The Jacobsthal Subcube of the Hypercube

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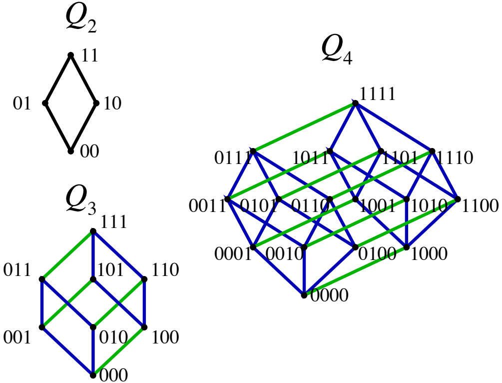
Cubes using the alphabet $\Sigma = \{0, 1\}$ with language $Q = \{0, 1\}$



Vertices: 2^n Edges: $n2^{n-1}$

Restrict to the language $A = \{0, 01, 11\}$

$A_0 = \{\emptyset\}$
$A_1 = \{0\}$
$A_2 = \{00, 01, 11\}$
$A_3 = \{000, 010, 110, 001, 011\}$
$A_4 = \{0000, 0100, 1100, 0010, 0110,$
$0001, 0101, 1101, \quad 0011, 0111, 1111\}$
$A_4 = A_3 0 \cup A_2 01 \cup A_2 11$
$|A_n| = |A_{n-1}| + 2|A_{n-2}|$

Jacobsthal numbers

$$J_0 = 1 \quad J_1 = 1$$
$$J_n = J_{n-1} + 2J_{n-2}$$

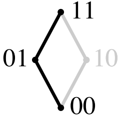
$1, 1, 3, 5, 11, 21, 43, 85, 171, 341, \ldots$

$$|A_0| = 1 \quad |A_1| = 1$$
$$|A_n| = |A_{n-1}| + 2|A_{n-2}|$$

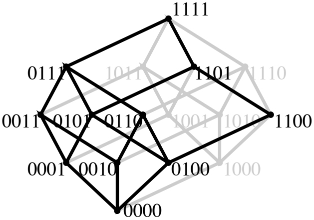
$J_n = \frac{2}{3} \cdot 2^n + \frac{1}{3} \cdot (-1)^n$

Jacobsthal subcube H_n

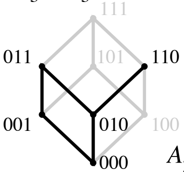
$A_2 = \{00, 01, 11\}$



$H_4 \subset Q_4$



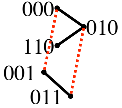
$H_3 \subset Q_3$

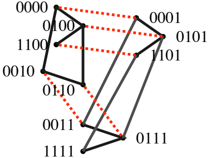


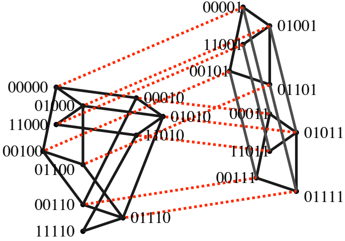
$A_3 = \{000, 010, 110, 001, 011\}$

Subcube recursion

$$A_n = A_{n-1}0 \cup A_{n-2}01 \cup A_{n-2}11$$

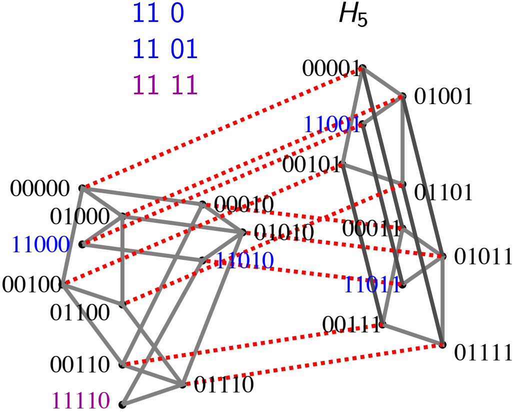






A string starts with an even number of 1s

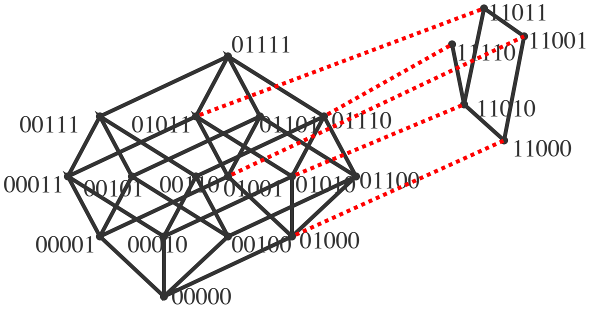
0
01
11 0
11 01
11 11



$A = \{0, 01, 11\}$

Subcube recursion with cubes

$$A_n = 0Q_{n-1} \cup 11A_{n-2}$$



Counting the edges

0	_	_	_	_	_	_	_	Q_7	$7 \cdot 2^6$
1	1	0	_	_	_	_	_	Q_5	$5 \cdot 2^4$
1	1	1	1	0	_	_	_	Q_3	$3 \cdot 2^2$
1	1	1	1	1	1	0	_	Q_1	$1 \cdot 2^0$

Count of edges between rows 2 and 3

0	_	_	_	_	_	_	_		Q_7	$7 \cdot 2^6$
1	1	0	$\underline{1}$	$\underline{0}$	_	_	_		Q_5	$5 \cdot 2^4$
1	1	1	1	0	_	_	_	2^3	Q_3	$3 \cdot 2^2$
1	1	1	1	1	1	0	_		Q_1	$1 \cdot 2^0$

Count of edges between rows

0	_	_	_	_	_	_	_		Q_7	$7 \cdot 2^6$
1	1	0	_	_	_	_	_	$1 \cdot 2^5$	Q_5	$5 \cdot 2^4$
1	1	1	1	0	_	_	_	$2 \cdot 2^3$	Q_3	$3 \cdot 2^2$
1	1	1	1	1	1	0	_	$3 \cdot 2^1$	Q_1	$1 \cdot 2^0$

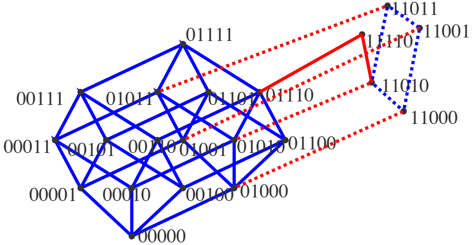
Total number of edges

0	_	_	_	_	_	_	_		Q_7	$7 \cdot 2^6$	$7 \cdot 2^6$
1	1	0	_	_	_	_	_	$1 \cdot 2^5$	Q_5	$5 \cdot 2^4$	$7 \cdot 2^4$
1	1	1	1	0	_	_	_	$2 \cdot 2^3$	Q_3	$3 \cdot 2^2$	$7 \cdot 2^2$
1	1	1	1	1	1	0	_	$3 \cdot 2^1$	Q_1	$1 \cdot 2^0$	$7 \cdot 2^0$
1	1	1	1	1	1	1	1				4

$$\sum_{k=1}^{\lfloor (n+1)/2 \rfloor} (n-1)2^{n-2k} = \left(\frac{n-1}{3}\right) 2^n \left[1 - \left(\frac{1}{4}\right)^{\lfloor \frac{n+1}{2} \rfloor}\right].$$

$$e_n = \left(\frac{n}{4}\right) [1 + (-1)^n] + \left(\frac{n-1}{3}\right) 2^n \left[1 - \left(\frac{1}{4}\right)^{\lfloor \frac{n+1}{2} \rfloor}\right].$$

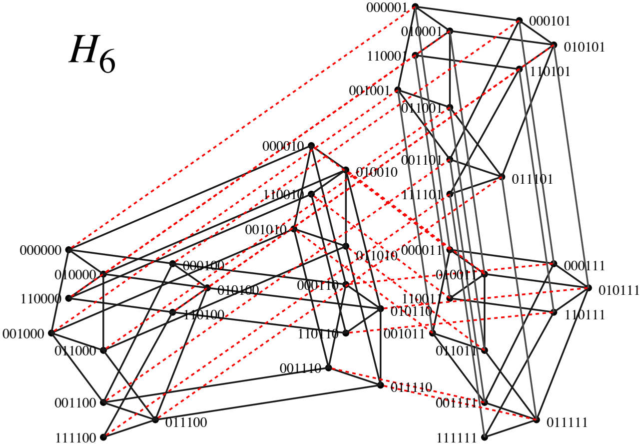
Direct count of edges in H_5



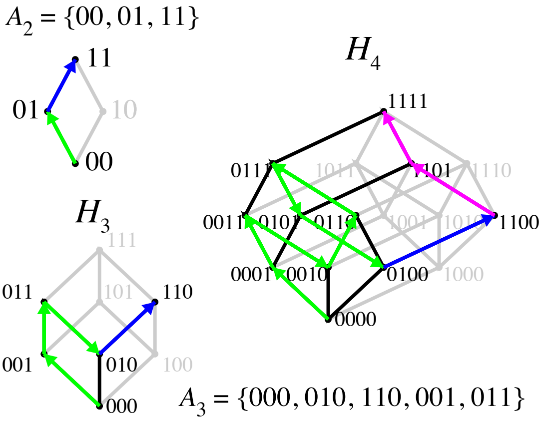
from \ to	(start 0)	(start 11)	(start 1111)
(start 0)	$4 \cdot 2^3$	$1 \cdot 2^2$	$1 \cdot 2^0$
(start 11)		$2 \cdot 2^1$	$1 \cdot 2^0$
(start 1111)			0
Total to	$4 \cdot 2^3$	$4 \cdot 2^1$	$4 \cdot 2^{-1}$
	32	8	2

42 edges in H_5

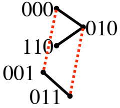
Does H_n have a Hamiltonian path?



Take the path through $0Q_{n-1}$ then move to $11H_{n-2}$



Summary



Verticies: $|H_n| = J_n = \frac{1}{3}(-1)^n + \frac{2}{3}(2)^n$

Edges: $e_n = \left(\frac{n}{4}\right)[1 + (-1)^n] + \left(\frac{n-1}{3}\right) 2^n \left[1 - \left(\frac{1}{4}\right)^{\left\lfloor \frac{n+1}{2} \right\rfloor}\right]$

Hamiltonian Paths!