Counting colorings of a regular graph

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An enumeration question for colorings

(Proper) q-colouring of graph G

- $C : V(G) \to \{1, \ldots, q\}$, $xy \in EG \Rightarrow C(x) \neq C(y)$
- $c_q(G) =$ number of q-colourings

An enumeration question

- Fix q and family $\mathcal{G}$ of graphs
- Which $G \in \mathcal{G}$ maximizes $c_q(G)$?

$\mathcal{G}$ the family of n-vertex, m-edge graphs

- Question raised by Wilf and Linial
- Partial results, but no complete answer or conjecture

Our family of interest

$\mathcal{G}(n,d)$ the family of n-vertex, d-regular graphs

Conjecture (G., Tetali) For $G \in \mathcal{G}(n,d)$ and all q

$$c_q(G) \leq c_q(K_{d,d})^{\frac{n}{2d}}$$

Remarks

- Tight when $2d|n$ (G the disjoint union of $n/2d$ copies of $K_{d,d}$)
- Conjecture true for *bipartite* $G \in \mathcal{G}(n,d)$ (G.-Tetali)
- Conjecture true for $q \geq n^8$ (G., Lazebnik, Zhao)

An asymptotic approach

An estimate for the conjectured upper bound, $q = 4$

$$\begin{aligned} c_4(K_{d,d}) &= 6 \cdot 4^d + O(3^d) \\ c_4(K_{d,d})^{\frac{n}{2d}} &= 2^n 6^{\frac{n(1+o_d(1))}{2d}} \end{aligned}$$

Best known upper bound to date for $G \in \mathcal{G}(n,d)$

$$c_4(G) \leq 2^n 6^{\frac{3n(1+o_d(1))}{4d}}$$

Theorem (G.) For $G \in \mathcal{G}(n,d)$,

$$c_4(G) \leq 2^n 6^{\frac{n(1+o_d(1))}{2d}}$$

- Similar bounds for all fixed $q \geq 3$
- In all cases correct in first two terms of exponent

Two sides to the coin

Two regimes in which we work

- G close to bipartite (there's an independent set of size $\approx n/2$)
- G far from bipartite (no large independent set)
- Independent set: set of mutually non-adjacent vertices

Theorem (G.) If G has independent set of size $n(1-\varepsilon)/2$

$$c_4(G) \leq 2^n 6^{\frac{n(1+\varepsilon)}{2d}}$$

- Uses entropy, and ideas from Kahn, Madiman-Tetali

Theorem (G.) If G has *no* independent set of size $n(1-\varepsilon)/2$

$$c_4(G) \leq 2^n \exp\left\{ O\left(n\sqrt{\frac{\log d}{d}} \right) - \Omega\left(\varepsilon n\right) \right\}$$

- Take $\varepsilon = C\sqrt{\log d/d}$ for large enough C to get the main result

A tool for dealing with small independent sets

Lemma (Sapozhenko) For each $G \in \mathcal{G}(n,d)$ there is $\mathcal{F} \subseteq 2^{V(G)}$ with

- Each independent set I satisfies $I \subseteq F$ for some $F \in \mathcal{F}$
- Each $F \in \mathcal{F}$ satisfies $|F| \leq \frac{n}{2}\left(1 + O\left(\sqrt{\frac{\log d}{d}}\right)\right)$
- $\mathcal{F}$ satisfies $|\mathcal{F}| \leq \exp\left\{O\left(n\sqrt{\frac{\log d}{d}}\right)\right\}$

Proof sketch

- Build "seed" S inside each I, by repeatedly adding vertices with $\geq \sqrt{d \log d}$ new neighbours
- Build "cover" F from S by taking all vertices not in $N(S)$ that have many neighbours to $N(S)$
- Seeds are small, so few seeds and few covers
- Construction of F ensures it really covers I
- $N(S)$ largish, so F smallish

Counting 4-colourings of regular graphs

A weak bound

- Colouring C partitions $V(G)$ into 4 independent sets (C_1, C_2, C_3, C_4)
- (C_1, C_2, C_3, C_4) "covered" by (F_1, F_2, F_3, F_4)
- At most $\exp\left\{O\left(n\sqrt{\frac{\log d}{d}}\right)\right\}$ covering 4-tuples
- Number of ways of reconstructing colouring from covering 4-tuples is

$$\prod_{v\in V(G)} a_v \ (a_v = \text{number of } F_i \text{ in which } v \text{ sits})$$

- AM-GM inequality:

$$\prod_{v\in V(G)} a_v \leq \left(\frac{1}{n}\sum_{v\in V(G)} a_v\right)^n \leq 2^n \exp\left\{O\left(n\sqrt{\frac{\log d}{d}}\right)\right\}$$

- Conclusion: for $G \in \mathcal{G}(n,d)$, $c_4(G) \leq 2^n \exp\left\{O\left(n\sqrt{\frac{\log d}{d}}\right)\right\}$

Incorporating the independent set information

Knowing G has no independent set of size $n/2(1-\varepsilon)$

- May assume all F_i's have at least $n/2$ vertices
- F_1 has no large independent sets so has $\Omega(\varepsilon n)$-sized matching
- For uv in matching, $a_u a_v$ in reconstruction replaced by $a_u a_v - 1$
- Dampens the count sufficiently to give main result

Final remarks

- All ok for even q; for odd q need to use that a_v's are integers
- Proof of tight bound remains to be found
- We've shown: for all $\varepsilon > 0$ there's $\delta > 0$ so that for $G \in \mathcal{G}(n,d)$ without independent sets of size $n(1-\varepsilon)/2$,

$$c_4(G) \leq (2-\delta)^n$$

How does δ depend on ε?