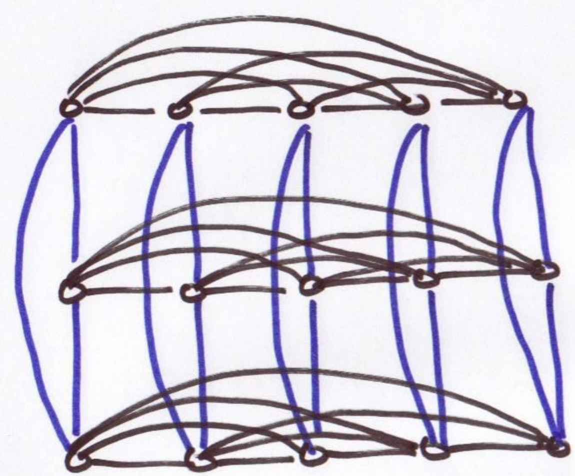
Team 3 misses opponents with total rankings

$$18 + 9 - 3 - 3 = 21$$

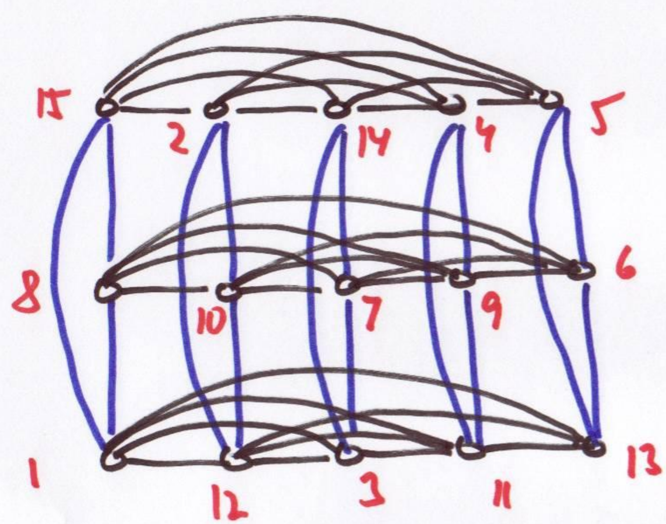
therefore plays opponents with total rankings

$$33 - 21 = 12$$

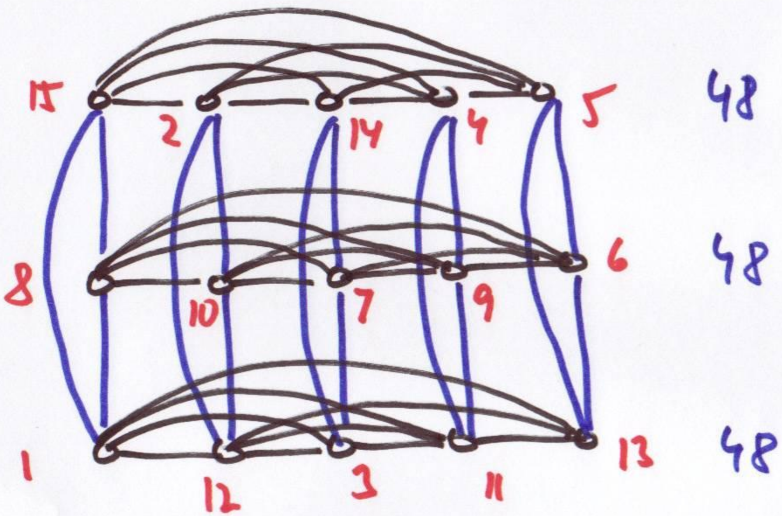
$K_3 \times K_5$



$K_3 \times K_5$



$K_3 \times K_5$



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Remove an *r*-factor *F* with
a distance-antimagic vertex labeling with diff = 2

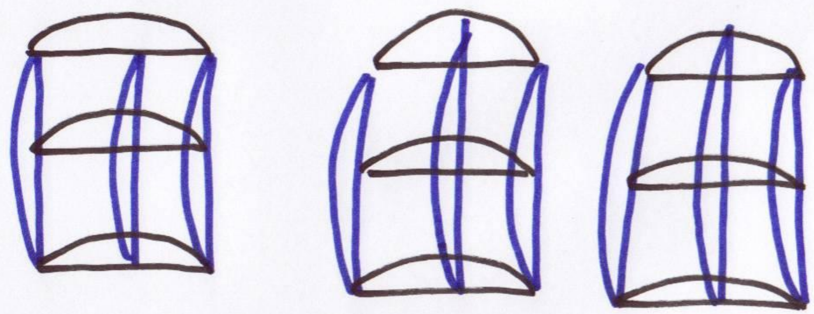
15	2	14	4	5	**40**
8	10	7	9	6	**40**
1	12	3	11	13	**40**
24	**24**	**24**	**24**	**24**	

$$K_3 \times K_5$$

Definition: A *magic rectangle* MR(a,b) is an $a \times b$ array with $a,b > 1$ in which the first ab positive integers are placed so that the sum over each column of MR(a,b) is $\sigma(a,b) = a(ab + 1)/2$ and the sum over each row is $\tau(a,b) = b(ab + 1)/2$.

Theorem: There is a magic rectangle MR(a,b) if and only if $a \equiv b \pmod 2$ and $2 \le a \le b$, except when $a = b = 2$.

$3 \; K_2 \times K_3$



Remove an r-factor F with *a distance-antimagic vertex labeling with diff = 2*

1	27	14	**42**	10	9	23	**42**	19	18	5
15	2	25	**42**	24	11	7	**42**	6	20	16
26	13	3	**42**	8	22	12	**42**	17	4	21
42	**42**	**42**		**42**	**42**	**42**		**42**	**42**	**42**

$$3\,K_3 \times K_3$$

Remove r-factor F with an

ordered distance-antimagic vertex labeling.

1	15	14	4	**34**	5	11	10	8
16	2	3	13	**34**	12	6	7	9
17	**17**	**17**	**17**		**17**	**17**	**17**	**17**

$$2\,K_2 \times K_4$$

Definition: A *magic rectangle set* MRS($a,b;c$) is a collection of c arrays $a\times b$ with $a,b > 1$ in which the first abc positive integers are placed so that the sum over each column of every MR(a,b) is $\sigma(a,b) = ac(abc + 1)/2$ and the sum over each row is $\tau(a,b) = bc(abc + 1)/2$.

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Theorem: There is a magic rectangle set MRS($a,b;c$) when $a \equiv b \pmod 2$, $c|a$ or $c|b$, $a,b > 1$and $ab > 4$.

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Theorem: Let $a \leq b$, $a \equiv b \equiv c \pmod 2$, and there exists $d \leq a$ such that $d|c$. Then there is a magic rectangle set MRS($a,b;c$).

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Theorem: Let $a \leq b$, $a \equiv b \equiv c \pmod 2$, and there exists $d \leq a$ such that $d|c$. Then there is a magic rectangle set MRS($a,b;c$).

It may be true even when exactly one of a,b,c is odd.

Example — MRS(7,11;15)

Take MR(7,11)

76	56	42	55	14	0	63	49	35	21	7	418
5	8	33	22	29	36	43	50	57	64	71	418
2	9	16	32	30	37	44	53	58	65	72	418
38	45	52	59	66	73	3	10	17	24	31	418
74	60	46	23	18	4	67	51	39	25	11	418
1	19	15	47	61	75	12	26	40	54	68	418
70	69	62	28	48	41	34	27	20	13	6	418
266	266	266	266	266	266	266	266	266	266	266	

Example — MRS(7,11;15)

Take MR(7,11)

76	56	42	55	14	0	63	49	35	21	7	418
5	8	33	22	29	36	43	50	57	64	71	418
2	9	16	32	30	37	44	53	58	65	72	418
38	45	52	59	66	73	3	10	17	24	31	418
74	60	46	23	18	4	67	51	39	25	11	418
1	19	15	47	61	75	12	26	40	54	68	418
70	69	62	28	48	41	34	27	20	13	6	418
266	266	266	266	266	266	266	266	266	266	266	

Multiply every element by 15

1140	840	630	825	210	0	945	735	525	315	105	6270
75	120	495	330	435	540	645	750	855	960	1065	6270
30	135	240	480	450	555	660	795	870	975	1080	6270
570	675	780	885	990	1095	45	150	255	360	465	6270
1110	900	690	345	270	60	1005	765	585	375	165	6270
15	285	225	705	915	1125	180	390	600	810	1020	6270
1050	1035	930	420	720	615	510	405	300	195	90	6270
3990	3990	3990	3990	3990	3990	3990	3990	3990	3990	3990	

Example — MRS(7,11;15)

Take KA(3,5)

0	1	2	3	4
3	4	0	1	2
3	1	4	2	0
6	6	6	6	6

Example — MRS(7,11;15)

Take KA(3,5)

0	1	2	3	4
3	4	0	1	2
3	1	4	2	0
6	6	6	6	6

Lift it

10	11	12	13	14	+10
8	9	5	6	7	+5
3	1	4	2	0	+0
21	21	21	21	21	

Example — MRS(7,11;15)

Take KA(3,5)

0	1	2	3	4
3	4	0	1	2
3	1	4	2	0
6	6	6	6	6

Lift it

10	11	12	13	14	+10
8	9	5	6	7	+5
3	1	4	2	0	+0
21	21	21	21	21	

Example — MRS(7,11;15)

Take KA(3,5)

0	1	2	3	4
3	4	0	1	2
3	1	4	2	0
6	6	6	6	6

Lift it

10	11	12	13	14	+10
8	9	5	6	7	+5
3	1	4	2	0	+0
21	21	21	21	21	

Pick a column to construct LS(3)

10	8	3									
8	3	10									
3	10	8									

Example — MRS(7,11;15)

Take KA(3,5)

0	1	2	3	4
3	4	0	1	2
3	1	4	2	0
6	6	6	6	6

Lift it

10	11	12	13	14	+10
8	9	5	6	7	+5
3	1	4	2	0	+0
21	21	21	21	21	

Pick a column to construct LS(3)

10	8	3	11	3	11	3	11	3	11	3	77
8	3	10	4	10	4	10	4	10	4	10	77
3	10	8	6	8	6	8	6	8	6	8	77

Expand rows

Example — MRS(7,11;15)

Take KA(3,5)

0	1	2	3	4
3	4	0	1	2
3	1	4	2	0
6	6	6	6	6

Lift it

10	11	12	13	14	+10
8	9	5	6	7	+5
3	1	4	2	0	+0
21	21	21	21	21	

Pick a column to construct LS(3)

10	8	3	11	3	11	3	11	3	11	3	77
8	3	10	4	10	4	10	4	10	4	10	77
3	10	8	6	8	6	8	6	8	6	8	77
11	4	6									
3	10	8									
11	4	6									
3	10	8									
49	49	49									

Expand rows and columns

Example — MRS(7,11;15)

Take KA(3,5)

0	1	2	3	4
3	4	0	1	2
3	1	4	2	0
6	6	6	6	6

Lift it

10	11	12	13	14	+10
8	9	5	6	7	+5
3	1	4	2	0	+0
21	21	21	21	21	

Pick a column to construct LS(3)

10	8	3	11	3	11	3	11	3	11	3	77
8	3	10	4	10	4	10	4	10	4	10	77
3	10	8	6	8	6	8	6	8	6	8	77
11	4	6	8	6	8	6	8	6	8	6	77
3	10	8	6	8	6	8	6	8	6	8	77
11	4	6	8	6	8	6	8	6	8	6	77
3	10	8	6	8	6	8	6	8	6	8	77
49	49	49	49	49	49	49	49	49	49	49	

Expand rows and columns

Fill the rest

Example — MRS(7,11;15)

10	8	3	11	3	11	3	11	3	11	3	77
8	3	10	4	10	4	10	4	10	4	10	77
3	10	8	6	8	6	8	6	8	6	8	77
11	4	6	8	6	8	6	8	6	8	6	77
3	10	8	6	8	6	8	6	8	6	8	77
11	4	6	8	6	8	6	8	6	8	6	77
3	10	8	6	8	6	8	6	8	6	8	77
49	49	49	49	49	49	49	49	49	49	49	

8	3	10									
3	10	8									
10	8	3									

Rotate the Latin Square

Example — MRS(7,11;15)

10	8	3	11	3	11	3	11	3	11	3	77
8	3	10	4	10	4	10	4	10	4	10	77
3	10	8	6	8	6	8	6	8	6	8	77
11	4	6	8	6	8	6	8	6	8	6	77
3	10	8	6	8	6	8	6	8	6	8	77
11	4	6	8	6	8	6	8	6	8	6	77
3	10	8	6	8	6	8	6	8	6	8	77
49	49	49	49	49	49	49	49	49	49	49	

8	3	10	4	10	4	10	4	10	4	10	77
3	10	8	6	8	6	8	6	8	6	8	77
10	8	3	11	3	11	3	11	3	11	3	77
4	6	11	3	11	3	11	3	11	3	11	77
10	8	3	11	3	11	3	11	3	11	3	77
4	6	11	3	11	3	11	3	11	3	11	77
10	8	3	11	3	11	3	11	3	11	3	77
49	49	49	49	49	49	49	49	49	49	49	

Rotate the Latin Square

Fill the rest as before

Example — MRS(7,11;15)

10	8	3	11	3	11	3	11	3	11	3	77
8	3	10	4	10	4	10	4	10	4	10	77
3	10	8	6	8	6	8	6	8	6	8	77
11	4	6	8	6	8	6	8	6	8	6	77
3	10	8	6	8	6	8	6	8	6	8	77
11	4	6	8	6	8	6	8	6	8	6	77
3	10	8	6	8	6	8	6	8	6	8	77
49	49	49	49	49	49	49	49	49	49	49	

8	3	10	4	10	4	10	4	10	4	10	77
3	10	8	6	8	6	8	6	8	6	8	77
10	8	3	11	3	11	3	11	3	11	3	77
4	6	11	3	11	3	11	3	11	3	11	77
10	8	3	11	3	11	3	11	3	11	3	77
4	6	11	3	11	3	11	3	11	3	11	77
10	8	3	11	3	11	3	11	3	11	3	77
49	49	49	49	49	49	49	49	49	49	49	

3	10	8	6	8	6	8	6	8	6	8	77
10	8	3	11	3	11	3	11	3	11	3	77
8	3	10	4	10	4	10	4	10	4	10	77
6	11	4	10	4	10	4	10	4	10	4	77
8	3	10	4	10	4	10	4	10	4	10	77
6	11	4	10	4	10	4	10	4	10	4	77
8	3	10	4	10	4	10	4	10	4	10	77
49	49	49	49	49	49	49	49	49	49	49	

Example — MRS(7,11;15)

Take KA(3,5)

0	1	2	3	4
3	4	0	1	2
3	1	4	2	0
6	6	6	6	6

Lift it

10	11	12	13	14	+10
8	9	5	6	7	+5
3	1	4	2	0	+0
21	21	21	21	21	

Pick a column to construct LS(3)

Example — MRS(7,11;15)

Take KA(3,5)

0	1	2	3	4
3	4	0	1	2
3	1	4	2	0
6	6	6	6	6

Lift it

10	11	12	13	14	+10
8	9	5	6	7	+5
3	1	4	2	0	+0
21	21	21	21	21	

Pick a column to construct LS(3)

14	7	0									
7	0	14									
0	14	7									

Example — MRS(7,11;15)

Take KA(3,5)

0	1	2	3	4
3	4	0	1	2
3	1	4	2	0
6	6	6	6	6

Lift it

10	11	12	13	14	+10
8	9	5	6	7	+5
3	1	4	2	0	+0
21	21	21	21	21	

Pick a column to construct LS(3)

14	7	0	14	0	14	0	14	0	14	0	77
7	0	14	0	14	0	14	0	14	0	14	77
0	14	7	7	7	7	7	7	7	7	7	77
14	0	7	7	7	7	7	7	7	7	7	77
0	14	7	7	7	7	7	7	7	7	7	77
14	0	7	7	7	7	7	7	7	7	7	77
0	14	7	7	7	7	7	7	7	7	7	77
49	49	49	49	49	49	49	49	49	49	49	

Fill the rest

Example — MRS(7,11;15)

Take KA(3,5)

0	1	2	3	4
3	4	0	1	2
3	1	4	2	0
6	6	6	6	6

Lift it

10	11	12	13	14	+10
8	9	5	6	7	+5
3	1	4	2	0	+0
21	21	21	21	21	

Pick a column to construct LS(3)

14	7	0	14	0	14	0	14	0	14	0	77
7	0	14	0	14	0	14	0	14	0	14	77
0	14	7	7	7	7	7	7	7	7	7	77
14	0	7	7	7	7	7	7	7	7	7	77
0	14	7	7	7	7	7	7	7	7	7	77
14	0	7	7	7	7	7	7	7	7	7	77
0	14	7	7	7	7	7	7	7	7	7	77
49	49	49	49	49	49	49	49	49	49	49	

Fill the rest

Repeat...

Example — MRS(7,11;15)

1140	840	630	825	210	0	945	735	525	315	105	6270
75	120	495	330	435	540	645	750	855	960	1065	6270
30	135	240	480	450	555	660	795	870	975	1080	6270
570	675	780	885	990	1095	45	150	255	360	465	6270
1110	900	690	345	270	60	1005	765	585	375	165	6270
15	285	225	705	915	1125	180	390	600	810	1020	6270
1050	1035	930	420	720	615	510	405	300	195	90	6270
3990	3990	3990	3990	3990	3990	3990	3990	3990	3990	3990	

Example — MRS(7,11;15)

1140	840	630	825	210	0	945	735	525	315	105	6270
75	120	495	330	435	540	645	750	855	960	1065	6270
30	135	240	480	450	555	660	795	870	975	1080	6270
570	675	780	885	990	1095	45	150	255	360	465	6270
1110	900	690	345	270	60	1005	765	585	375	165	6270
15	285	225	705	915	1125	180	390	600	810	1020	6270
1050	1035	930	420	720	615	510	405	300	195	90	6270
3990	3990	3990	3990	3990	3990	3990	3990	3990	3990	3990	

8	3	10	4	10	4	10	4	10	4	10	77
3	10	8	6	8	6	8	6	8	6	8	77
10	8	3	11	3	11	3	11	3	11	3	77
4	6	11	3	11	3	11	3	11	3	11	77
10	8	3	11	3	11	3	11	3	11	3	77
4	6	11	3	11	3	11	3	11	3	11	77
10	8	3	11	3	11	3	11	3	11	3	77
49	49	49	49	49	49	49	49	49	49	49	

Example — MRS(7,11;15)

1140	840	630	825	210	0	945	735	525	315	105	6270
75	120	495	330	435	540	645	750	855	960	1065	6270
30	135	240	480	450	555	660	795	870	975	1080	6270
570	675	780	885	990	1095	45	150	255	360	465	6270
1110	900	690	345	270	60	1005	765	585	375	165	6270
15	285	225	705	915	1125	180	390	600	810	1020	6270
1050	1035	930	420	720	615	510	405	300	195	90	6270
3990	3990	3990	3990	3990	3990	3990	3990	3990	3990	3990	

8	3	10	4	10	4	10	4	10	4	10	77
3	10	8	6	8	6	8	6	8	6	8	77
10	8	3	11	3	11	3	11	3	11	3	77
4	6	11	3	11	3	11	3	11	3	11	77
10	8	3	11	3	11	3	11	3	11	3	77
4	6	11	3	11	3	11	3	11	3	11	77
10	8	3	11	3	11	3	11	3	11	3	77
49	49	49	49	49	49	49	49	49	49	49	

1148	843	640	829	220	4	955	739	535	319	115	6347
78	130	503	336	443	546	653	756	863	966	1073	6347
40	143	243	491	453	566	663	806	873	986	1083	6347
574	681	791	888	1001	1098	56	153	266	363	476	6347
1120	908	693	356	273	71	1008	776	588	386	168	6347
19	291	236	708	926	1128	191	393	611	813	1031	6347
1060	1043	933	431	723	626	513	416	303	206	93	6347
4039	4039	4039	4039	4039	4039	4039	4039	4039	4039	4039	

Thank you!