Upper bounds on the size of 4- and 6-cycle-free subgraphs of the hypercube

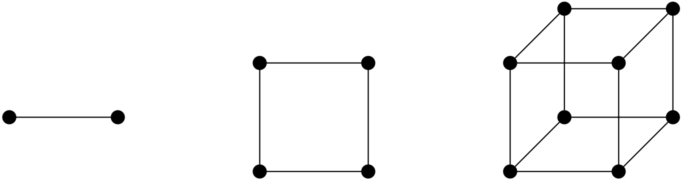
Ping Hu
Joint work with József Balogh, Bernard Lidický and Hong Liu

University of Illinois at Urbana-Champaign

MIGHTY LII - April 28, 2012

Hypercube

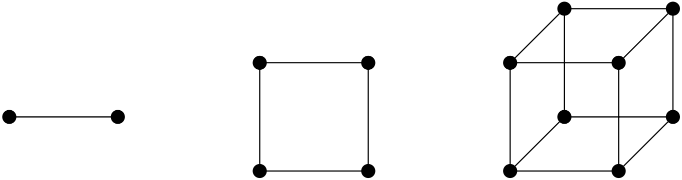
- $\mathcal{Q}_n$ is n-dimensional hypercube (n-cube)



$\mathcal{Q}_1$ $\mathcal{Q}_2$ $\mathcal{Q}_3$

Hypercube

- $\mathcal{Q}_n$ is n-dimensional hypercube (n-cube)

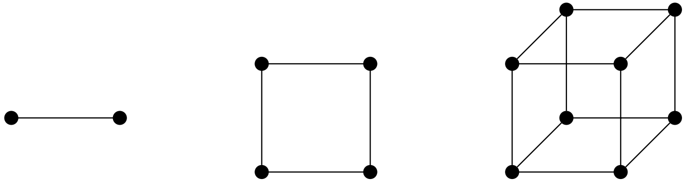


$\mathcal{Q}_1$ $\mathcal{Q}_2$ $\mathcal{Q}_3$

- $e(G) := |E(G)|$

Hypercube

- $\mathcal{Q}_n$ is n-dimensional hypercube (n-cube)

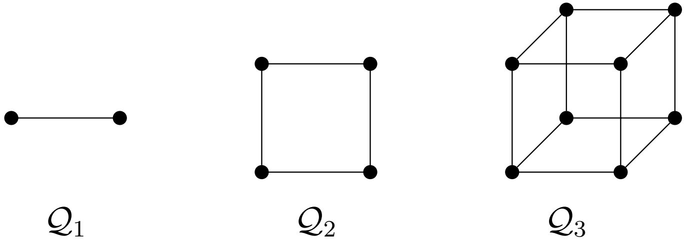


$\mathcal{Q}_1$ $\mathcal{Q}_2$ $\mathcal{Q}_3$

- $e(G) := |E(G)|$
- $\text{ex}_{\mathcal{Q}}(n, F) :=$ the maximum number of edges of a F-free subgraph of $\mathcal{Q}_n$

Hypercube

- $\mathcal{Q}_n$ is n-dimensional hypercube (n-cube)



- $e(G) := |E(G)|$
- $\text{ex}_{\mathcal{Q}}(n, F) :=$ the maximum number of edges of a F-free subgraph of $\mathcal{Q}_n$
- $\pi_{\mathcal{Q}}(F) = \lim_{n\to\infty} \frac{\text{ex}_{\mathcal{Q}}(n, F)}{e(\mathcal{Q}_n)}$

$\pi_{\mathcal{Q}}(C_{2t})$

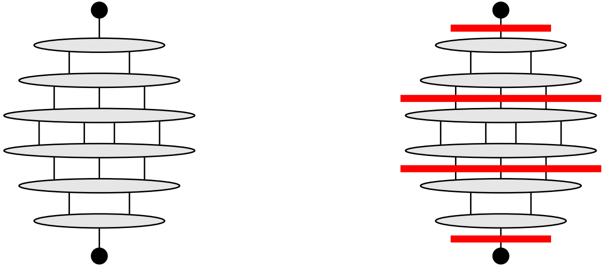
Conjecture (Erdős [1984])

$\pi_{\mathcal{Q}}(C_4) = 1/2$, $\pi_{\mathcal{Q}}(C_{2t}) = 0$ *for* $t > 2$

$\pi_{\mathcal{Q}}(C_{2t})$

Conjecture (Erdős [1984])

$\pi_{\mathcal{Q}}(C_4) = 1/2$, $\pi_{\mathcal{Q}}(C_{2t}) = 0$ *for* $t > 2$



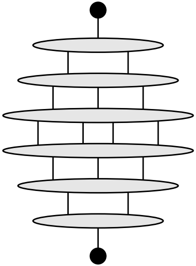
$\mathcal{Q}_7$ $\qquad$ $\mathcal{Q}_7$

$$\pi_{\mathcal{Q}}(C_4) \geq 1/2$$

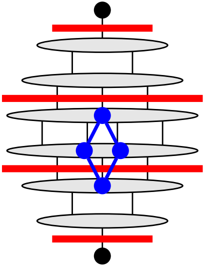
$\pi_{\mathcal{Q}}(C_{2t})$

Conjecture (Erdős [1984])

$\pi_{\mathcal{Q}}(C_4) = 1/2$, $\pi_{\mathcal{Q}}(C_{2t}) = 0$ for $t > 2$



$\mathcal{Q}_7$



$\mathcal{Q}_7$

$$\pi_{\mathcal{Q}}(C_4) \geq 1/2$$

$\pi_{\mathcal{Q}}(C_{2t})$

Conjecture (Erdős [1984])

$\pi_{\mathcal{Q}}(C_4) = 1/2$, $\pi_{\mathcal{Q}}(C_{2t}) = 0$ *for* $t > 2$

Theorem (Chung [1992], Brouwer–Dejter–Thomassen [1993])

$\pi_{\mathcal{Q}}(C_6) \geq 1/4$

$\pi_{\mathcal{Q}}(C_{2t})$

Conjecture (Erdős [1984])

$\pi_{\mathcal{Q}}(C_4) = 1/2$, $\pi_{\mathcal{Q}}(C_{2t}) = 0$ *for* $t > 2$

Theorem (Chung [1992], Brouwer–Dejter–Thomassen [1993])

$\pi_{\mathcal{Q}}(C_6) \geq 1/4$

Theorem (Conder [1993])

$\pi_{\mathcal{Q}}(C_6) \geq 1/3$

$\pi_{\mathcal{Q}}(C_{2t})$

Conjecture (Erdős [1984])

$\pi_{\mathcal{Q}}(C_4) = 1/2$, $\pi_{\mathcal{Q}}(C_{2t}) = 0$ for $t > 2$.

Theorem (Chung [1992])

$\pi_{\mathcal{Q}}(n, C_{2t}) = 0$ for even $t \geq 4$.

Theorem (Füredi–Özkahya [2009])

$\pi_{\mathcal{Q}}(C_{2t}) = 0$ for odd $t \geq 7$.

$\pi_{\mathcal{Q}}(C_{2t})$

Conjecture (Erdős [1984])

$\pi_{\mathcal{Q}}(C_4) = 1/2$, $\pi_{\mathcal{Q}}(C_{2t}) = 0$ *for* $t > 2$.

Theorem (Chung [1992])

$\pi_{\mathcal{Q}}(n, C_{2t}) = 0$ *for even* $t \geq 4$.

Theorem (Füredi–Özkahya [2009])

$\pi_{\mathcal{Q}}(C_{2t}) = 0$ *for odd* $t \geq 7$.

if $\pi_{\mathcal{Q}}(C_{10}) = 0$ is still open.

$\pi_{\mathcal{Q}}(C_4)$

Theorem (Brass–Harborth–Nienborg [1995])

$ex_{\mathcal{Q}}(n, C_4) \geq \frac{1}{2}(1 + \frac{1}{\sqrt{n}})e(\mathcal{Q}_n)$ *(valid when n is a power of 4)*

$\pi_{\mathcal{Q}}(C_4)$

Theorem (Brass–Harborth–Nienborg [1995])

$ex_{\mathcal{Q}}(n, C_4) \geq \frac{1}{2}(1 + \frac{1}{\sqrt{n}})e(\mathcal{Q}_n)$ (valid when n is a power of 4)

Theorem (Chung [1992])

$\pi_{\mathcal{Q}}(C_4) \leq 0.62284$.

$\pi_{\mathcal{Q}}(C_4)$

Theorem (Brass–Harborth–Nienborg [1995])

$ex_{\mathcal{Q}}(n, C_4) \geq \frac{1}{2}(1 + \frac{1}{\sqrt{n}})e(\mathcal{Q}_n)$ *(valid when n is a power of 4)*

Theorem (Chung [1992])

$\pi_{\mathcal{Q}}(C_4) \leq 0.62284.$

Theorem (Thomason–Wagner [2009])

$\pi_{\mathcal{Q}}(C_4) \leq 0.62256.$

$\pi_{\mathcal{Q}}(C_4)$

Theorem (Brass–Harborth–Nienborg [1995])

$ex_{\mathcal{Q}}(n, C_4) \geq \frac{1}{2}(1 + \frac{1}{\sqrt{n}})e(\mathcal{Q}_n)$ *(valid when n is a power of 4)*

Theorem (Chung [1992])

$\pi_{\mathcal{Q}}(C_4) \leq 0.62284.$

Theorem (Thomason–Wagner [2009])

$\pi_{\mathcal{Q}}(C_4) \leq 0.62083.$

$\pi_{\mathcal{Q}}(C_4)$

Theorem (Brass–Harborth–Nienborg [1995])

$ex_{\mathcal{Q}}(n, C_4) \geq \frac{1}{2}(1 + \frac{1}{\sqrt{n}})e(\mathcal{Q}_n)$ *(valid when n is a power of 4)*

Theorem (Chung [1992])

$\pi_{\mathcal{Q}}(C_4) \leq 0.62284.$

Theorem (Thomason–Wagner [2009])

$\pi_{\mathcal{Q}}(C_4) \leq 0.62083.$

Theorem (Balogh–Hu–Lidický–Liu, ind. Baber [2012+])

$\pi_{\mathcal{Q}}(C_4) \leq 0.6068.$

$\pi_{\mathcal{Q}}(n, C_6)$

Theorem (Conder [1993])

$\pi_{\mathcal{Q}}(C_6) \geq 1/3$.

$\pi_{\mathcal{Q}}(n, C_6)$

Theorem (Conder [1993])

$\pi_{\mathcal{Q}}(C_6) \geq 1/3$.

Theorem (Chung [1992])

$\pi_{\mathcal{Q}}(C_6) \leq \sqrt{2} - 1 \approx 0.41421$.

$\pi_{\mathcal{Q}}(n, C_6)$

Theorem (Conder [1993])

$\pi_{\mathcal{Q}}(C_6) \geq 1/3.$

Theorem (Chung [1992])

$\pi_{\mathcal{Q}}(C_6) \leq \sqrt{2} - 1 \approx 0.41421.$

Theorem (Balogh–Hu–Lidický–Liu, ind. Baber [2012+])

$\pi_{\mathcal{Q}}(C_6) \leq 0.3755.$

Flag Algebras

Definition

$p(H,G)$: the probability that a random $|V(H)|$-set U in $V(G)$ induces $G[U]$ isomorphic to H.

Razborov [2007] developed flag algebras. Let $\mathcal{G}$ be the family of graphs forbidding some structures, then flag algebras can be used to bound

$$\lim_{G\in\mathcal{G},|V(G)|\to\infty} p(H,G).$$

Results using Flag Algebras

Results using Flag Algebras

Theorem (Hladký–Kráľ–Norine [2009])

Every n-vertex digraph with minimum outdegree at least $0.3465n$ contains a triangle.

Results using Flag Algebras

Theorem (Hladký–Král'–Norine [2009])

Every n-vertex digraph with minimum outdegree at least $0.3465n$ contains a triangle.

Theorem (Hatami–Hladký–Král'–Norine–Razborov [2011], Grzesik [2011])

The number of C_5s in a triangle-free graph of order n is at most $(n/5)^5$.

Results using Flag Algebras

Theorem (Hladký–Král'–Norine [2009])

Every n-vertex digraph with minimum outdegree at least $0.3465n$ contains a triangle.

Theorem (Hatami–Hladký–Král'–Norine–Razborov [2011], Grzesik [2011])

The number of C_5s in a triangle-free graph of order n is at most $(n/5)^5$.

Theorem (Falgas-Ravry–Vaughan [2011])

$\pi(K_4^-, C_5, F_{3,2}) = 12/49, \pi(K_4^-, F_{3,2}) = 5/18.$

$F_{3,2} : \{123, 145, 245, 345\}, C_5 : \{123, 234, 345, 451, 512\}.$

Proof by an Example

Example

$\pi_{\mathcal{Q}}(C_4) \leq 2/3$

Proof by an Example

Example

$\pi_{\mathcal{Q}}(C_4) \leq 2/3$

Bound infinite problem by a finite piece.

Proof by an Example

Example

$\pi_{\mathcal{Q}}(C_4) \leq 2/3$

Bound infinite problem by a finite piece.

Definition

$\mathcal{H}_n$: the family of spanning subgraphs of $\mathcal{Q}_n$ not containing C_4.

Let $H \in \mathcal{H}_s, G \in \mathcal{H}_n, s < n$, $p(H, G)$ is the probability that a random s-hypercube vertex set in G induces H.

$\rho(G) = e(G)/e(\mathcal{Q}_n)$.

Proof by an Example

Example

$\pi_{\mathcal{Q}}(C_4) \leq 2/3$

Bound infinite problem by a finite piece.

Definition

$\mathcal{H}_n$: the family of spanning subgraphs of $\mathcal{Q}_n$ not containing C_4.

Let $H \in \mathcal{H}_s, G \in \mathcal{H}_n, s < n$, $p(H, G)$ is the probability that a random s-hypercube vertex set in G induces H.

$\rho(G) = e(G)/e(\mathcal{Q}_n)$.

$$\rho(G) = \sum_{H \in \mathcal{H}_s} \rho(H) p(H, G)$$

Proof by an Example

Example

$\pi_{\mathcal{Q}}(C_4) \leq 2/3$

Bound infinite problem by a finite piece.

Definition

$\mathcal{H}_n$: the family of spanning subgraphs of $\mathcal{Q}_n$ not containing C_4.

Let $H \in \mathcal{H}_s, G \in \mathcal{H}_n, s < n$, $p(H, G)$ is the probability that a random s-hypercube vertex set in G induces H.

$\rho(G) = e(G)/e(\mathcal{Q}_n)$.

$$\rho(G) = \sum_{H \in \mathcal{H}_s} \rho(H) p(H, G)$$

$$\rho(G) \leq \max_{H \in \mathcal{H}_s} \rho(H)$$

Proof by an Example

Example

$\pi_{\mathcal{Q}}(C_4) \leq 2/3$

Bound infinite problem by a finite piece.

Definition

$\mathcal{H}_n$: the family of spanning subgraphs of $\mathcal{Q}_n$ not containing C_4.

Let $H \in \mathcal{H}_s, G \in \mathcal{H}_n, s < n$, $p(H, G)$ is the probability that a random s-hypercube vertex set in G induces H.

$\rho(G) = e(G)/e(\mathcal{Q}_n)$.

$$\rho(G) = \sum_{H \in \mathcal{H}_s} \rho(H) p(H, G)$$

$$\pi_{\mathcal{Q}}(C_4) \leq \max_{H \in \mathcal{H}_s} \rho(H)$$

Is the bound good?

$$\rho(G) = \sum_{H \in \mathcal{H}_s} \rho(H) p(H, G)$$

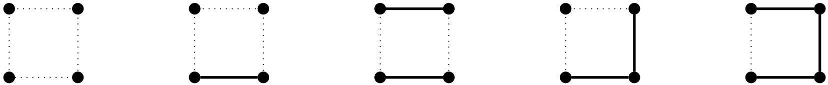
$$\pi_{\mathcal{Q}}(C_4) \leq \max_{H \in \mathcal{H}_s} \rho(H)$$

Is the bound good?

$$\rho(G) = \sum_{H \in \mathcal{H}_s} \rho(H) p(H, G)$$

$$\pi_{\mathcal{Q}}(C_4) \leq \max_{H \in \mathcal{H}_s} \rho(H)$$

$\mathcal{H}_2$



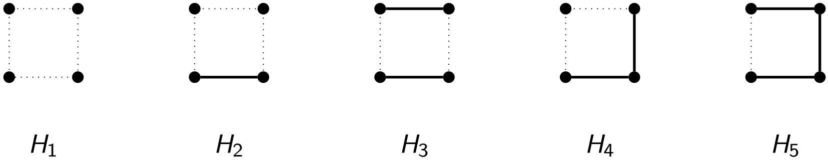
H_1 H_2 H_3 H_4 H_5

Is the bound good?

$$\rho(G) = \sum_{H \in \mathcal{H}_s} \rho(H) p(H, G)$$

$$\pi_{\mathcal{Q}}(C_4) \leq \max_{H \in \mathcal{H}_s} \rho(H)$$

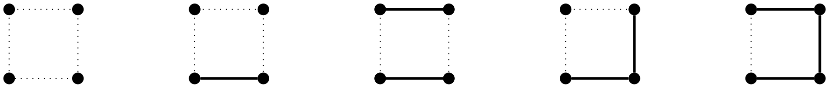
$$\mathcal{H}_2$$



$$\pi_{\mathcal{Q}}(C_4) \leq \max \rho(H_i) = \rho(H_5) = 3/4$$

Is the bound good?

$$\rho(G) = \sum_{H \in \mathcal{H}_s} \rho(H) p(H, G)$$



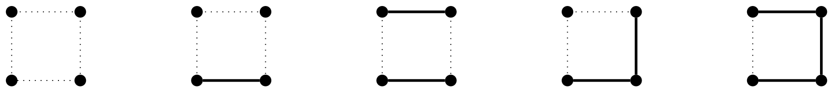
H_1 H_2 H_3 H_4 H_5

$$\pi_{\mathcal{Q}}(C_4) \leq \max \rho(H_i) = \rho(H_5) = 3/4$$

If $0 \leq \sum_i c_{H_i} p(H_i, G)$, then

Is the bound good?

$$\rho(G) = \sum_{H \in \mathcal{H}_s} \rho(H) p(H, G)$$



H_1 H_2 H_3 H_4 H_5

$$\pi_{\mathcal{Q}}(C_4) \leq \max \rho(H_i) = \rho(H_5) = 3/4$$

If $0 \leq \sum_i c_{H_i} p(H_i, G)$, then

$$\rho(G) \leq \sum_i \left(\rho(H_i) + c_{H_i}\right) p(H_i, G)$$

$$\pi_{\mathcal{Q}}(C_4) \leq \max_i \left(\rho(H_i) + c_{H_i}\right)$$

c_{H_i} might be negative

OPTIMIZE c_{H_i}

Let M be a positive semidefinite 2-by-2 matrix.

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

OPTIMIZE c_{H_i}

Let M be a positive semidefinite 2-by-2 matrix.

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

$$\begin{aligned}
\rho(H_1) + c_{H_1} &= 0 + m_{11} \\
\rho(H_2) + c_{H_2} &= 1/4 + m_{11}/2 + m_{12}/2 \\
\rho(H_3) + c_{H_3} &= 1/2 + m_{12} \\
\rho(H_4) + c_{H_4} &= 1/2 + m_{11}/4 + m_{12}/2 + m_{22}/4 \\
\rho(H_5) + c_{H_5} &= 3/4 + m_{12}/2 + m_{22}/2
\end{aligned}$$

$$\pi_{\mathcal{Q}}(C_4) \leq \max_i(\rho(H_i) + c_{H_i})$$

Solution

Take

$$M = \begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 1/6 \end{pmatrix},$$

then

$$\max_i(\rho(H_i) + c_{H_i}) = 2/3$$

Results

Theorem (Balogh–Hu–Lidický–Liu, ind. Baber [2012+])

$\pi_{\mathcal{Q}}(C_4) \leq 0.6068.$

Theorem (Balogh–Hu–Lidický–Liu, ind. Baber [2012+])

$\pi_{\mathcal{Q}}(C_6) \leq 0.3755.$

By using $\mathcal{H}_3$.

Results

Theorem (Balogh–Hu–Lidický–Liu, ind. Baber [2012+])

$\pi_{\mathcal{Q}}(C_4) \leq 0.6068.$

Theorem (Balogh–Hu–Lidický–Liu, ind. Baber [2012+])

$\pi_{\mathcal{Q}}(C_6) \leq 0.3755.$

By using $\mathcal{H}_3$.

Almost surely can be improved by waiting.

Thank you for your attention!