Extending graph choosability results to paintability

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Joint work with
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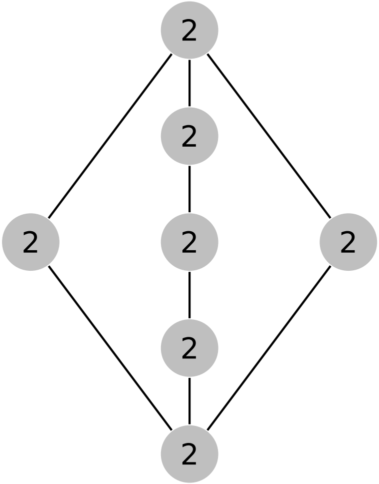

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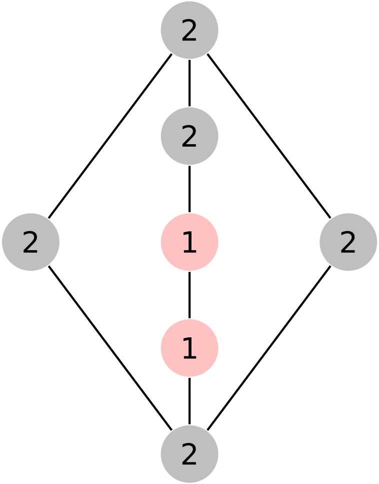

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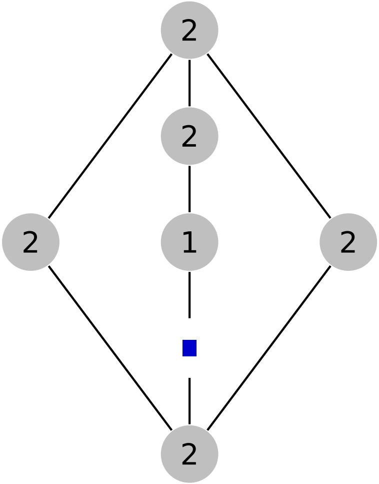

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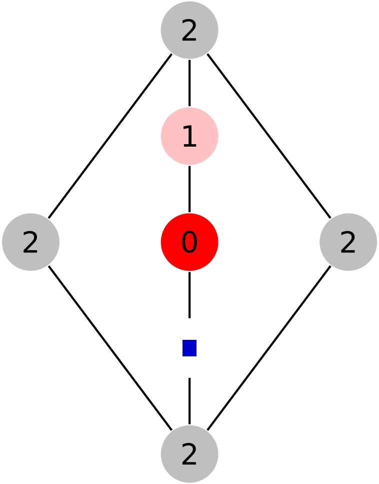

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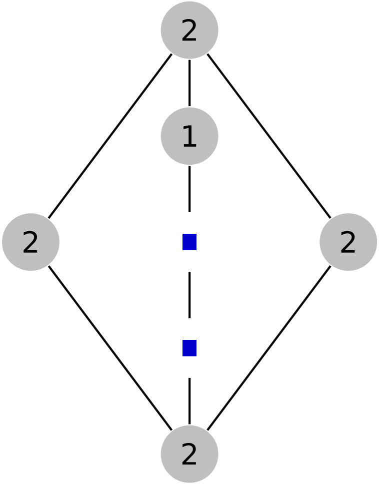

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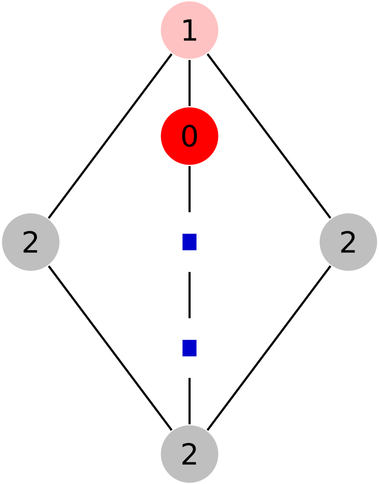

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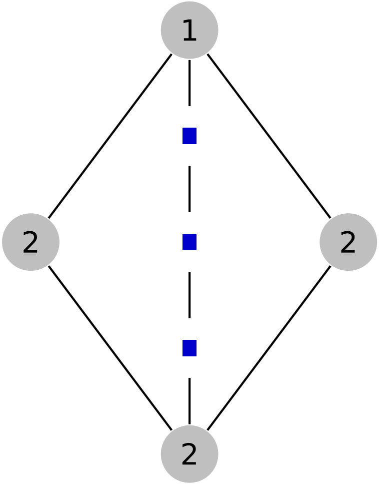

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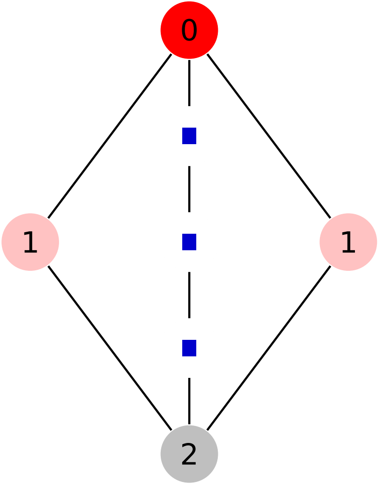

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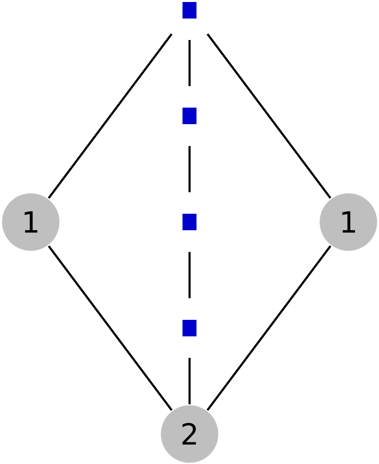

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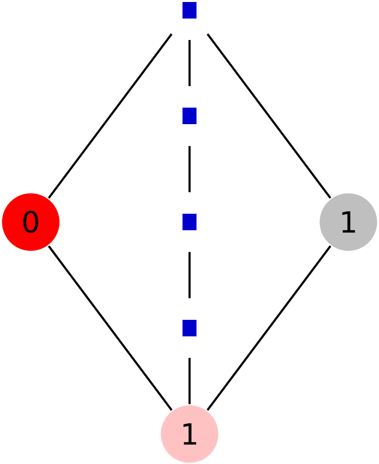

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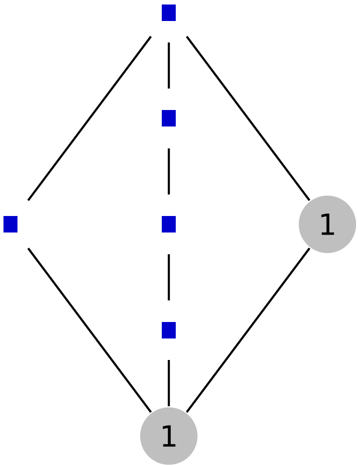

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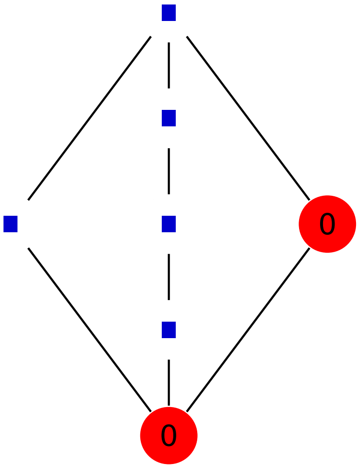

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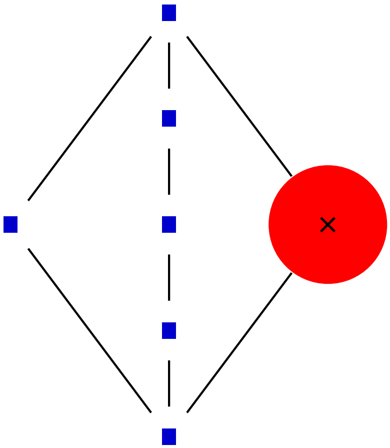

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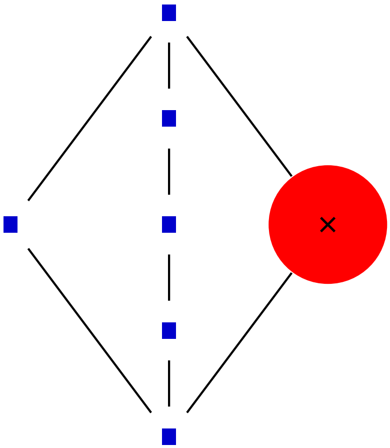

Conclude: Marker has a winning strategy on this graph when each vertex has 2 tokens.

Definitions

Def. Given a graph G and a function $f : V(G) \to \mathbb{N}$ assigning tokens, we say G is f-paintable if Remover has a winning strategy in this game.

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Def. If $f(v) = k$ for all $v \in V(G)$ and G is f-paintable, then we say G is k-paintable.

Def. The least such k for which this is true is the paintability or paint number of G and is denoted $\chi_p(G)$.

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Obs. Sets removed by Remover form a proper coloring.

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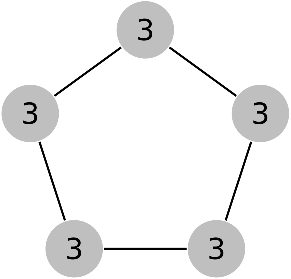
Prop. $\chi(C_5) = 3$.

Ex. Consider the following strategy for Marker:



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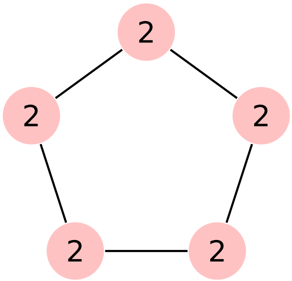

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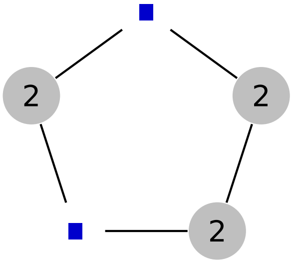

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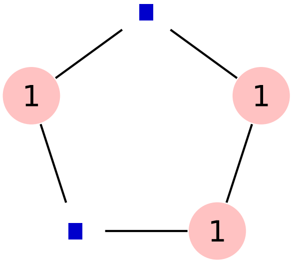

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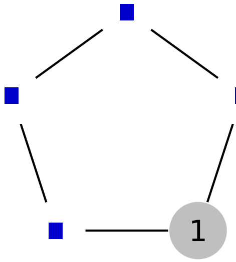

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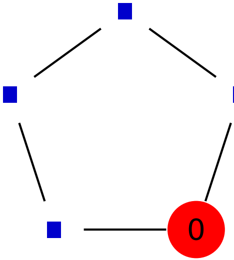

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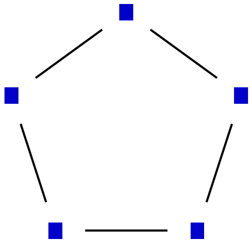

Obs. If Marker always marks all available vertices, then the least k such that Remover can win against this strategy is $\chi(G)$.

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Choosability Example

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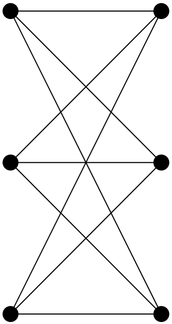

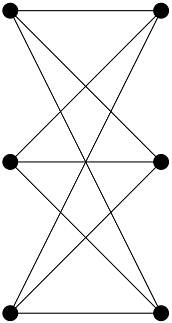

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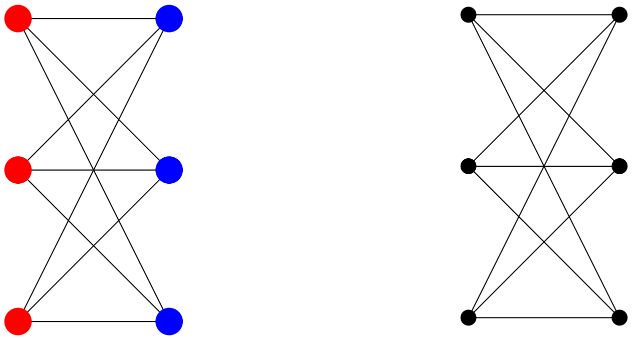

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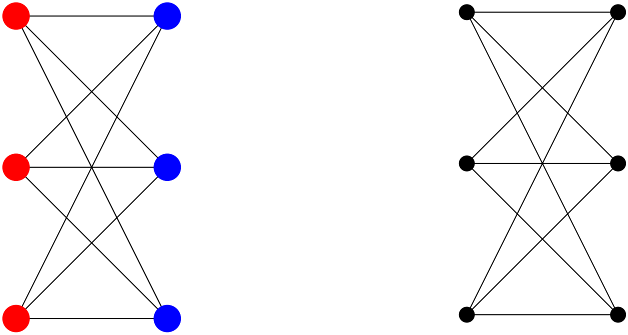

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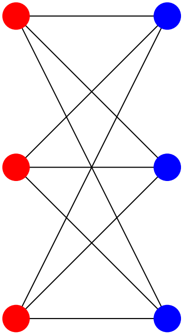

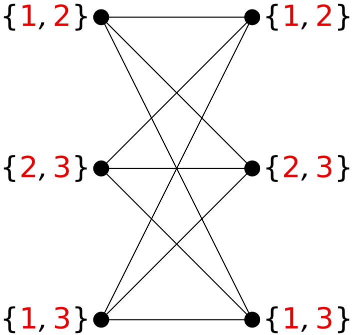

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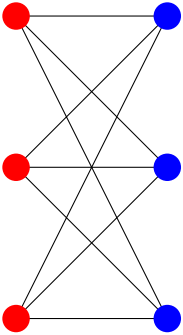

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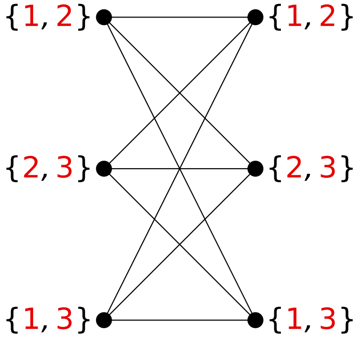

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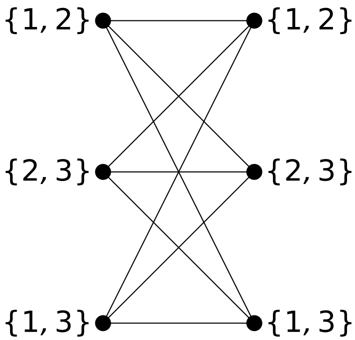

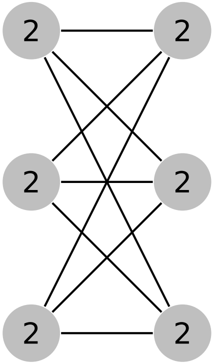

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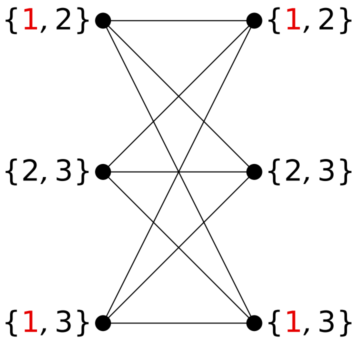

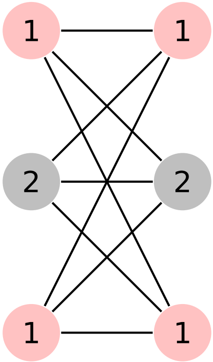

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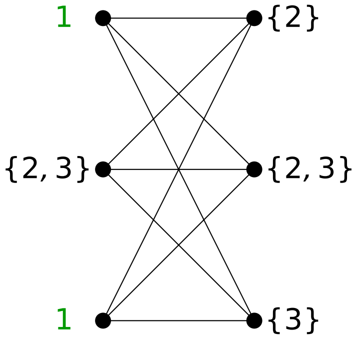

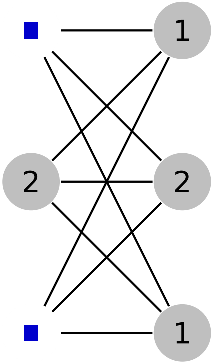

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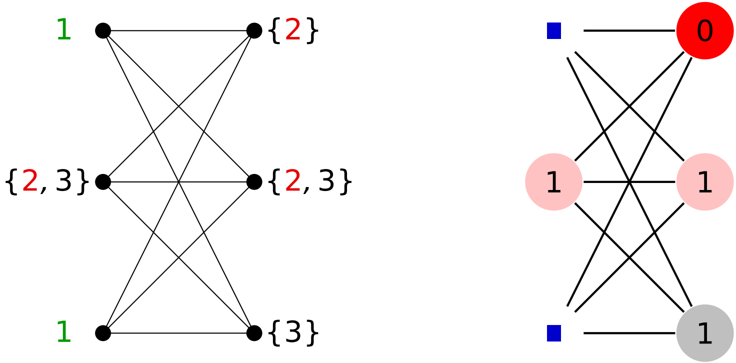

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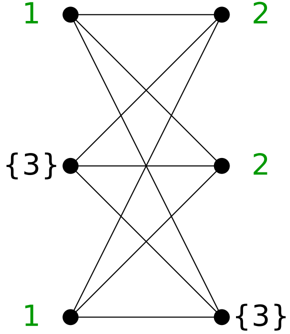

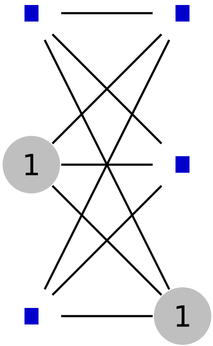

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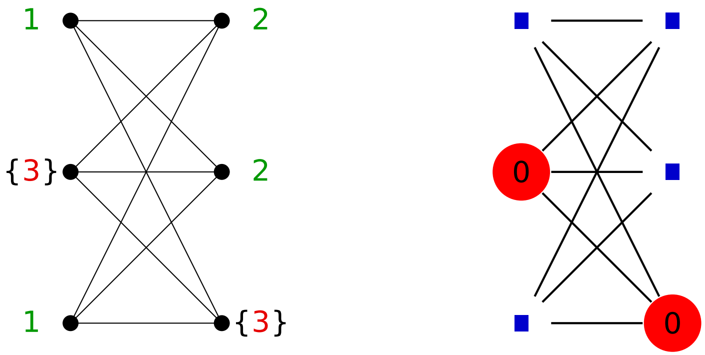

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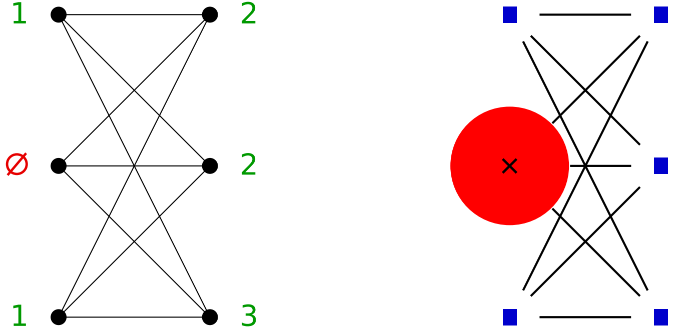

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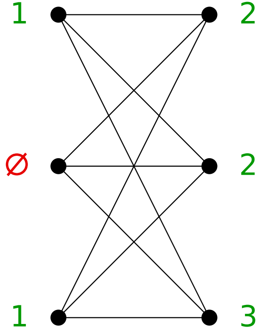
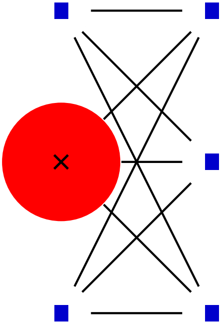
Obs. If Marker's strategy mimics list assignments by marking vertices whose list has color i on the ith round, then the least k such that Remover has a winning strategy against all L having list of size k is $\chi_\ell(G)$.





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Obs. The “list maker” could make all lists the same, but it’s not the only option.

Obs. Similarly, Marker could list moves ahead of time, but an adaptive strategy may be better.

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- Brooks' Theorem: $\chi(G) \leq \Delta(G)$ (usually...) (Brooks [1941], Vizing [1976], Hladký–Král–Schauz [2010])
- Alon–Tarsi Theorem: Exact same hypothesis gives same bound for paintability. (Alon–Tarsi [1992], Schauz [2010])

Background

Paintability and the Marker/Remover game were introduced by Schauz [2009].

Independently introduced by Zhu [2009] as on-line list coloring.

Some coloring upper bounds also hold for list coloring, strengthening the result. Similarly, some list coloring upper bounds hold for paintability.

- Brooks' Theorem: $\chi(G) \le \Delta(G)$ (usually...) (Brooks [1941], Vizing [1976], Hladký–Král–Schauz [2010])
- Alon–Tarsi Theorem: Exact same hypothesis gives same bound for paintability. (Alon–Tarsi [1992], Schauz [2010])
- Planar Graphs: $\chi_\ell(G) \le \chi_p(G) \le 5$ if G is planar (Thomassen [1994], Schauz [2009])

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Ques. What analogous bounds hold for paintability?

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Cor. (KKLZ [2012]) For all r, K_{2*r} is chromatic-paintable.

Thm. For any graph G, there exists $t_0 \in \mathbb{N}$ such that if $t > t_0$, then $G \vee K_t$ is chromatic-paintable.

Complete Bipartite Graphs

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Obs. Determining when $K_{\ell,r}$ is $(\ell - 1)$-paintable is different and more complicated than $(\ell - 1)$-choosable.

Sum-Choosability and Sum-Paintability

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Obs. The natural inequality $\chi_{sc}(G) \leq \chi_{sp}(G)$ holds.

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Lem. Adding a leaf to G increases $\chi_{sp}(G)$ by 2.
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Cor. These Lemmas determine the sum-paintability of generalized theta-graphs.

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Obs. Adding leaves and ears of length at least 3 to an sp-greedy graph creates another sp-greedy graph.

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Cor. $\chi_{sp}(G) - \chi_{sc}(G)$ can be arbitrarily large.

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Ques. What other choosability results hold for paintability?