$K_7$ in the torus: a long story

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Rotation systems

To define an orientable embedding for a graph $G$, we only need to give for each vertex $v$ a cyclic order of the edges incident to $v$, that would be induced by an orientation of the embedding surface. The collection of all these cyclic orders is called a rotation system. To see the embedding surface associated with a rotation system, just thicken each vertex to a disk, thicken each edge to a band and attach around each vertex-disk by the order given by the rotation. The result is a thickening of the graph to a surface with boundary.
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The idea of specifying an embedding for a given graph $G$ this way is due to Heffter (1895) and Edmonds (1956). Important observation: any graph automorphism that respects the rotation (cyclic order at each vertex) induces an automorphism of the embedding (takes faces to faces).
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We can trace out the faces: start at vertex 0, go out on 1, coming into vertex 1 on $-1$, follow rotation to $-3$ and leave to vertex $-2$, arriving there on 3, follow rotation to 2, go out returning to 0 arriving on $-2$, and follow rotation back to 1.
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Automorphisms

We already know that $CM(Z_7, (1, 3, 2, -1, -3, -2)$ is vertex transitive by looking at “left addition” (remember we are looking at $Z_7$ additively).

Now consider multiplication by 3, which is an additive automorphism of $Z_7$. It respects the rotation so it is a map automorphism. This means our map has rotational 6-fold symmetry at every vertex, making it orientably regular. Orientably regular maps are analogous to the Platonic solids, having full rotation symmetry at every vertex, every face-center, and every edge-midpoint.
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Reflexible maps for other $K_n$

So there are chiral maps for $K_n$. Are they ALL chiral?

**Theorem**

*(Biggs, James and Jones, Wilson). The only reflexibly regular maps with underlying graph $K_n$ are for $n = 3, 4, 6$ and for $n = 6$ the map must be non-orientable.*

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Classifying all orientably regular maps for $K_n$

First, the $K_7$ map generalizes to any finite fields $GF(q)$, where $q = p^n$. Let $x$ generate the cyclic multiplicative group. The additive group is abelian $A = Z_p^n$. The Cayley map $CM(A, (1, x, x^2, x^3, \ldots x^{q-2})$ is is regular.
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It is chiral for the same reason as before for $p > 2$: any reflection fixing 1 will fix the additive $p$-cycle generated by 1, which means other edges at 0 are fixed besides 1 and $-1$. 

Theorem

The only orientably regular maps with underlying $K_n$ are the finite field maps.
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Proof

Suppose that $M$ is an orientably regular map with underlying graph $K_n$. Then $Aut^+(M)$ acts transitively on the vertex set such that no element fixes two vertices (otherwise it contains a reflection), making it a Frobenius group.
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By a classic theorem of Frobenius, \( Aut^+(M) \) contains a normal subgroup \( A \) that acts regularly on the vertex set and the stabilizer of a vertex, acting by conjugation on \( A \) injects into \( Aut(A) \).
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Since the stabilizer of a vertex is cyclic generated by a rotation $y$ around that vertex, we have that conjugation by $y$ gives an automorphism of $A$ that cyclically permutes the non-identity elements of $A$.

Thus every element of $A$ has the same order, which therefore must be a prime $p$; and the only characteristic subgroups are trivial, making $A$ abelian. So $A = Z_p^n$ and mult by $y$ is linear transformation with irred minimal poly of degree $n$ etc.
Comments on chirality for finite field maps

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Since reflection takes $x$ to multiplicative inverse (rotation is $(1, x, x^2, x^3, \cdots x^{q-2})$), we have that $x \rightarrow x^{-1}$ is an additive automorphism on nonzero elements: $(1 + x)^{-1} = 1 + x^{-1}$ so $x = (1 + x)^2$
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That happens only for $q = 3, 2^2$. That gives $K_3$ and $K_4$ (tetrahedron).
New proof: no algebra

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**Theorem**

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Note this handles also the non-orientable case of $K_6$. Also it says far more. And the proof is almost trivial!
Angle measure

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Suppose that the cyclic order of vertices adjacent to \( v \) is \( u_1, u_2, \ldots, u_d \), where \( d \) is the valence of \( v \). Then we call \( u_i v u_j \) an angle at \( v \) with measure \( m(u_i v u_j) \) either \( |i - j| \) or \( d - |i - j| \), whichever is smaller. In particular, \( m(u_i v u_j) \leq d/2 \).
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We are assuming here that the underlying graph $G$ has no multiple edges. The definition easily extends using the cyclic order of incident edges rather than adjacent vertices.
The proof

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It follows that if $uvw$ is triangle (3-cycle), then the reflection at $v$ means $m(vuw) = m(wvu)$. Since this is true at each vertex, the triangle $uvw$ is equiangular, namely $m(uvw) = m(vwu) = m(wuv)$.
The case $a + b + c = d$

Suppose now that $u, v, w, x$ induce $K_4$. There are three angles at $u$. Suppose their measures are:

$$m(vuw) = a, m(wux) = b, m(xuv) = c,$$
where $a \leq b \leq c$. Then either $a + b + c = d$ or $c = a + b$. Suppose first that $a + b + c = d$. Then in the tetrahedron $u, v, w, x$, there are four triangles: one has all angles $a$, one $b$, and one $c$. The last has angles $d - (a + b)$, $d - (b + c)$, $d - (c + a)$. Since all triangles are equiangular, we have $a = b = c = d/3$. 

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Suppose now that $u, v, w, x$ induce $K_4$. There are three angles at $u$. Suppose their measures are:

$$m(vuw) = a, m(wux) = b, m(xuv) = c, \text{ where } a \leq b \leq c.$$  

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The last has angles $d - (a + b) = c, d - (b + c) = a, d - (c + a) = b$. Since all triangles are equiangular, we have $a = b = c = d/3$. 
Consequences of $a = b = c$

We have all $K_4$ subgraphs are symmetrically situated at every vertex making angles $a = b = c = d/3$. In particular, each edge can be in one and only one $K_4$. 
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So \( a = b, 2c + a = d, a = d/5, c = 2d/5 \)
Consequences of $a = b = d/5, c = 2d/5$

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By the circular symmetry around a vertex, we must have a \( K_6 \) making all angles \( \frac{d}{5} \) at any vertex. Thus here \( K_4 \) implies \( K_6 \).

Again every edge is in one and only one \( K_6 \), so \( G \) has no \( K_7 \) and \( G \) has a \( K_6 \) factorization
Non-orientability for the $K_6$ case

Let $B \subseteq Aut(M)$ be the subgroup stabilizing a $K_6$ subgraph $H \subseteq G$. By the 5-fold dihedral symmetry at each vertex of $H$, we have $|H| = 10 \cdot 6 = 60$. But if $M$ were orientable, the orientation-preserving elements of $B$ would form a subgroup of index two, a contradiction (Note: $H$ contains reflections, so $H$ is not orientation-preserving.)
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Thus if $G$ has clique number $n > 2$, then $A$ is vertex-transitive. Now choose any vertex and any automorphism $f$ generating the index two cyclic subgroup of $A_v$. For each other vertex $v$ choose an automorphism $g(v) = u$ and use $gfg^{-1}$ to define a cyclic order around $u$. 
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This cyclic order at each vertex can now be used to define an angle measure that is invariant under $A$, allowing us to apply the previous map argument.
Examples of maps for $K_4$ and $K_6$

For $K_4$, there is the family of groups from Conder, Širáň, Tucker (JEMS 2010):

$G(3, 3, n) = \langle X, Y : X^{3n} = Y^{3n} = (XY)^2 = 1, X^{12} Y^{12} = 1 \rangle$

Marston Conder has found, with the help of Magma, a family of infinitely many examples for $K_4$ where the underlying graph has no multiple edges. Same for $K_6$. 
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