K_7 in the torus: a long story

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I am mostly interested in the last item.

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A Cayley graph $C(A, X)$ for group A and generating set X has A as vertex set and an edge (directed and colored x) from a to ax for all $a \in A$ and $x \in X$. If $x^2 = 1$, we often identify the pair of directed edges (a, ax) and (ax, a) to a single undirected edge.
Main fact; the action of A given by left multiplication by b is a graph isomorphism: $a \to ax$ goes to $ba \to bax$. Thus A acts **regularly** (transitively without fixed points) on the vertex set of $C(A, X)$.

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Rotation systems

To define an orientable embedding for a graph G, we only need to give for each vertex v a cyclic order of the edges incident to v, that would be induced by an orientation of the embedding surface. The collection of all these cyclic orders is called a **rotation system**. To see the embedding surface associated with a rotation system, just thicken each vertex to a disk, thicken each edge to a band and attach around each vertex-disk by the order given by the rotation. The result is a thickening of the graph to a surface with boundary.

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The idea os specifying an embedding for a given graph G this way is due to Heffter (1895) and Edmonds(1956). Important

observation: any graph automorphism that respects the rotation (cyclic order at each vertex) induces an automorphism of the embedding (takes faces to faces).

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Call this the **Cayley map** $CM(Z_7, (1, 3, 2, -1, -3, -2)$, namely a Cayley graph together with a cyclic order of $X \cup X^{-1}$.
We can trace out the faces: start at vertex 0, go out on 1, coming into vertex 1 on -1, follow rotation to -3 and leave to vertex -2, arriving there on 3, follow rotation to 2, go out returning to 0 arriving on -2, and follow rotation back to 1.

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Orientably regular maps are analagous to the Platonic solids, having full rotation symmetry at every vertex, every face-center, and every edge-midpoint.

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Reflexible maps for other K_n

So there are chiral maps for K_n. Are they ALL chiral?

Theorem

(Biggs, James and Jones, Wilson). The only reflexibly regular maps with underlying graph K_n are for $n = 3, 4, 6$ and for $n = 6$ the map must be non-orientable.

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3. Deal with non-orientable maps

Classifying all orientably regular maps for K_n

First, the K_7 map generalizes to any finite fields $GF(q)$, where $q = p^n$. Let x generate the cyclic multiplicative group. The additive group is abelian $A = Z_p^n$. The Cayley map $CM(A, (1, x, x^2, x^3, \cdots x^{q-2}$ is is regular.

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Theorem

The only orientably regular maps with underlying K_n are the finite field maps.

Proof

Suppose that M is an orientably regular map with underlying graph K_n. Then $Aut^+(M)$ acts transitively on the vertex set such that no element fixes two vertices (otherwise it contains a reflection), making it a **Frobenius group**.

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Since the stabilizer of a vertex is cyclic generated by a rotation y around that vertex, we have that conjugation by y gives an automorphism of A that cyclically permutes the non-identity elements of A.
Thus every element of A has the same order, which therefore must be a prime p; and the only characteristic subgroups are trivial, making A abelian. So $A = Z_p^n$ and mult by y is linear transformation with irred minimal poly of degree n etc.

Comments on chirality for finite field maps

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Since reflection takes x to multiplicative inverse (rotation is $(1, x, x^2, x^3, \cdots x^{q-2})$), we have that $x \to x^{-1}$ is an additive automophism on nonzero elements: $(1 + x)^{-1} = 1 + x^{-1}$ so $x = (1 + x)^2$

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That happens only for $q = 3, 2^2$. That gives K_3 and K_4 (tetrahedron).

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Note this handles also the non-orientable case of K_6. Also it says far more. And the proof is almost trivial!

Angle measure

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Suppose that the cyclic order of vertices adjacent to v is $u_1, u_2, \cdots u_d$, where d is the valence of v. Then we call $u_i v u_j$ an **angle** at v with **measure** $m(u_i v u_j)$ either $|i-j|$ or $d-|i-j|$, whichever is smaller. In particular, $m(u_i v u_j) \leq d/2$.

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We are assuming here that the underlying graph G has no multiple edges. The definition easily extends using the cyclic order of incident edges rather than adjacent vertices.

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Since the action of $Aut(M)$ is naturally dihedral at vertices, every angle uvw has an angle reflection, namely an automorphism f fixing v and interchanging u and w.

It follows that if uvw is triangle (3-cycle), then the reflection at v means $m(vuw) = m(wvu)$. Since this is true at each vertex, the triangle uvw is equiangular, namely $m(uvw) = m(vwu) = m(wuv)$.

The case $a + b + c = d$

Suppose now that u, v, w, x induce K_4. There are three angles at u. Suppose their measures are:

$$m(vuw) = a, m(wux) = b, m(xuv) = c, \text{ where } a \leq b \leq c.$$

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$d - (a + b) = c, d - (b + c) = a, d - (c + a) = b$. Since all triangles are equiangular, we have $a = b = c = d/3$.

Consequences of $a = b = c$

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This means there is no K_5 and that G has a K_4 factorization.

Consequences of $a = b = c$

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So $a = b, 2c + a = d, a = d/5, c = 2d/5$

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Again every edge is in one and only one K_6, so G has no K_7 and G has a K_6 factorization

Non-orientability for the K_6 case

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But if M were orientable, the orientation-preserving elements of B would form a subgroup of index two, a contradiction (Note: H contains reflections, so H is is not orientation-preserving.)

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Thus if G has clique number $n > 2$, then A is vertex-transitive. Now choose any vertex and any automorphism f generating the index two cyclic subgroup of A_v. For each other vertex v choose an automorphism $g(v) = u$ and use gfg^{-1} to define a cyclic order around u.

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Now choose any vertex and any automorphism f generating the index two cyclic subgroup of A_v. For each other vertex v choose an automorphism $g(v) = u$ and use gfg^{-1} to define a cyclic order around u.
This cyclic order at each vertex can now be used to define an angle measure that is invariant under A, allowing us to apply the previous map argument.

Examples of maps for K_4 and K_6

For K_4, there is the family of groups from Conder, Širà ň, Tucker (JEMS 2010):
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Marston Conder has found, with the help of Magma, a family of infinitely many examples for K_4 where the underlying graph has no multiple edges. Same for K_6.