Dynamic Programming (Weighted Interval Scheduling)

Arash Rafiey

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2. We use the basic idea of divide and conquer. Dividing the problem into a number of subproblems.

- The solution to the original problem can be easily computed from the solution to the subproblems (e.g. sum the solutions to the subproblems and get the solution to the original).

- The idea is to solve each subproblem only once. Once the solution to a given subproblem has been computed, it is stored or memorized: (the next time the same solution is needed, it is simply looked up). This way we reduce the number of computations.
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Problem Statement:

1. We have a resource and many people request to use the resource for periods of time (an interval of time).
2. Each interval (request) $i$ has a start time $s_i$ and finish time $f_i$.
3. Each interval (request) $i$, has a value $v_i$.

Conditions:

- the resource can be used by at most one person at a time.
- we can accept only compatible intervals (requests) (overlap-free).

Goal: a set of compatible intervals (requests) with a maximum total value.
What if all the values are the same?

[Diagram of intervals]

We start with the one which has the smallest finish time and add it into our solution and remove the ones that have intersection with it and repeat!
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\(v_1 = 2\)

\(v_2 = 4\)

\(v_3 = 4\)

\(v_4 = 7\)

\(v_5 = 2\)

\(v_6 = 1\)
\[ v_1 = 2 \]

1

\[ v_2 = 4 \]

2

\[ v_3 = 4 \]

3

\[ v_4 = 7 \]

4

\[ v_5 = 2 \]

5

\[ v_6 = 1 \]

6

\[ v_1 = 2 \]

1

\[ v_2 = 4 \]

2

\[ v_3 = 4 \]

3

\[ v_4 = 7 \]

4

\[ v_5 = 5 \]

5

\[ v_6 = 1 \]

6
We order the intervals (requests) based on their finishing time (non-increasing order of finishing time):

\[ f_1 \leq f_2 \leq \cdots \leq f_n \]
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For every interval \( j \) let \( p(j) \) be the largest index \( i < j \) such that intervals \( i, j \) do not overlap (have more than one points in common)

\[ p(j) = 0 \text{ if there is no interval before } j \text{ and disjoint from } j. \]
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<table>
<thead>
<tr>
<th>Interval</th>
<th>Value</th>
<th>( p(j) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( v_1 = 2 )</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>( v_2 = 4 )</td>
<td>0</td>
</tr>
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$$OPT(j) = \max\{OPT(j - 1), v_j + OPT(p(j))\}$$
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Try not to implement using recursive call because the running time would be exponential!

Recursive function is easy to implement but time consuming!
Iterative-Compute-Opt()
1. $M[0] := 0$;
2. for $j = 1$ to $n$
3. \[ M[j] = \max\{v_j + M[p(j)], M[j - 1]\} \]
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Find-Solution(j)
1. if \( j = 0 \) then return
2. else
3. if \( (v_j + M[p(j)] \geq M[j - 1]) \)
4. print \( j \) and print " "
5. Find-Solution(p(j))
6. else
7. Find-Solution(j-1)
Subset Sums (Problem Statement):

1. We are given \( n \) items \( \{1, 2, \ldots, n\} \) and each item \( i \) has weight \( w_i \)
2. We are also given a bound \( W \)

Goal: select a subset \( S \) of the items so that:

1. \( \sum_{i \in S} w_i \leq W \)
2. \( \sum_{i \in S} w_i \) is maximized
A greedy approach won’t work if it is based on picking the biggest value first. Suppose we have a set \( \{ \frac{W}{2} + 1, \frac{W}{2}, \frac{W}{2} \} \) of items. If we choose \( \frac{W}{2} + 1 \) then we can not choose anything else. However the optimal is \( \frac{W}{2} + \frac{W}{2} \).
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Exhaustive search! Produce all the subset and check which one satisfies the constraint ($\leq W$) and has maximum size. Running time $O(2^n)$. 

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If $w_i > w$ then $OPT[i, w] = OPT[i - 1, w]$ otherwise

$$OPT[i, w] = \max \{OPT[i - 1, w], w_i + OPT[i - 1, w - w_i]\}$$
Subset-Sum(n, W)
1. Array $M[0, .., n, 0, .., W]$
2. for ( $i = 1$ to $W$ ) $M[0, i] = 0$
3. for ( $i = 1$ to $n$ )
4. for ( $w = 1$ to $W$ )
5. if ( $w < w_i$ ) then $M[i, w] = M[i - 1, w]$
6. else
7. if ( $w_i + M[i - 1, w - w_i] > M[i - 1, w]$ )
8. $M[i, w] = w_i + M[i - 1, w - w_i]$
9. else $M[i, w] = M[i - 1, w]$

Time complexity $O(nW)$
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Suppose $W = 6$ and $n = 3$ and the items of sizes $w_1 = w_2 = 2$ and $w_3 = 3$. Fill out matrix $M$. 
Problem: We are given \( n \) jobs \( \{J_1, J_2, \ldots, J_n\} \) where each \( J_i \) has a processing time \( p_i > 0 \) (an integer).

We have two identical machines \( M_1, M_2 \) and we want to execute all the jobs.

Schedule the jobs so that the finish time is minimized.
Knapsack Problem:

1. We are given $n$ items $\{1, 2, \ldots, n\}$ and each item $i$ has weight $w_i$.
2. Each item has value $v_i$.
3. We are also given a bound $W$.

Goal: select a subset $S$ of the items so that:

1. $\sum_{i \in S} w_i \leq W$.
2. $\sum_{i \in S} v_i$ is maximized.
Let \( OPT[i, w] \) be the optimal maximum value of a set \( S \subset \{1, 2, \ldots, i\} \) where the total value of the items in \( S \) is at most \( w \).
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\text{OPT}[i, w] = \max\{\text{OPT}[i - 1, w], v_i + \text{OPT}[i - 1, w - w_i]\}
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