Rainbow Edge-coloring and Rainbow Domination

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slides available on DBW preprint page

Joint work with Timothy D. LeSaulnier

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edge-coloring: cover $E(G)$ with matchings — $\chi'(G)$

domination: cover $V(G)$ with disjoint stars — $\gamma(G)$

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If the edge-coloring is rainbow, then $\hat{\chi}'(G) = \chi'(G)$.

If the edge-coloring is proper, then $\hat{\gamma}(G) = \gamma(G)$.

Large Rainbow Matchings

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With Pfender: $\hat{\alpha}'(G) \geq \hat{\delta}(G)$ when $n \geq 5.5(\hat{\delta}(G))^2$.

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classical		generalized
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$\gamma(G) \leq \frac{1}{2}n$	Ore [1962] (no isol.)	$\hat{\gamma}(G) \leq \frac{t}{t+1}n$
$\gamma(G) \leq \frac{1+\ln(\delta(G)+1)}{\delta(G)+1}n$	Arnautov [1974] Payan [1975]	$\hat{\gamma}(G) \leq \frac{1+\ln k}{k}n$

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Thm. When G is t-tolerant (and no isolated vertices),
$\hat{\gamma}(G) = \frac{t}{t+1}n \iff$ each component is a t-flare
(or monochr. C_3 ($t=2$) or properly edge-colored C_4 ($t=1$)).

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Lem. For $t \in \mathbb{N}$ and $c \in \mathbb{R}$ with $c > 0$, every t-tolerant edge-colored G with average color degree $\geq c$ has a t-tolerant edge-colored subgraph H with $\hat{\delta}(H) > \frac{c}{t+1}$.

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Pf. t-tolerant $\Rightarrow$ $\hat{d}_G(v) \geq d_G(v)/t$. With degree-sum $2m$, the average color degree is $\geq 2m/(nt)$. The lemma yields H with $\hat{\delta}(H) > \frac{2m}{nt(t+1)}$. Now $\hat{\alpha}'(H) \geq \left\lceil \frac{m}{nt(t+1)} \right\rceil$. ■

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It remains to bound j and $|E(F_j)|$.

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Note $a_i\binom{n}{2} = |E(F_{i-1})| - |M_{i-1}| \leq a_{i-1}\binom{n}{2}\left(1 - \frac{1}{nt(t+1)}\right)$.

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We have $a_j \leq \frac{2t(t+1)}{n-1} = \frac{2t(t+1)}{n-1} < a_{j-1} < e^{\frac{-j+1}{nt(t+1)}}$.

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Note $a_i\binom{n}{2} = |E(F_{i-1})| - |M_{i-1}| \leq a_{i-1}\binom{n}{2}\left(1 - \frac{1}{nt(t+1)}\right)$.

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Finally, we compute

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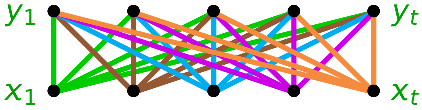

Arnautov–Payan bound — Generalization

Thm. If G is an n-vertex t-tolerant edge-colored graph, then $\hat{\gamma}(G) \leq \frac{1+\ln k}{k} n$, where $k = \frac{\delta(G)}{t} + 1$.

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Thus $\mathbb{E}(|B|) \le n/k$. We conclude $\mathbb{E}(|A \cup B|) \le \frac{(1+\ln k)}{k} n$. ■

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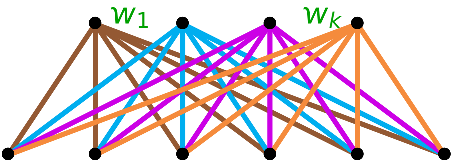

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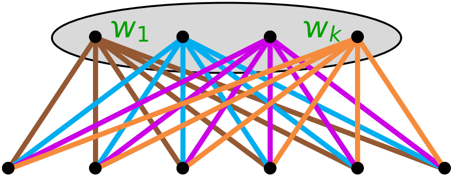

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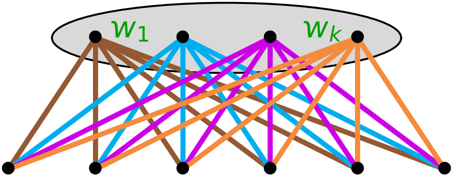

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(No rainbow star covers two vertices of U.) ∎

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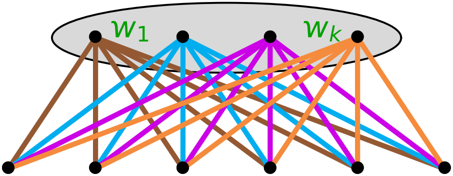

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Note: $\hat{\gamma}(G)/n \to 1$, but $t/n \to 1$.

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Add 1-vertex stars; now $\hat{\gamma}(G) \le n - s$. Note that $s \ge k$.

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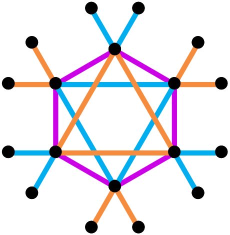
Thus $\hat{\gamma}(G) \le n - s \le \frac{t}{t+1} n$, since $s \ge \frac{1}{t+1} n$. ∎

Characterization of Equality

Def. The t-corona $H \circ t$ is formed by adding t pendant edges at each vertex of H. A t-flare is an edge-colored t-corona $H \circ t$ that is t-tolerant and, for each vertex of H, has the same color on all t new pendant edges there.



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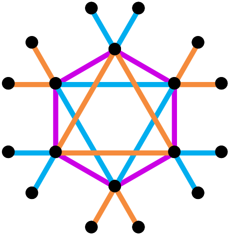

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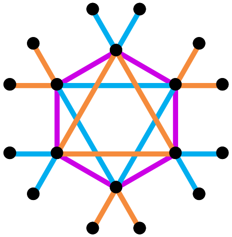

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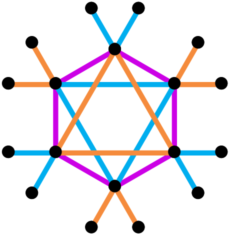

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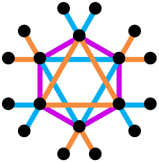
- For $t = 1$ (where $\hat{\gamma}(G) = \gamma(G)$), Payan–Xuong [1982] and Fink–Jacobson–Kinch–Roberts [1985] char'zd $\gamma(G) = n/2$.

Sketch of Characterizing Equality in $\hat{\gamma}(G) \leq \frac{t}{t+1}n$



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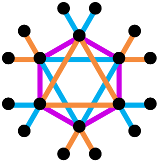

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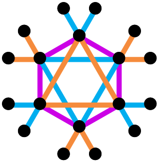

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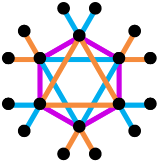

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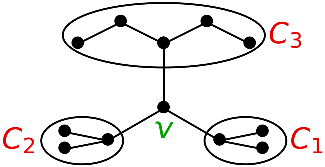

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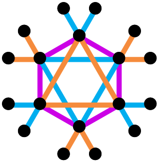

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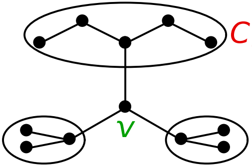

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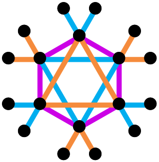

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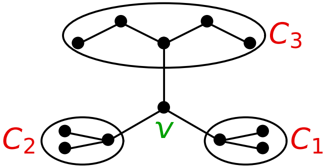

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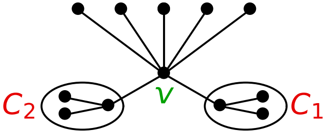
Now $(t+1) \nmid n$, and again the inequality is strict for G.

Idea, continued

$\therefore$ v has leaf nbr(s), say ℓ of them, with k colors.



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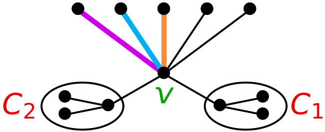

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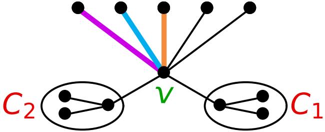

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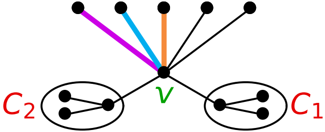

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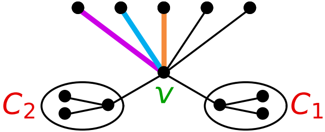

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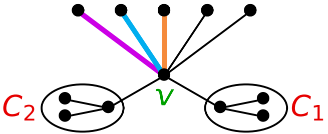

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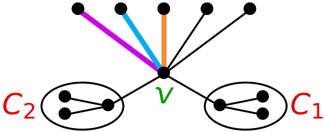

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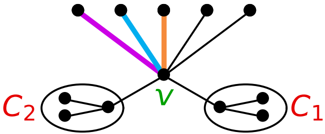

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∴ Every spanning tree is a t-flare.

Claim: No other edges at leaves of a spanning tree T.
(Otherwise, some spanning tree is not a t-flare.)

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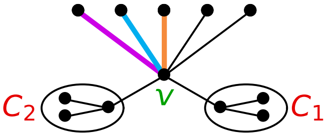

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(The exceptions: monochr. C_3 and properly colored C_4.) ■

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Reference:
T. D. LeSaulnier and D. B. West,
Rainbow edge-coloring and rainbow domination,
Discrete Math. (2012), DOI: 10.1016/j.disc.2012.03.014.