Odd Cycle Transversal in P_k-Free Graphs

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Abstract

The Odd Cycle Transversal (OCT) problem, which asks for a minimum subset of vertices whose removal renders a graph bipartite, is a central problem in algorithmic graph theory. It is known to be NP-complete even on P_k-free graphs for $k \geq 6$. Furthermore, assuming the Unique Games Conjecture (UGC), OCT does not admit a constant-factor approximation algorithm on general graphs.

Motivated by these hardness results, we investigate the approximability of OCT on P_k-free graphs. We first establish that the problem becomes polynomial-time solvable on specific subclasses of P_k-free graphs, most notably (P_6, C_3)-free graphs, by exploiting a structural decomposition into rings of bipartite graphs. Leveraging these tractable substructures as a basis, we present a constant-factor approximation algorithm for OCT on general P_k-free graphs. We achieve an approximation ratio of $k-2$ when k is odd and $k-3$ when k is even. These results provide the first nontrivial constant-factor approximations for this class dependent on k, aligning with the UGC implication that no approximation factor independent of k is likely to exist.

1 Introduction

The Odd Cycle Transversal (OCT) problem, also known as Graph Bipartization, is a fundamental vertex-deletion problem: given a graph $G = (V, E)$, find a minimum set $S \subseteq V$ such that $G - S$ is bipartite. Equivalently, S must intersect every odd cycle of G. OCT is a canonical representative of "transversal to a hereditary class" problems and is closely related to finding a largest induced bipartite subgraph.

OCT is NP-complete, already as a special case of vertex-deletion problems to hereditary graph classes [11]. From the approximation viewpoint, the best currently known polynomial-time guarantee on general graphs is an $O(\sqrt{\log n})$-approximation, obtained via approximation algorithms for Min-UnCut / Min-2CNF Deletion [1]; this remains essentially the state of the art for OCT in unrestricted graphs. Furthermore, under the Unique Games Conjecture (UGC), OCT becomes dramatically harder: it is NP-hard to approximate within any constant factor. This UGC-based inapproximability is commonly attributed to [3] and is explicitly used as a hardness benchmark in subsequent OCT literature, including kernelization work [10]. Taken together, these results make OCT a compelling testbed for investigating whether—and how—structural restrictions on the input graph can restore approximability beyond what is possible in general graphs.

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Why P_k-free graphs? A major line of research in algorithmic graph theory studies hard problems on hereditary classes defined by forbidding induced subgraphs. In particular, forbidding long induced paths (i.e., working in P_k-free graphs) often yields strong structure and enables efficient algorithms for problems that are intractable in general. For OCT, however, this problem exhibits a sharp threshold: the problem is polynomial-time solvable on P_5-free graphs [2], while it becomes NP-hard already on P_6-free graphs [7]. Thus, in contrast to many classical problems, excluding long induced paths does *not* automatically restore tractability for OCT once $k \geq 6$. This leaves open a natural and practically relevant algorithmic question: even when exact optimization is hard on P_k-free graphs, can one obtain approximation guarantees whose quality depends only on k (and not on size of the input graph G)?

Since OCT is precisely the distance to 2-colorability, it is natural to place our work in the broader context of coloring graphs with forbidden induced paths. The class of P_k-free graphs is central to the theory of χ-boundedness and often exhibits strong structural constraints; see the survey of Scott and Seymour [13]. On the algorithmic side, several sharp thresholds are known: for every fixed q, q-Colorability is solvable in polynomial time on P_5-free graphs [8], and 3-Colorability is polynomial-time solvable on P_6-free graphs [12]. More generally, a near-complete complexity classification of k-Coloring on P_t-free graphs, including NP-completeness of 4-Coloring on P_7-free graphs and of 5-Coloring on P_6-free graphs was initiated in [9]. The remaining borderline case of 4-Coloring on P_6-free graphs was later resolved by a polynomial-time algorithm [5, 6]. Together, these results illustrate that forbidding long induced paths can create fine-grained complexity transitions for coloring, and they motivate our study of OCT within the same P_k-free hierarchy, where exact solvability breaks at $k = 6$ but meaningful approximation guarantees may be achievable.

Our focus and contributions. Motivated by (i) the UGC-based barrier against constant-factor approximation on general graphs [3, 10], and (ii) the sharp P_5 versus P_6 tractability threshold [2, 7], we study the approximability of OCT on P_k-free graphs for $k \geq 6$.

Our Contributions.

1. **Polynomial-time solvability on structured subclasses.** We identify a tractable "backbone" subclass of P_k-free graphs obtained by additionally excluding all odd cycles up to a length determined by k. We show that graphs in this subclass admit a decomposition into a cyclic arrangement of bipartite interactions, and we reduce OCT on these ring-like instances to a sequence of Minimum Vertex Cover computations in bipartite graphs, solvable in polynomial time. The case $k = 6$ is more difficult than the case $k > 6$, as it requires to established several structural properties for (P_6, C_3)-free graphs. It is known that OCT is polynomial time solvable on (P_6, C_3)-free graphs by showing that they have bounded clique-width [4]. However, we have a direct proof that doesn't rely on polynomial of OCT on bounded clique-width graphs and the core Lemma developed in the proof are used for the case where $k > 6$.

2. **k-dependent constant-factor approximation on P_k-free graphs.** Building on the tractable backbone, we give the first polynomial-time approximation guarantees for OCT problem on P_k-free graphs whose ratio depends only on k. The algorithm greedily packs short odd cycles and removes them, after which the remaining instance falls into the tractable subclass handled exactly. This yields an approximation ratio of $k - 2$ for

odd k and ratio $k-3$ for even k. The running time of our algorithm is a polynomial whose degree is independent of k.

These guarantees are qualitatively consistent with the UGC-based hardness for general graphs: while no constant independent of the instance size is expected in full generality, forbidding induced P_k allows approximation factors that scale only with parameter k.

2 Preliminaries and Notations

Let G be a graph. We denote the vertex and edge sets of G by $V(G)$ and $E(G)$, respectively. For simplicity, we denote the edge (x, y) in G as xy. Unless otherwise stated, we assume all graphs are finite with no multiple edges or self-loops. For an edge uv in G, u is a neighbor of v, and v is a neighbor of u. For a vertex $x \in V(G)$, $N(x)$ denotes the set of neighbors of x, and for a subset $X \subset V(G)$, $N(X)$ is the set of vertices in $V(G) \setminus X$ that have at least one neighbor in X. For two disjoint subsets X and Y of $V(G)$, we say X is complete to Y if for every $x \in X$ and $y \in Y$, xy is an edge of G. We say X is independent of (or anti-complete to) Y if there is no edge of G between X and Y. A complete bipartite graphs is also called a biclique.

For a set $S \subset V(G)$, let $G[S]$ be the subgraph induced by the vertices of S; if xy is an edge of G with $x, y \in S$, then $xy \in E(G[S])$. Let P_k denote an induced path on k vertices; it has vertices $v_1, v_2, \ldots, v_k$ and edges v_iv_{i+1} for $1 \le i \le k-1$. Similarly, C_k is an induced cycle on k vertices, which has exactly k edges. Let $V_0, V_1, \ldots, V_{k-1}$ be disjoint subsets of $V(G)$. We say $V_0, V_1, \ldots, V_{k-1}$ form a *k-ring* W if every edge of $G[V_0 \cup V_1 \cup \cdots \cup V_{k-1}]$ is between V_i and V_{i+1}, $0 \le i \le k-1$ (with indices modulo k). We say k-ring W is complete if $G[V_i \cup V_{i+1}]$ is complete bipartite graph (biclique). Notice that by definition each V_i is an independent set. For two distinct graphs X and Y, we say G is (X, Y)-free if there is no induced subgraph of G isomorphic to X or Y. For example, if G is a (P_6, C_3)-free graph, then there is no induced P_6 in G and there is no triangle in G. *Chain bipartite graph:* A bipartite graph $H = (X, Y, E)$ is a *chain bipartite graph* (or simply a *chain graph*) if the neighborhoods of the vertices in X can be linearly ordered by inclusion. That is, there exists an ordering of the vertices in X, say $x_1, x_2, \ldots, x_{|X|}$, such that $N(x_1) \subseteq N(x_2) \subseteq \cdots \subseteq N(x_{|X|})$. By symmetry, this condition implies that the neighborhoods of the vertices in Y are also linearly ordered by inclusion. Equivalently, a bipartite graph is a chain graph if and only if it does not contain an induced $2K_2$ (two disjoint edges with no cross-edges between their endpoints).

3 (P_6, C_3)-free graphs

Basic Properties and partition of vertices in G: Let $G = (V, E)$ be a (P_6, C_3)-free graph containing an induced 5-cycle $C = \{a, b, c, d, e\}$. Let A_1, B_1, C_1, D_1, E_1 be the sets of vertices adjacent to exactly one vertex on the C_5. Specifically:

- $A_1 = \{v \in V(G) \setminus V(C_5) : N(v) \cap V(C) = \{a\}\}$
- B_1, C_1, D_1, E_1 are defined similarly for vertices b, c, d, e respectively. We often refer to these sets with X_1 and Y_1. For $X_1 \neq Y_1$ from $\{A_1, B_1, C_1, D_1, E_1\}$, we say X_1 and Y_1 are consecutive if $(X_1, Y_1) = (A_1, B_1), (B_1, C_1), (C_1, D_1), (D_1, E_1), (E_1, A_1)$ otherwise we say they are not consecutive.

Observation 3.1. *Any vertex $v \in V(G) \setminus V(C)$ is adjacent to at most two vertices on $V(C)$, and these neighbors must be non-consecutive.*

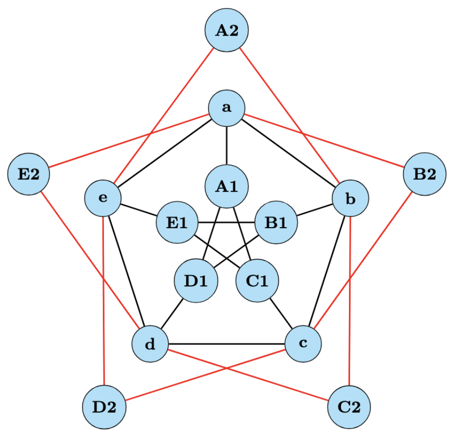
We have the following lemma for adjacencies within and between the sets in $\{A_1, B_1, C_1, D_1, E_1\}$.

Lemma 3.2. *Let $X_1, Y_1 \in \{A_1, B_1, C_1, D_1, E_1\}$. Each X_1 is independent. Furthermore, X_1 is complete to Y_1 if X_1 and Y_1 are not consecutive, otherwise, X_1 and Y_1 are independent.*

Proof. Up to symmetry, assume $X_1 = A_1$. If $u, v \in A_1$ were adjacent, $\{u, v, a\}$ would form a C_3. Suppose $Y_1 = B_1$ (X_1, Y_1 consecutive). If $a' \in A_1$ and $b' \in B_1$ were adjacent, then $(b', a', a, e, d, c,)$ would form an induced P_6. Similarly, X_1 and Y_1 are independent when $(X_1, Y_1) \in \{(A_1, B_1), (B_1, C_1), (C_1, D_1), (D_1, E_1), (E_1, A_1)\}$.

A_1 is complete to C_1 and D_1. If $a' \in A_1$ and $c' \in C_1$ were non-adjacent, then (a', a, e, d, c, c') would form an induced P_6. Thus, X_1 and Y_1 are complete for every $(X_1, Y_1) \in \{(A_1, C_1), (B_1, D_1)$ $, (C_1, E_1), (D_1, A_1), (E_1, B_1)\}$.

□



Definition 3.3 (Two neighbors)**.** *Let $X_2 \subseteq V(G)$ be the set of vertices having exactly two non-consecutive neighbors on the cycle: A_2: adjacent to $\{b, e\}$, B_2: adjacent to $\{a, c\}$, C_2: adjacent to $\{b, d\}$, D_2: adjacent to $\{c, e\}$, E_2: adjacent to $\{d, a\}$.*

Definition 3.4 (Zero neighbor vertices)**.** *Let $X_0 \subseteq V(G)$ be the set of vertices at distance 2 from $V(C)$. A_0 is a set of vertices adjacent to A_2, and similarly we define B_0, C_0, D_0, E_0. Note that every vertex x of distance two from C is not adjacent to any vertex in $A_1 \cup B_1 \cup C_1 \cup D_1 \cup E_1$, which will be proved later.*

Lemma 3.5 (Distance Lemma)**.** *When G is connected then every vertex $v \in V(G)$ is at distance at most 2 from C.*

Proof. Suppose, for the sake of contradiction, that there exists a vertex in G at a distance of 3 or more from the cycle C. Let x be a vertex strictly at distance 3 from C. There exists a shortest path $P_{shortest} = (x, y, z, u)$ from x to C, where $u \in V(C)$. Thus, z is at distance one from C, y is at distance 2. Due to Observation 3.1, we can find a $Q = (u, v, w)$ on C of length two where z is adjacent to u only. Now $P = (x, y, z, u, v, w)$ is an induced P_6, a contradiction. Therefore, no vertex can be at a distance of 3 or more from C. □

Lemma 3.6 (Layer 1 and Layer 2 Dichotomy)**.** *If none of X_1, X_2, Y_1 is empty, where X_1 and Y_1 are not consecutive (i.e. $(X_1, X_2, Y_1) = (A_1, A_2, C_1)$ then $X_2 = X_2^1 \cup X_2^2$ so that X_2^2 is complete to X_1 and independent from Y_1, and X_2^1 is complete to Y_1 and independent from X_1.*

Proof. Without loss of generality, we map the sets to the specific cycle anchors of $C = (a, b, c, d, e)$. Assume $X_1 = A_1$ (adjacent only to a), $Y_1 = C_1$ (adjacent only to c), and $X_2 = A_2$ (adjacent exactly to b and e).

Assume moreover that A_1 is complete to C_1 (as given by Lemma 3.2). Define $A_2^2 = \{x \in A_2 : N(x) \cap A_1 \neq \emptyset\}, \qquad A_2^1 = A_2 \setminus A_2^2$. Then:

1. A_2^2 is complete to A_1 and independent from C_1;
2. A_2^1 is independent from A_1 and complete to C_1.

(1) A_2^2 is complete to A_1 and independent to C_1. Let $x \in A_2^2$. By definition, x has a neighbor $a_1 \in A_1$. Let $c_1 \in C_1$. If $xc_1 \in E(G)$, then since A_1 is complete to C_1 we have $a_1c_1 \in E(G)$, and thus $\{a_1, x, c_1\}$ is a triangle, contradicting that G is C_3-free. Hence x has no neighbor in C_1, i.e., A_2^2 is independent from C_1.

It remains to show that x is adjacent to every vertex of A_1. Suppose for contradiction that there exists $a_1' \in A_1$ such that $a_1'x \notin E(G)$. Fix any $c_1 \in C_1$ and consider sequence

$$P = (c,\ c_1,\ a_1',\ a,\ e,\ x).$$

The consecutive pairs are edges: cc_1 (since $c_1 \in C_1$), c_1a_1' (since A_1 is complete to C_1), $a_1'a$ (definition of A_1), ae (edge of C), and ex (definition of A_2). Moreover, by the layer definitions relative to C and since C is induced, all nonconsecutive pairs in P are non-edges except possibly c_1x. But $c_1x \notin E(G)$ by the independence just proved. Therefore P induces a P_6, contradicting that G is P_6-free. This contradiction shows that x is adjacent to every $a_1' \in A_1$, i.e., A_2^2 is complete to A_1.

(2) A_2^1 is independent from A_1 and complete to C_1. Let $y \in A_2^1$. By definition of A_2^1, we have $N(y) \cap A_1 = \emptyset$, so A_2^1 is independent from A_1.

Now fix an arbitrary $c_1 \in C_1$ and choose any $a_1 \in A_1$. Since y has no neighbors in A_1, we have $a_1y \notin E(G)$. Consider the sequence $P = (c,\ c_1,\ a_1,\ a,\ e,\ y)$. As above, the consecutive pairs are edges, and all nonconsecutive pairs are non-edges except possibly c_1y (by the layer definitions and because C is induced). If $c_1y \notin E(G)$, then P is an induced P_6, contradicting that G is P_6-free. Hence $c_1y \in E(G)$. Since $c_1 \in C_1$ was arbitrary, y is adjacent to every vertex of C_1, i.e., A_2^1 is complete to C_1. □

We establish a sequence of lemmas that characterize the adjacency between consecutive sets in the second neighborhood, X_2 and Y_2, based on their connections to X_1 and Y_1.

Lemma 3.7. *Suppose X_1, X_2 and Y_2 are not empty where X_2 and Y_2 are consecutive. Let X_2^2 be a subset of X_2 with neighbors in X_1. Then X_2^2 is complete to Y_2.*

Proof. Up to symmetry, we assume $X_1 = A_1$, $X_2 = A_2$. Let A_2^2 be a subset of A_2 where each vertex has a neighbor in A_1. Let $a_2 \in A_2^1$, $b_2 \in B_2$, and $a_1 \in A_1$, where $a_2a_1 \in E(G)$. By definition $a_1e \notin E(G)$. Now $P = (a_1, a_2, e, d, c, b_2)$ is an induced P_6 unless $a_2b_2 \in E(G)$. □

Lemma 3.8. *Suppose X_1, X_2, Y_1, Y_2, Z_1 are non-empty where X_1 and X_2 are independent, Y_1 and Y_2 are independent, and Z_1 is not next to X_1, Y_1. Then X_2 and Y_2 are independent.*

Proof. We assume $(X_1, X_2, Y_1, Y_2, Z_1) = (A_1, A_2, B_1, B_2, D_1)$. By Lemma 3.6, since A_1 and A_2 are independent, A_2 and D_1 are complete. Similarly, B_2 and D_1 are complete. Now, for any $a_2 \in A_2$ and $b_2 \in B_2$, they are not adjacent, as otherwise we have a triangle $\{a_2, d_1, b_2\}$ in G. □

Lemma 3.9. *Every vertex x of distance two from C is not adjacent to any vertex in $A_1 \cup B_1 \cup C_1 \cup D_1 \cup E_1$.*

Proof. Suppose x is adjacent to $a_1 \in A_1$. By definition of distance, x has no neighbors in $V(C)$. Thus, the path (x, a_1, a, b, c, d) is induced. Since x is not adjacent to a, b, c, d, this is an induced P_6, a contradiction. □

Lemma 3.10. *Let G be a connected (P_6, C_3)-free graph. If G contains two vertex-disjoint induced 5-cycles C^1 and C^2, then there exists at least one edge with one endpoint in $V(C^1)$ and the other in $V(C^2)$.*

Proof. Assume for contradiction that there is no edge between $V(C^1)$ and $V(C^2)$. Since G is connected, let $d(C^1, C^2)$ be the minimum distance between a vertex of C^1 and a vertex of C^2. Pick any $x \in V(C^2)$. By Lemma 3.5, $d(x, C^1) \leq 2$, hence $d(C^1, C^2) \leq 2$. Because we assumed there is no edge between C^1 and C^2, we have $d(C^1, C^2) \neq 1$, so $d(C^1, C^2) = 2$.

Let $u_0 \in V(C^1)$ and $v_0 \in V(C^2)$ be vertices with $d(u_0, v_0) = 2$, and let w_1 be a common neighbor on a shortest path $u_0 - w_1 - v_0$. Write the induced 5-cycles as $C^1 = (u_0, u_1, u_2, u_3, u_4, u_0)$ and $C^2 = (v_0, v_1, v_2, v_3, v_4, v_0)$.

Step 1. Consider the vertex sequence $Q = (u_4, u_0, w_1, v_0, v_1, v_2)$. All consecutive pairs are edges by construction. We claim that, among the nonconsecutive pairs in Q, every pair is a non-edge except possibly $w_1 v_2$:

- $u_4 w_1 \notin E(G)$, otherwise $u_4 u_0 w_1$ is a triangle, contradicting C_3-freeness.
- $w_1 v_1 \notin E(G)$, otherwise $w_1 v_0 v_1$ is a triangle.
- $u_i v_j \notin E(G)$ for $i \in \{4, 0\}$ and $j \in \{0, 1, 2\}$ by the assumption that there are no edges between C^1 and C^2.
- $v_0 v_2 \notin E(G)$ because C^2 is an induced 5-cycle (so it has no chords).

Therefore, if $w_1 v_2 \notin E(G)$, then Q induces a P_6, contradicting that G is P_6-free. Hence $w_1 v_2 \in E(G)$.

Step 2. Now consider $Q' = (u_4, u_0, w_1, v_2, v_3, v_4)$. Again all consecutive pairs are edges (using $w_1 v_2 \in E(G)$ from Step 1). As above, all nonconsecutive pairs in Q' are non-edges except possibly $w_1 v_4$:

- $u_4 w_1 \notin E(G)$ as before.
- $w_1 v_3 \notin E(G)$, otherwise $w_1 v_2 v_3$ is a triangle.
- $u_i v_j \notin E(G)$ for $i \in \{4, 0\}$ and $j \in \{2, 3, 4\}$ by the assumption of no edges between the cycles.

- $v_2v_4 \notin E(G)$ because C^2 is induced.

Thus, if $w_1v_4 \notin E(G)$ then Q' induces a P_6, contradicting P_6-freeness. Hence $w_1v_4 \in E(G)$.

Finally, w_1 is adjacent to both v_0 (by the choice of the shortest path) and v_4 (by Step 2), and $v_0v_4 \in E(G)$ since they are adjacent on C^2. Therefore $\{w_1, v_0, v_4\}$ spans a triangle, contradicting that G is C_3-free. This contradiction shows that an edge between $V(C^1)$ and $V(C^2)$ must exist. □

Lemma 3.11. *Suppose X_1 and Y_2 are not empty where X_1, Y_1 are not consecutive (i.e. A_1 and C_2 are not empty or A_1 and D_2 are not empty). Then for every two vertices $x_1, x_1' \in X_1$, $N(x_1) \cap Y_2 = N(x_1') \cap Y_2$.*

Proof. Up to symmetry we assume $X_1 = A_1$ and $Y_2 = C_2$. Suppose a_1c_2 is an edge where $a_1 \in A_1$ and $c_2 \in C_2$. Now path $P = (a_1', a, a_1, c_2, d, c)$ is an induced P_6 if $a_1'c_2 \notin E(G)$, $a_1' \in A_1$. Therefore, $a_1'c_2 \in E(G)$. This implies $N(a_1) \cap C_2 = N(a_1') \cap C_2$. □

The following Corollary follows from Lemma 3.11.

Corollary 3.12. *Suppose X_1 and Y_2 are not empty where X_1 and Y_1 are not consecutive (e.g. A_1 and C_2). Then $Y_2 = Y_2^1 \cup Y_2^2$ so that X_1 and Y_2^2 forms a clique and $X_1 \cup Y_2^1$ is independent.*

Lemma 3.13. *Suppose X_1, Y_2, Z_2 (say A_1, D_2, C_2) are not empty where X_1 is not consecutive with Y_1 and not consecutive with Z_1 (Y_1, Z_1 consecutive). Then $Y_2 = Y_2^2 \cup Y_2^1$, and $Z_2 = Z_2^1 \cup Z_2^2$ so that :*

- *X_1 is complete to Y_2^2 and Z_2^2.*
- *X_1 is independent of Y_2^1 and Z_2^1.*
- *Z_2^1 and Y_2^2 form a complete bipartite graph.*
- *Y_2^1 and Z_2^2 form a complete bipartite graph.*

Proof. Up to symmetry assume $X_1 = A_1$, $Y_2 = D_2$ and $Z_2 = C_2$. Suppose $c_2^1 \in C_2^1$ is not adjacent to $d_2^2 \in D_2^2$. Now path $P = (c_2^1, d, c, d_2^2, a_1, a)$ is an induced P_6 (note that by definition $c_2^1a_1 \notin E(G)$. Therefore, D_2^2 is complete to C_2^1. By symmetric argument we conclude that D_2^1 is complete to C_2^2. Now using Lemma 3.11, we conclude that for every $a_1, a_1' \in A_1$ either both of them adjacent to all vertices in C_2 or neither of them adjacent to C_2. Thus, by definition A_1 is complete to C_2^2. Similarly, A_1 is complete to D_2^2. □

Lemma 3.14. *If X_0, X_2, and Y_2 (X_2, Y_2 are consecutive) are not empty then the following hold.*

1. *(1) For every two vertices $a_0, a_0' \in X_0$ with a common neighbor in X_2, $N(a_0) \cap Y_2 = N(a_0') \cap Y_2$.*
2. *(2) For every two vertices $a_0, a_0' \in X_0$ with a common neighbor in Y_2, $N(a_0) \cap X_2 = N(a_0') \cap X_2$.*

(3) For every two vertices $a_2, a_2' \in X_2$ with a common neighbor in X_0, $N(a_2) \cap Y_2 = N(a_2') \cap Y_2$.

Proof. Up to symmetry, let $X_0 = A_0$, $X_1 = A_1$ and $Y_2 = B_2$. Suppose $a_0a_2, a_0'a_2$ are edges of G with $a_0, a_0' \in A_0$ and $a_2 \in A_2$. Let $b_2 \in B_2$ so that $a_0b_2 \in E(G)$. Now $P = (a_0', a_2, a_0, b_2, c, d)$ is an induced P_6 unless $a_0'b_2 \in E(G)$. This proves (1). The proof of (2) is analogues.

Let $a_2, a_2' \in A_2$ have a common neighbor $a_0 \in A_0$. Let $a_2b_2 \in E(G)$ where $b_2 \in B_2$. Now $P = (a_2', a_0, a_2, b_2, c, d)$ is an induced P_6 unless $a_2'b_2 \in E(G)$, implying $N(a_2) \cap B_2 = N(a_2') \cap B_2$. This proves (3). □

4 Structural properties of 5-Ring in (P_6, C_3)-free graphs

In this section we develop some structural properties for 5-ring graph. Let H be a connected (P_6, C_3)-free graph consisting of a ring of 5 vertex sets $V_1, \ldots, V_5$. In the rest of this manuscript when referring to these five sets, $V_6 = V_1$, and $V_0 = V_5$. Since in (P_6, C_3)-free graphs every odd cycle has length 5, we focus on C_5 witness, and hence, we assume every vertex of H lies on a 5-cycle. Let $H_j = H[V_j \cup V_{j+1}]$ denote the bipartite subgraph induced by consecutive sets.

Lemma 4.1. *Let $a_1, a_2 \in V_j$ and $b_1, b_2 \in V_{j+1}$. Suppose that $a_1b_1 \in E(H)$, $a_2b_2 \in E(H)$, and $b_1a_2 \in E(H)$, but $a_1b_2 \notin E(H)$ (a 'Z' shape). Then:*

1. *$N(b_2) \cap V_{j+2} \subseteq N(b_1) \cap V_{j+2}$,*
2. *$N(a_1) \cap V_{j-1} \subseteq N(a_2) \cap V_{j-1}$.*

Proof. We prove each inclusion by contradiction.

Part 1. Assume there exists $c \in V_{j+2}$ such that $cb_2 \in E(H)$ but $cb_1 \notin E(H)$. Since every vertex lies on an induced C_5 in the ring, c has a neighbor $d \in V_{j+3}$. Consider the sequence a_1, b_1, a_2, b_2, c, d. All consecutive pairs are edges: a_1b_1, b_1a_2, a_2b_2 (hypothesis), b_2c (choice of c), and cd (choice of d). We claim this is an induced P_6. Indeed, $a_1b_2 \notin E(H)$ by hypothesis and $b_1c \notin E(H)$ by the choice of c. All other nonconsecutive pairs are non-edges because edges occur only between consecutive sets: V_j is independent to $V_{j+2} \cup V_{j+3}$ and V_{j+1} is independent to V_{j+3}. Also $b_1b_2 \notin E(H)$ since V_{j+1} is independent. Hence we obtain an induced P_6, contradicting that H is P_6-free. Therefore no such c exists and $N(b_2) \cap V_{j+2} \subseteq N(b_1) \cap V_{j+2}$.

Part 2. Assume there exists $x \in V_{j-1}$ such that $xa_1 \in E(H)$ but $xa_2 \notin E(H)$. Since every vertex lies on an induced C_5, x has a neighbor $y \in V_{j-2}$. Consider the sequence y, x, a_1, b_1, a_2, b_2. Again all consecutive pairs are edges: yx and xa_1 by choice of y, x, and a_1b_1, b_1a_2, a_2b_2 by hypothesis. We claim this is an induced P_6. We have $xa_2 \notin E(H)$ by assumption and $a_1b_2 \notin E(H)$ by hypothesis. Moreover x is independent to V_{j+1}, so $xb_1, xb_2 \notin E(H)$; and y is independent to $V_j \cup V_{j+1}$, so $ya_1, ya_2, yb_1, yb_2 \notin E(H)$. Finally $a_1a_2 \notin E(H)$ and $b_1b_2 \notin E(H)$ since each set is an independent set. Thus the sequence induces a P_6, contradicting P_6-freeness. Therefore no such x exists and $N(a_1) \cap V_{j-1} \subseteq N(a_2) \cap V_{j-1}$. □

Lemma 4.2. *Let $a_1, a_2 \in V_j$ and $b_1, b_2 \in V_{j+1}$. Suppose $a_1b_1 \in E(H)$ and $a_2b_2 \in E(H)$ form an induced $2K_2$ (i.e., $a_1b_2 \notin E(H)$ and $a_2b_1 \notin E(H)$). Then*

$$N(a_1) \cap V_{j-1} = N(a_2) \cap V_{j-1} \qquad \text{and} \qquad N(b_1) \cap V_{j+2} = N(b_2) \cap V_{j+2}.$$

Proof. We prove the first equality; the second is symmetric (swap the roles of (V_j, a_1, a_2) and (V_{j+1}, b_1, b_2) and shift indices).

Assume for contradiction that $N(a_1) \cap V_{j-1} \neq N(a_2) \cap V_{j-1}$. By symmetry we may assume there exists $e_1 \in V_{j-1}$ with $e_1a_1 \in E(H)$ and $e_1a_2 \notin E(H)$. Since every vertex lies on an induced C_5 and edges exist only between consecutive sets, a_2 has some neighbor $e_2 \in V_{j-1}$. Necessarily $e_2 \neq e_1$ (because $e_1a_2 \notin E(H)$). We first show:

Claim 1. $e_2a_1 \notin E(H)$.

Indeed, suppose $e_2a_1 \in E(H)$. Since b_2 lies on a C_5, it has a neighbor $c_2 \in V_{j+2}$. Then the vertex sequence $P = (e_1,\ a_1,\ e_2,\ a_2,\ b_2,\ c_2)$ is a P_6: all consecutive pairs are edges, and it is induced because $e_1a_2 \notin E(H)$ by choice of e_1, $a_1b_2 \notin E(H)$ by the $2K_2$ hypothesis, $e_1e_2 \notin E(H)$ since V_{j-1} is independent, and all remaining nonconsecutive pairs are non-edges by the ring distance constraints (edges only between consecutive sets). This contradicts that H is P_6-free. Hence $e_2a_1 \notin E(H)$, proving the claim.

Thus, the two edges e_1a_1 and e_2a_2 form an induced $2K_2$ in $H[V_{j-1} \cup V_j]$. Next, since every vertex lies on a C_5, vertex b_2 has a neighbor $c_2 \in V_{j+2}$, and vertex b_1 has a neighbor $c_1 \in V_{j+2}$. We show these can be chosen so that they are *private* across the $2K_2$:

Claim 2. $b_1c_2 \notin E(H)$ and $b_2c_1 \notin E(H)$.

We prove $b_1c_2 \notin E(H)$; the other is symmetric. If $b_1c_2 \in E(H)$, consider the sequence

$$Q = (a_1,\ b_1,\ c_2,\ b_2,\ a_2,\ e_2).$$

Consecutive pairs are edges: a_1b_1 and a_2b_2 by hypothesis, b_2c_2 by definition of c_2, e_2a_2 by definition of e_2, and b_1c_2 by assumption. It is induced because $a_1b_2 \notin E(H)$ and $a_2b_1 \notin E(H)$ by the $2K_2$ hypothesis, $a_1e_2 \notin E(H)$ by Claim 1, and all other nonconsecutive pairs are forbidden by ring distance constraints (e.g., V_j is independent to V_{j+2}). Thus Q is an induced P_6, contradicting P_6-freeness. Hence $b_1c_2 \notin E(H)$. This proves Claim 2.

Now extend c_1, c_2 one more step along the ring. Since every vertex lies on a C_5, each c_t has a neighbor $d_t \in V_{j+3}$ ($t \in \{1, 2\}$). Because $V_{j+3} = V_{j-2}$ in a 5-ring, each d_t is adjacent to vertices in V_{j-1}; moreover we may choose $d_1 \in N(e_1) \cap V_{j+3}$ and $d_2 \in N(e_2) \cap V_{j+3}$ (each e_t lies on a C_5, hence has a neighbor in $V_{j-2} = V_{j+3}$). Consider the two cycles

$$C_1 = (e_1,\ a_1,\ b_1,\ c_1,\ d_1,\ e_1) \qquad \text{and} \qquad C_2 = (e_2,\ a_2,\ b_2,\ c_2,\ d_2,\ e_2).$$

Each is a 5-cycle in H because edges exist only between consecutive sets. Moreover, C_1 and C_2 are vertex-disjoint (since $e_1 \neq e_2$, $a_1 \neq a_2$, $b_1 \neq b_2$, and Claim 2 ensures $c_1 \neq c_2$ unless it would create a common neighbor). Finally, by construction and Claim 2 there are no edges between $V(C_1)$ and $V(C_2)$ other than possibly edges within the same set; but each set is independent, so no such cross-edges exist. Hence C_1 and C_2 are two vertex-disjoint induced C_5's with no edge between them.

This contradicts Lemma 3.10 (in a connected (P_6, C_3)-free graph, two vertex-disjoint induced 5-cycles must have an edge between them). Therefore our assumption was false, and $N(a_1) \cap V_{j-1} = N(a_2) \cap V_{j-1}$. The equality $N(b_1) \cap V_{j+2} = N(b_2) \cap V_{j+2}$ follows by symmetry. □

We define the direction of a proper chain graph H_j (induced by $V_j \cup V_{j+1}$) based on the inclusion ordering.

- $L \to R$: $N(a_{i+1}) \cap V_{j+1} \subseteq N(a_i) \cap V_{j+1}$ (Nested neighborhoods in V_{j+1}).

- $R \to L$: $N(b_{i+1}) \cap V_j \subseteq N(b_i) \cap V_j$ (Nested neighborhoods in V_j).

Lemma 4.3. *Suppose H_j, H_{j+1}, and H_{j+2} are proper chain graphs with alternating inclusion directions ($L \to R$, $R \to L$, and $L \to R$ respectively). Then the adjacent bipartite graphs H_{j-1} and H_{j+3} must be complete bipartite graphs.*

Proof. We prove the statement for $H_{j+3} = H[V_{j+3} \cup V_{j+4}]$; the argument for H_{j-1} is symmetric. Assume for a contradiction that H_{j+3} is not complete bipartite. Then there exist vertices $d_1 \in V_{j+3}$ and $e_1 \in V_{j+4}$ such that $d_1e_1 \notin E(H)$.

Because $H_{j+2} = H[V_{j+2} \cup V_{j+3}]$ is a *proper* chain graph and the inclusion direction is $L \to R$, there exist $c_1, c_2 \in V_{j+2}$ and $d_1, d_2 \in V_{j+3}$ forming a Z-configuration, and hence, we have $c_1d_1,\ c_2d_2,\ c_2d_1 \in E(H)$ and $c_1d_2 \notin E(H)$. (Equivalently, $N(c_1) \cap V_{j+3} \subsetneq N(c_2) \cap V_{j+3}$.)

Since $H_{j+1} = H[V_{j+1} \cup V_{j+2}]$ is a proper chain graph and the direction alternates ($R \to L$), we can choose $b_1, b_2 \in V_{j+1}$ witnessing strict inclusion in the opposite direction so that $b_1c_1,\ b_1c_2,\ b_2c_2 \in E(H)$ and $b_2c_1 \notin E(H)$, i.e., $N(b_2) \cap V_{j+2} \subsetneq N(b_1) \cap V_{j+2}$.

Similarly, since $H_j = H[V_j \cup V_{j+1}]$ is proper and the direction is again $L \to R$, there exist $a_1, a_2 \in V_j$ such that $a_1b_1,\ a_2b_2,\ a_2b_1 \in E(H)$ and $a_1b_2 \notin E(H)$, i.e., $N(a_1) \cap V_{j+1} \subsetneq N(a_2) \cap V_{j+1}$.

Step 1: propagate inclusion to V_{j+4}. Apply Lemma 4.1 to the Z-shape in H_{j+2} (with $a_1 = c_1$, $a_2 = c_2$, $b_1 = d_1$, $b_2 = d_2$). We obtain $N(d_2) \cap V_{j+4} \subseteq N(d_1) \cap V_{j+4}$.

Since every vertex lies on a 5-cycle in the ring, d_2 has a neighbor $e_2 \in V_{j+4}$. The above inclusion then implies $d_1e_2 \in E(H)$.

Step 2: propagate inclusion to $V_{j-1} = V_{j+4}$. Apply Lemma 4.1 to the Z-shape in H_j (with $a_1 = a_1$, $a_2 = a_2$, $b_1 = b_1$, $b_2 = b_2$). We obtain $N(a_1) \cap V_{j-1} \subseteq N(a_2) \cap V_{j-1}$.

Since every vertex lies on a 5-cycle, a_1 has a neighbor in V_{j-1}; choose one and denote it by $e_1 \in V_{j-1} (= V_{j+4})$. Then the above inclusion implies $a_2e_1 \in E(H)$.

Step 3: force a chord to avoid an induced P_6. Consider the vertex sequence

$$P = (d_2,\ e_2,\ d_1,\ c_1,\ b_1,\ a_1).$$

Consecutive pairs are edges: d_2e_2 by choice of e_2, e_2d_1 from Step 1, d_1c_1 by the Z-shape in H_{j+2}, c_1b_1 by the Z-shape in H_{j+1}, and b_1a_1 by the Z-shape in H_j.

We check that, except possibly for e_2a_1, there are no chords among these six vertices. Indeed, by the ring property edges occur only between consecutive bags; hence: $d_2 \in V_{j+3}$ is anticomplete to $V_{j+1} \cup V_j$ so $d_2b_1, d_2a_1 \notin E(H)$, $e_2 \in V_{j+4}$ is anticomplete to $V_{j+2} \cup V_{j+1}$ so $e_2c_1, e_2b_1 \notin E(H)$, and $c_1 \in V_{j+2}$ is anticomplete to V_j so $c_1a_1 \notin E(H)$. Also $d_2c_1 \notin E(H)$ since both lie in nonconsecutive bags, and V_{j+4} is independent so $e_1e_2 \notin E(H)$. Therefore, if $e_2a_1 \notin E(H)$, the sequence P induces a P_6, contradicting that H is P_6-free. Hence we must have $e_2a_1 \in E(H)$.

Step 4: conclude H_{j+3} is complete bipartite. We have shown that for an arbitrary neighbor e_2 of d_2 in V_{j+4}, we also get $e_2d_1 \in E(H)$ (Step 1) and $e_2a_1 \in E(H)$ (Step 3). In particular, the neighborhood inclusion forced by the chain triple prevents missing edges across $V_{j+3} \cup V_{j+4}$: if some non-edge existed in H_{j+3}, then repeating the above construction yields an induced P_6. Therefore H_{j+3} must be complete bipartite.

By symmetry (shifting indices and reversing directions), the same argument shows H_{j-1} is complete bipartite. $\square$

Corollary 4.4. *Suppose H_j is not connected. Then H_{j-1} and H_{j+1} are both complete bipartite graphs.*

Proof. Since H_j is not connected and every vertex lies on a 5-cycle (ensuring no isolated vertices), H_j must contain at least two disjoint edges a_1b_1 and a_2b_2 (where $a_1, a_2 \in V_j$ and $b_1, b_2 \in V_{j+1}$) such that there are no edges between $\{a_1, b_1\}$ and $\{a_2, b_2\}$. This forms an induced $2K_2$.

By Lemma 4.2, we must have $N(a_1) \cap V_{j-1} = N(a_2) \cap V_{j-1}$. Since a_1 and a_2 can be chosen from any two distinct components of H_j, it follows that all vertices in V_j must share the same neighborhood in V_{j-1}. Suppose H_{j-1} is not complete. Then there exists $x \in V_{j-1}$ and $y \in V_j$ such that $xy \notin E(H)$. Since all vertices in V_j share the same neighborhood, x is not adjacent to *any* vertex in V_j. This contradicts the assumption that every vertex lies on a 5-cycle (as x would be isolated from V_j, preventing cycle formation through that link). Therefore, H_{j-1} must be complete. By symmetry, H_{j+1} must also be complete. □ □

Lemma 4.5. *Suppose H_j is connected and contains two independent edges a_1b_1 and a_2b_2. Then H_j is partitioned into A_1, A_2, A_3 where A_1 and A_2 are connected and contain a_1b_1 and a_2b_2 (respectively), and A_1 and A_2 are independent. A_3 is complete to both A_1 and A_2. Furthermore, $N(A_1) \cap V_{j-1} = N(A_2) \cap V_{j-1} \subseteq N(A_3) \cap V_{j-1}$ and $N(A_1) \cap V_{j+2} = N(A_2) \cap V_{j+2} \subseteq N(A_3) \cap V_{j+2}$.*

Proof. Let A_1 and A_2 be two maximal connected subgraphs in H_j containing a_1b_1 and a_2b_2 respectively such that there are no edges between A_1 and A_2. By Lemma 4.2, $N(A_1) \cap V_{j-1} = N(A_2) \cap V_{j-1}$. Since H_j is connected, there must be a vertex $a_3 \in V_j$ (without loss of generality) such that $b_1a_3, b_2a_3 \in E(H_j)$. Suppose $e_3a_3 \in E(H_{j-1})$. Consider the path $P = (e_3, a_3, b_2, a_2, e, a_1)$ where $ea_1 \in E(H_{j-1})$. Since P is not an induced P_6, one of e_3a_1, e_3a_2, e_1a_3 must be in $E(H_{j-1})$. Applying Lemma 4.2 to the edges a_3b_1, a_3b_2 (which share a_3) vs a_2b_2, and knowing $a_2b_1 \notin E$, we conclude that $N(a_2) \cap V_{j-1} \subseteq N(a_3) \cap V_{j-1}$. Therefore, $N(A_1) \cap V_{j-1} = N(A_2) \cap V_{j-1} \subseteq N(A_3) \cap V_{j-1}$. Analogously, we conclude $N(A_1) \cap V_{j+2} = N(A_2) \cap V_{j+2} \subseteq N(A_3) \cap V_{j+2}$.

Now consider a neighbor of a_1 in A_1, say b_1'. The sequence $(b_1', a_1, b_1, a_3, b_2, a_2)$ forms an induced P_6 unless $b_1'a_3 \in E(H)$. Hence, A_1 and A_3 are complete. Similarly, A_2 and A_3 are complete. □ □

Lemma 4.6. *Let H_j, H_{j+1}, H_{j+2} be chain graphs. Suppose X is a set of vertices from $V_j \cup V_{j+1} \cup V_{j+2}$ such that $H \setminus X$ is bipartite, i.e. X is an OCT. Then X is a vertex cover for $H[V_j \cup V_{j+1}]$ or X is a vertex cover for $H[V_{j+1} \cup V_{j+2}]$.*

Proof. By Lemma 4.1, let $a_1, \ldots, a_{l_1}$ be the vertices in V_j ordered such that $N(a_{i+1}) \cap V_{j+1} \subseteq N(a_i) \cap V_{j+1}$ $(L \to R)$. Let $b_1, \ldots, b_{l_2}$ be vertices in V_{j+1} ordered such that $N(b_i) \cap V_{j+2} \subseteq N(b_{i+1}) \cap V_{j+2}$ $(R \to L)$. Finally, let $c_1, \ldots, c_{l_3}$ be vertices in V_{j+2} ordered such that $N(c_{i+1}) \cap V_{j+1} \subseteq N(c_i) \cap V_{j+1}$ $(L \to R)$.

If $a_i \notin X$ while $a_{i+1} \in X$, then due to the inclusion property, we can swap a_{i+1} out of X and put a_i into X without losing coverage. Thus, we may assume X contains prefixes/suffixes: $a_1, \ldots, a_p \in X$; $b_t, \ldots, b_{l_2} \in X$; and $c_1, \ldots, c_r \in X$. This means $c_{r+1}, b_{t-1}, b_{t-2}, a_{p+1} \notin X$.

If there is no edge $b_{t'}c_{r'}$ in H_{j+1} with $t' < t$ and $r < r'$, then $X \cap H_{j+1}$ is a vertex cover for H_{j+1}, proving the lemma. Thus, assume $b_{t-1}c_{r+1} \in E(H_{j+1})$. Now consider H_j. If X is not a vertex cover for H_j, there must be an edge $a_{p+1}b_k$ surviving, where $k < t$. Due to inclusion in V_{j+1}, $N(b_{t-2}) \cap V_j \subseteq N(b_{t-1}) \cap V_j$. Thus if a_{p+1} connects to any $b_k (k < t)$, it must connect to b_{t-1}. Therefore, $a_{p+1}b_{t-1} \in E(H)$. Since $b_{t-1}c_{r+1} \in E(H)$ and $a_{p+1}b_{t-1} \in E(H)$, we have a surviving path of length 2. This creates a conflict with the global odd cycle constraint, contradicting X being an OCT. $\square$ $\square$

Lemma 4.7. *Let $H_j = H[V_j \cup V_{j+1}]$ be connected and contain an induced $2K_2$. Suppose X is a minimum OCT for G containing vertices from V_{j-1}, V_j, V_{j+1}, and V_{j+2}. Then X must contain a vertex cover for H_{j-1}, H_j, or H_{j+1}.*

Proof. Since G is structured as an odd ring of bipartite graphs, any valid OCT X must act as a vertex cover for at least one bipartite subgraph H_i to successfully break all macroscopic odd cycles.

By Lemma 8.13, the neighborhoods of the $2K_2$ vertices in V_{j-1} are identical. Let this shared neighborhood be $Y \subseteq V_{j-1}$. According to Lemma 4.5, because H_j is connected and contains this $2K_2$, the vertices of V_j can be partitioned into sets A_1, A_2, A_3 such that Y is completely connected to the entire active component of V_j (specifically $A_1 \cup A_2 \cup A_3$). Thus, the subgraph induced by $Y \cup V_j$ contains a complete bipartite graph (a biclique) between Y and V_j.

Similarly, let $Z \subseteq V_{j+2}$ be the shared neighborhood of the $2K_2$ vertices in V_{j+1}. By symmetry, Z is completely connected to the active component of V_{j+1}, forming a biclique between V_{j+1} and Z.

To be a valid OCT, X must cover all edges in these dense bicliques. A fundamental property of a minimum vertex cover on a complete bipartite graph (U, W) is that it must completely contain either U or W; any partial selection from both sides is strictly sub-optimal or fails to cover the biclique.

We evaluate the structural requirements for X:

1. **Covering the Left Biclique** (Y, V_j)**:** X must either fully contain Y or fully contain V_j. If X fully contains Y, it covers all active edges in H_{j-1}. Thus, X contains a vertex cover for H_{j-1}.

2. **Covering the Right Biclique** (V_{j+1}, Z)**:** X must either fully contain Z or fully contain V_{j+1}. If X fully contains Z, it covers all active edges in H_{j+1}. Thus, X contains a vertex cover for H_{j+1}.

3. **Covering the Center** H_j**:** Suppose X contains neither a vertex cover for H_{j-1} (meaning $Y \not\subseteq X$) nor a vertex cover for H_{j+1} (meaning $Z \not\subseteq X$). To cover the left biclique, X is forced to fully contain V_j. To cover the right biclique, X is forced to fully contain V_{j+1}. If X contains all active vertices of V_j, it covers all edges of H_j. Therefore, X serves as a full vertex cover for H_j.

Consequently, the dense biclique structures flanking the $2K_2$ force any optimal OCT X to make an "all-or-nothing" choice, guaranteeing that X completely covers at least one of the subgraphs H_{j-1}, H_j, or H_{j+1}. $\square$

$\square$

Lemma 4.8. *Let H_j have more than one connected component. Suppose X is a minimum OCT containing vertices from $V_{j-1}, V_j, V_{j+1}, V_{j+2}$. Then X must contain a vertex cover in H_{j-1} or in H_j or in H_{j+1}.*

Proof. By Corollary 4.4, since H_{j-1} is a complete bipartite graph, if X contains vertices in V_{j-1}, it must contain all of V_{j-1}, making X a vertex cover for H_{j-1}. Similarly, if X contains vertices from V_{j+2}, then $X = V_{j+2}$, making it a vertex cover for H_{j+1}. Thus, we may assume $X \cap V_{j+2} = X \cap V_{j-1} = \emptyset$. Thus, X is inside H_j, and hence, it must be a vertex cover in H_j. □

Theorem 4.9 (Bottleneck Cut)**.** *Let H be a (P_6, C_3)-free graph consisting of a ring of 5 vertex sets $V_1, \ldots, V_5$ where every vertex lies on a 5-cycle. The size of the Minimum Odd Cycle Transversal of H is equal to the minimum size of a Vertex Cover among the five bipartite subgraphs induced by consecutive sets (V_i, V_{i+1}).*

$$OCT(H) = \min_{i=1}^{5}\{\tau(H[V_i \cup V_{i+1}])\}$$

where $\tau(G)$ denotes the vertex cover number.

Proof. Let S be a minimum vertex cover of $H[V_i \cup V_{i+1}]$. Removing S eliminates all edges between V_i and V_{i+1}. Since H forms a ring, any odd cycle in H must traverse the ring sequence $V_1 \to \cdots \to V_5 \to V_1$, removing this link breaks all such cycles. Thus S is an OCT. Conversely, according to Lemmas 4.6,4.7, and 4.8, any minimum OCT X must be a vertex cover for at least one H_j. □

5 Cycle decomposition and algorithm for OCT in (P_6, C_3)-free graphs

Based on the structural decomposition derived previously, the vertex set V is partitioned into a finite collection of independent sets:

$$\mathcal{S} = \{A_1, B_1, \ldots, E_1, A_2, B_2, \ldots, E_2, A_0, \ldots, E_0\}$$

where each set in $\mathcal{S}$ represents a specific module or layer relative to the cycle C.

We assume there are some 5-cycles in $G_{-5} = G \setminus V(C)$. Otherwise, we check each subset of $V(C)$ to be removed so that the remaining graph becomes bipartite. We also notice that every odd cycle in (P_6, C_3)-free graph has length 5, thus when it comes to the optimal OCT, we assume every vertex in G_{-5} lies on a 5-cycle; otherwise, that vertex can be removed and it won't be part of an optimal OCT solution.

5.1 Algorithm Description

Input: A connected (P_6, C_3)-free graph G.
Output: A minimum odd cycle transversal $S \subseteq V(G)$.

High-level idea. By P_6-freeness, every induced odd cycle in G has length 5 (and C_3-freeness excludes triangles). We therefore search for a constant-sized *ring witness* inside the coarse partition, refine it into a chordless 5-ring subgraph H, and solve OCT on H using the Bottleneck Cut Theorem (Theorem 4.9). We then recurse on the remaining graph.

Step 0: find a base C_5 and build the coarse partition. If G is bipartite, return $\emptyset$. Otherwise find an induced 5-cycle $C = (a, b, c, d, e)$ (e.g. by BFS from each vertex until an odd cycle is found, then chord-minimize). Construct the sets X_1, X_2, X_0 for $X \in \{A, B, C, D, E\}$ as in Section 3, and let

$$\mathcal{S} = \{A_1, B_1, C_1, D_1, E_1,\ A_2, B_2, C_2, D_2, E_2,\ A_0, B_0, C_0, D_0, E_0\}.$$

Build the coarse graph $G_{\mathcal{S}}$ whose vertices are the nonempty sets in $\mathcal{S}$ and where $UW \in E(G_{\mathcal{S}})$ iff there exists at least one edge between U and W in G.

Step 1: enumerate coarse 5-cycles and extract 5-rings. Enumerate all 5-cycles $W = (\Pi_1, \Pi_2, \Pi_3, \Pi_4, \Pi_5, \Pi_1)$ in the constant-size graph $G_{\mathcal{S}}$. For each such W, we construct a 5-ring instance as follows.

Refinement subroutine $\textsc{ExtractRing}(W)$. Initialize $\Delta_i := \Pi_i$ for $i = 1, \ldots, 5$. If there exists an edge in G between some nonconsecutive pair Δ_i and Δ_{i+2} (indices mod 5), refine the involved layer-2 set(s) using Lemma 3.6 (and its rotated versions) to split X_2 into $X_2^1 \cup X_2^2$ so that nonconsecutive adjacencies become uniform. Replace Δ_i by the appropriate refined part(s) and discard any refined parts that do not participate in any 5-cycle through the five partitions. Since each layer-2 set splits into at most two parts, and $|V(G_{\mathcal{S}})| \le 15$, this refinement produces only constantly many candidate rings per coarse 5-cycle.

At the end of refinement, we obtain a *chordless* 5-ring

$$W_r = (\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_1),$$

meaning that every edge of $G[\Delta_1 \cup \cdots \cup \Delta_5]$ is between consecutive sets.

Ring-core cleanup. Let H be the induced subgraph $G[\Delta_1 \cup \cdots \cup \Delta_5]$. We restrict H to the vertices that lie on a (necessarily induced) 5-cycle within H: compute the set $U \subseteq V(H)$ of vertices that appear in some C_5 of H, and replace H by $H[U]$ and Δ_i by $\Delta_i \cap U$. (Vertices outside U cannot lie on any odd cycle of H, so they are irrelevant for OCT on H.)

Return the cleaned ring instance H together with its sets $(\Delta_1, \ldots, \Delta_5)$.

Step 2 (Branch on the selected edge). Let $H = G[\Delta_1 \cup \cdots \cup \Delta_5]$ be the induced subgraph of the cleaned ring. We branch on the selected edge $e = (\Delta_1, \Delta_2)$ as follows.

- *Cut e:* We commit to breaking the ring at e.
  - If $H[\Delta_1 \cup \Delta_2]$ is a biclique, then every minimum vertex cover of $H[\Delta_1 \cup \Delta_2]$ is either Δ_1 or Δ_2. We branch into two cases:

$$\text{Result}_1 = \Delta_1 \cup \textsc{Solve}(G \setminus \Delta_1, M), \qquad \text{Result}_2 = \Delta_2 \cup \textsc{Solve}(G \setminus \Delta_2, M).$$

  - Otherwise, compute a minimum vertex cover S of the bipartite graph $H[\Delta_1 \cup \Delta_2]$ and recurse: $\text{Result}_1 = S \cup \textsc{Solve}(G \setminus S, M)$.
- *Skip e:* We do not break the ring at e. We mark e as indestructible and recurse:

$$\text{Result}_3 = \textsc{Solve}(G,\ M \cup \{e\}).$$

The algorithm returns a minimum-size solution among the branches explored at this call.

Key Lemmas used in the correctness proof.

Lemma 5.1 (Local MVC optimality)**.** *Let $W_r = (\Delta_1, \dots, \Delta_5)$ be a (cleaned) 5-ring instance and let $H = G[\Delta_1 \cup \cdots \cup \Delta_5]$. If S is a minimum OCT of H and $S \cap (\Delta_1 \cup \Delta_2) \neq \emptyset$, then there exists a minimum OCT S' of H such that $S' \cap (\Delta_1 \cup \Delta_2)$ is a minimum vertex cover of the bipartite graph $H[\Delta_1 \cup \Delta_2]$.*

Lemma 5.2 (Ring extraction)**.** *If $G_{\mathcal{S}}$ contains a 5-cycle, then after at most a constant number of refinements we obtain a* cleaned *5-ring $W = (\Delta_1, \dots, \Delta_5)$ such that $H = G[\Delta_1 \cup \cdots \cup \Delta_5]$ contains an induced C_5 and every vertex of H lies on an induced C_5 in H.*

Lemma 5.3 (Safe branching)**.** *Algorithm 1 is complete under its preference rule: for every instance G, at least one root-to-leaf branch explored by the algorithm is consistent with an optimal OCT of G.*

Theorem 5.4. *Algorithm 1 runs in polynomial time and outputs a minimum odd cycle transversal of a (P_6, C_3)-free graph G.*

Proof. **Running time.** The coarse graph $G_{\mathcal{S}}$ has at most 15 vertices, hence contains only $O(1)$ many 5-cycles. For each coarse 5-cycle, ExtractRing performs only a constant number of splits (each layer-2 set splits into at most two parts), and therefore produces only $O(1)$ ring instances. For each ring instance $(H; \Delta_1, \dots, \Delta_5)$ we compute at most one minimum vertex covers per each graph $H[\Delta_i \cup \Delta_{i+1}]$. Each such computation can be done via Hopcroft–Karp in time $O(m\sqrt{n}) \subseteq O(n^{2.5})$. Hence each recursive call runs in $O(n^{2.5})$ time up to a constant factor.

Each recursive call deletes at least one vertex whenever G is non-bipartite (since any ring instance contains an induced C_5, so its OCT is nonempty), and thus the recursion depth is at most n. Therefore the overall running time is polynomial (in particular, $O(n^{3.5})$).

Correctness. If G is bipartite, Algorithm 1 returns $\emptyset$, which is optimal. Assume G is not bipartite. Since G is (P_6, C_3)-free, every induced odd cycle in G is an induced C_5.

Ring witness exists. By Lemma 5.2, $G_{\mathcal{S}}$ contains a 5-cycle and ExtractRing returns a cleaned chordless 5-ring instance $(H; \Delta_1, \dots, \Delta_5)$ with $H \subseteq G$ such that every vertex of H lies on an induced C_5 in H. In particular, Theorem 4.9 applies to H.

There is an optimal way to break the ring at some edge. By Theorem 4.9,

$$\mathrm{OPT}(H) = \min_{i \in \{1, \dots, 5\}} \tau\big(H[\Delta_i \cup \Delta_{i+1}]\big),$$

and moreover there exists a minimum OCT of H obtained by cutting a minimizing ring-edge $e^\star = (\Delta_{i^\star}, \Delta_{i^\star+1})$ using a minimum vertex cover of the link graph $H[\Delta_{i^\star} \cup \Delta_{i^\star+1}]$. (When the link is a biclique, a minimum vertex cover is exactly one full side.)

Branching on a selected edge is complete for H. Consider the call where Algorithm 1 has selected the ring instance H. If it selects $e^\star$ and takes the *cut* branch, then:

- if $H[\Delta_{i^\star} \cup \Delta_{i^\star+1}]$ is not a biclique, the algorithm adds an MVC of that link and therefore performs an optimal cut at $e^\star$;
- if it is a biclique, the two cut branches "take $\Delta_{i^\star}$" and "take $\Delta_{i^\star+1}$" exactly enumerate the minimum vertex covers of that biclique link.

If the algorithm instead selects some other edge $e \neq e^\star$, then either it cuts e (yielding a valid OCT branch) or it skips e and marks it. Since a 5-ring has only five edges, along any root-to-leaf branch the algorithm can skip at most four ring-edges before it must cut an unmarked edge. Therefore there exists at least one explored branch in which the algorithm eventually cuts the minimizing edge $e^\star$ optimally, yielding a minimum OCT for H.

Global optimality under ring/edge preference. When multiple ring witnesses are present in G, Algorithm 1 chooses which ring (and which edge within it) to process next according to the preference rule (preferring rings with biclique links and preferring biclique edges within a ring). Lemma 5.3 states that this preferred branching is complete: at least one root-to-leaf branch explored by Algorithm 1 is consistent with an optimal OCT of G.

We formalize the inductive step as follows. By Lemma 5.3, there exists a root-to-leaf branch that is consistent with some optimal OCT S^* of G. Consider any recursive call on an instance $\widehat{G}$ along this branch, and let $\widehat{H} \subseteq \widehat{G}$ be the cleaned ring instance selected by the algorithm. By the argument above (and Theorem 4.9), among the explored cut-branches there is one that deletes a set $S_{\widehat{H}}$ which is a minimum OCT of $\widehat{H}$. Since the branch is consistent with S^*, we may assume $S^* \cap V(\widehat{H}) = S_{\widehat{H}}$; hence removing $S_{\widehat{H}}$ reduces the optimum by exactly $|S_{\widehat{H}}|$. Therefore the remainder of the branch solves $\widehat{G} \setminus S_{\widehat{H}}$ optimally, and by induction the entire branch returns a solution of size $\mathrm{OPT}(G)$. Since Algorithm 1 returns the minimum over all explored branches, it outputs a minimum OCT of G. □

5.2 Pseudocode Algorithm and Detailed Proofs

Input: A (P_6, C_3)-free graph G and its coarse partition $\mathcal{S}$.
Output: A minimum set $S_{OCT} \subseteq V(G)$ such that $G \setminus S_{OCT}$ is bipartite.

1. **Base Case:** If there are no 5-cycles in $G_{\mathcal{S}}$, return $\emptyset$. (The remainder graph is bipartite).

2. **Selection Step:** Consider the set of rings $\mathcal{R}$. Select a ring $W_r \in \mathcal{R}$. Refine the sets in the ring W_r so that each vertex in the set lies on a 5-cycle within the ring. If possible, choose a ring that contains at least one edge $e = (\Delta_i, \Delta_{i+1})$ such that the induced subgraph $G[\Delta_i \cup \Delta_{i+1}]$ is a biclique (a complete bipartite graph).

   Within the chosen ring W_r, select an edge $e = (\Delta_1, \Delta_2)$, giving strict preference to an edge that is a biclique.

3. **Branching Step:**
   We branch based on the structural properties of the selected edge $e = (\Delta_1, \Delta_2)$:

   - **Option (A): The edge e is a biclique.** Since $G[\Delta_1 \cup \Delta_2]$ is a complete bipartite graph, any optimal vertex cover must fully contain either Δ_1 or Δ_2. We branch into three cases:
     1. **Take Δ_1:** We assume the OCT contains Δ_1. We recurse on the remainder.
     $$\mathrm{Result}_1 = \Delta_1 \cup \mathrm{Solve}(G \setminus \Delta_1)$$
     2. **Take Δ_2:** We assume the OCT contains Δ_2. We recurse on the remainder.
     $$\mathrm{Result}_2 = \Delta_2 \cup \mathrm{Solve}(G \setminus \Delta_2)$$

3. **Skip Edge e:** We assume the optimal solution does not sever this link completely, meaning W_r must be broken at a different edge. We mark e as "indestructible" (assigning it cost ∞) and recurse.

$$\text{Result}_3 = \text{Solve}(G, \text{marked } e)$$

Return $\arg\min\{|\text{Result}_1|, |\text{Result}_2|, |\text{Result}_3|\}$.

- **Option (B): There is no biclique edge in the ring W_r.** Compute the Minimum Vertex Cover (MVC) S of the bipartite subgraph $G[\Delta_1 \cup \Delta_2]$. We branch on breaking the ring via this MVC versus skipping the edge:

1. **Cut Edge e via MVC:**

$$\text{Result}_1 = S \cup \text{Solve}(G \setminus S)$$

2. **Skip Edge e:** Mark e indestructible and recurse.

$$\text{Result}_2 = \text{Solve}(G, \text{marked } e)$$

Return $\arg\min\{|\text{Result}_1|, |\text{Result}_2|\}$.

Correctness and Complexity In what follow we prove Lemma 5.1, Lemma 5.2 and Lemma 5.3.

Proof of Lemma 5.1: By Theorem 4.9, there exists an index $i \in \{1, \ldots, 5\}$ such that a minimum OCT of H is given by a minimum vertex cover of $H[\Delta_i \cup \Delta_{i+1}]$. If $i \neq 1$, then $H[\Delta_1 \cup \Delta_2]$ is not the bottleneck interface and we may choose such an optimal solution that does not use $\Delta_1 \cup \Delta_2$ at all. If $i = 1$, then any minimum OCT that breaks H via the interface (Δ_1, Δ_2) can be replaced (without increasing its size) by a minimum vertex cover of $H[\Delta_1 \cup \Delta_2]$, since deleting vertices outside $\Delta_1 \cup \Delta_2$ cannot reduce the number of vertices needed to cover all edges of the link graph. Thus, whenever an optimal OCT intersects $\Delta_1 \cup \Delta_2$, there exists an optimal OCT whose intersection with $\Delta_1 \cup \Delta_2$ is an MVC. $\square$

Proof of Lemma 5.2: We analyze the possible 5-cycles in $\mathcal{G}$ based on the layers involved (X_1, X_2, X_0).

Case A: Standard Cycles ($W_1 - W_4$)

- $W_1 = (A_1, D_1, B_1, E_1, C_1)$: This forms a 5-ring because consecutive X_1 sets (e.g., A_1, B_1) are strictly independent by definition.
- $W_2 = (A_2, B_2, C_2, D_2, E_2)$: This forms a 5-ring. Non-consecutive sets in Layer 2 are independent. First notice that W_2 is a ring, as there is no edge from X_2 to $Y_2 \cup Z_2$, where Y_2, Z_2 are not consecutive with X_2, as otherwise, we get a triangle in G.
- $W_3 = (A_2, D_1, B_1, E_1, C_1)$: A_2 is independent of B_1 and E_1. If A_2 were connected to B_1, then for $a_2 \in A_2, b_1 \in B_1$, the triangle $\{a_2, b_1, b\}$ would exist (impossible). Thus, W_3 is a 5-ring.

Case B: Cycles with chords (W_5, W_6) and not having X_0

- $W_4 = (A_2, B_2, B_1, D_1, A_1)$: A_1 is independent of B_1 and B_2. A_2 is independent of B_1. However, there could be a chord as there are some possible edges between D_1 and A_2. In this case by Lemma 3.6, $B_2 = B_2^1 \cup B_2^2$ so that D_1 is complete to B_1^2, and independent from B_2^2. Similarly $A_2 = A_2^2 \cup A_2^1$ so D_1 is complete to A_2^1 and independent from A_2^2. Notice that B_2^2 is complete to A_2^1 as otherwise, $b_2^2, c, d, e, a_2^1, a_1$ for $b_2^2 \in B_2^2$, $a_1 \in A_1$ and $a_2^1 \in A_2^1$ is an induced P_6. Similarly, A_2^2 is complete to B_2^1, as otherwise, $a_1, a_2^2, e, d, c, b_2^1$ (for $a_2^2 \in A_2^2$ and $b_2^1 \in B_2^1$) is an induced P_6. Now $(B_2^2, A_2^1, B_2^1, B_1, D_1, B_2^2)$ is a 5-ring because there is no edge between A_2^2 and D_1 and there is no edge between B_1 and B_2^2.

- $W_5 = (A_2, A_1, D_1, D_2, E_2, A_2)$: A_1 is independent of E_2 (otherwise triangle via a). Similarly, D_1 is independent of A_2. There could be some edges between A_1 and D_2 (possibly between D_1 and A_2). By Lemma 3.6, $D^2 = D_2^1 \cup D_2^2$ so that A_1 is complete to D_2^1 and independent from D_2^2. Similar to what we have shown in case W_4, we have D_2^2 is complete to E_2 (consider path $P = (d_1, d_2^2, c, b, a, e_2)$ for details). Notice that $A_2 = A_2^1 \cup A_2^2$ so that A_1 is complete to A_2^2 and independent from A_2^1 and D_1 is complete to A_2^1 and independent from A_2^2. By similar argument, we also conclude that A_2^2 is complete to E_2. Now we have a 5-ring $(A_2^2, A_1, D_1, D_2^2, E_2, A_2^2)$. To see that it is enough to observe that D_1 is independent from A_2^2 and D_1 is independent from E_2. Note that A_1 and D_2^1 are independent and D_2^2 is independent from A_2^2 (otherwise we get a triangle).

  Note that if there are some edges between D_2^1 and E_2, then we have a ring

  $$(A_1, D_2^1, E_2, D_2^2, D_1, A_1).$$

  This is because A_1 is independent from D_2^2, and A_1 is independent from E_2, furthermore, D_1 is independent from D_2^1 and D_1 is independent from E_2.

- $W_6 = (A_2, A_1, D_2, C_2, B_2, A_2)$: Potential chords exist between A_1 and C_2. If edges exist between A_1 and C_2, we refine the partition. By Lemma 3.13, D_2 is partitioned into $D_2^2 \cup D_2^1$ and $C_2 = C_2^2 \cup C_1^2$ so that A_1 is independent from $D_2^1 \cup C_2^1$ and it is complete to $D_2^2 \cup C_2^2$. Furthermore, C_2^2 is complete to D_2^1 and D_2^2 is complete to C_2^1. In order to have a 5-cycle we must have some edges between D_2^1 and C_2^1. Now by definition $R_1 = (A_1, D_2^2, C_2^1, D_2^1, C_2^2, A_1)$ is a 5-ring.

Case C: Cycles involving Distance-2 vertices:

- $W_7 = (A_0, A_2, B_2, C_2, D_2)$. If such a cycle exists, we apply a **Re-centering Argument**. Let $C' = (a_0, a_2, b_2, c_2, d_2)$ be a specific instance of this cycle in G. We can redefine our coordinate system (sets $A'_1, B'_1 \dots$) relative to C'. In this new reference frame:

  1. The vertex e from the original cycle becomes a member of A'_2 (adjacent to d_2, a_2).
  2. The vertex b becomes a member of C'_2.
  3. The original sets map to a configuration containing a cycle of type W_6 or similar.

Since we have already shown how to handle W_6 (via refinement) and W_5, this case reduces to previously solved cases. Thus, we assume that both cycles $(A_2, D_1, D_2, C_2, C_1, A_1)$ (as W_5) and $(B_0, B_2, A_2, E_2, D_2, B_0)$ exists. Suppose there is a 5-cycle C'' in $B_0 \cup B_2 \cup A_2$. Thus by using Lemma 3.14, this cycle has one vertex in B_0, two vertices in A_2 and two vertices in B_2. $C'' = (b_0, b_2^2, a_2^1, b_2^1, a_2^2, b_0)$. Thus, we may assume there exists 5-cycle $W' = (B_0, B_2^2, A_2^1, B_2^1, A_2^2, B_0)$ in $\mathcal{G}$. Notice that B_2^2 are vertices of B_2 that have neighbors in B_0 and A_2^2 are vertices of A_2 with neighbors in B_0. Since path $(d_1, d, c, b, a_2^2, b_0)$ is not induced P_6, we have $d_1 b_2^2 \in E(G)$. This implies D_1 is complete to A_2^2. Since path b_0, b_2^2, a, e, d, d_1 is not induced P_6 we have $d_1 b_2^2 \in E(G)$, implying that D_1 is complete to B_2^2. Thus, there is no edge from B_2^2 to A_2^2 as otherwise we have a triangle in G. Thus, W' forms a 5-ring.

- $W_8 = (A_0, A_2^1, B_2^1, A_2^2, B_2^2, A_0)$ which is handled using the re-centering argument as in the previous case.

□

Proof of Lemma 5.3: Suppose G contains two (cleaned) 5-rings W_1 and W_2 that share exactly one sets Δ_1, and let $e_{12} = (\Delta_1, \Delta_2)$ be an edge of W_1 and $e_{13} = (\Delta_1, \Delta_3)$ be an edge of W_2 such that:

- W_1 does not use the edge e_{13}, and W_2 does not use the edge e_{12};
- Algorithm 1 gives preference to branching on a ring that contains a biclique link, and within such a ring, gives preference to a biclique link.

Then there exists an optimal OCT S^* consistent with the branch decision taken on e_{12} and with the subsequent optimal decision on e_{13}.

Case 1. Up to symmetry, assume $\Delta_1 = A_1$. Now if $\Delta_2 = X_1$ (C_1 or D_1) then we know that A_1 is complete to X_1 and hence, MVC contain entire A_1 and by removing A_1 we break both rings containing A_1. Similarly of $\Delta_3 = C_1$ (or D_1) So we may assume that $\Delta_2, \Delta_3 \in \{C_2, A_2, D_2\}$. First suppose $\Delta_2 = A_2$ and $\Delta_3 = C_2$.

By Corollary 3.12 A_1 is complete to $N(A_1) \cap C_2$. Thus, MVC contain the entire A_1 and hence, it would be part of optimal OCT. This implies that we also break W_1.

Case 2. Up to symmetry $\Delta_1 = A_2$.

Subcase 2.1. $\Delta_2 = A_1$. If $\Delta_3 = C_1$ then since, A_1 is complete to C_1, $A_2 = A_2^1 \cup A_2^2$, so that A_2^1 have neighbors in A_1 and not to C_1 and A_2^2 have neighbors in C_1 and no neighbor in A_1. Thus, MVC for $G[\Delta_1 \cup \Delta_2]$ is disjoint from MVC for $G[\Delta_1 \cup \Delta_3]$. So their contribution to OCT is disjoint.

Next suppose $\Delta_3 = B_2$. Now by Lemma 3.7, the part of A_2, say A_2^2 that have neighbor in A_1 is complete to B_2. Thus MVC for $G[\Delta_1 \cup \Delta_3]$ contains A_2^2 and consequently it should be part of OCT and choosing A_2^2 would break W_1.

Subcase 2.2 $\Delta_1 = A_2$, $\Delta_2 = C_1$ and $\Delta_3 = D_1$. Next suppose $\Delta_3 = C_1$.

Let $A_{2,c}$ be a set of vertices in A_2 that have a neighbor in C_1 and $A_{2,d}$ be a set of vertices in A_2 that have neighbor in D_1. By Corollary 3.12, C_1 is complete to $A_{2,c}$ and D_1 is complete to $A_{2,d}$. Let $a_2, a_2' \in A_2$. Let $c_1 \in C_1$ so that $a_2 c_1 \in E(G)$. Let $d_1 \in D_1$ be a neighbor of a_2'.

Now consider $P_1 = (d_1, a_2', e, a_2, c_1, c)$ and $P_2 = (c_1, a_2, b, a_2', d_1, d)$. Since neither of P_1 and P_2 is induced P_6 we have $d_1a_2 \in E(G)$ or $c_1a_2' \in E(G)$. This implies that

$A_{2,c} \subset A_{2,d}$ or $A_{2,d} \subseteq A_{2,c}$. Suppose $A_{2,c} \subseteq A_{2,d}$. Thus, MVC for $G[\Delta_1 \cup \Delta_3]$ contain entire A_2 (because it is a clique) and hence it is part of OCT and it also break ring W_1.

Subcase 3. $\Delta_2 = C_1$, and $\Delta_3 = E_2$. Let A_2^1 be the set of of A_2 which has neighbor in A_2. We notice that by Lemma 3.11, C_1 and A_2^1 form a biclique. Therefore, if MVC for $G[\Delta_1 \cup \Delta_2]$ contains some vertices of A_2 then it must contain entire A_2^1 and hence, it must be include to OCT. Now the MVC for $G[\Delta_2 \cup \Delta_3]$ is independent from the choice of A_2^1, and it does not affect the OCT.

Subcase 4. $\Delta_2 = A_0$ and $\Delta_3 = B_2$.

Observation I Suppose there exists $a_0 \in A_0$ so that $a_2a_0 \in E(G)$. Let a_2' be a vertex in A_2 that has a neighbor b_2 in B_2. Now $P = (a_0, a_2, e, a_2', b_2, c)$ is an induced P_6 unless $a_2b_2 \in E(G)$.

Observation II Suppose there exists $a_0 \in A_0$ so that $a_2a_0, a_2'a_0 \in E(G)$. Let $b_2 \in B_2$ so that $a_2b_2 \in E(G)$. Now $a_2', a_0, a_2, b_2, c, d$ is an induced P_6 unless $a_2'b_2 \in E(G)$. This means $N(a_2) \cap B_2 = N(a_2') \cap B_2$.

Let $A0, 2$ be a set of vertices in A_2 with some neighbor in A_0 (concerning W_1 ring). Let $A_{2,2}$ be a set of vertices in A_2 that have some neighbor in B_2 (concerning W_2 ring). Let $A_{2,0,2}$ be the set of vertices in A_2 that have some neighbor in A_0 and have some neighbor in B_2. We also observe that $A_{2,0,2}$ is complete to $N(A_{2,0,2}) \cap B_2$. Furthermore, there is no path from $A' = A_{0,2} \setminus A_{2,0,2}$ to $A_{2,0,2}$ because otherwise by Observation II, A' is empty.

However, we conclude that MVC for $G[B_2 \cup A_2]$ must include $A_{2,0,2}$ if it contains some vertices in A_2 and hence it must be part of OCT. Now the selection of $A_{2,0,2}$ does not affect the MVC for $G[A' \cup (N(A') \cap A_0)]$.

Subcase 5. $\Delta_2 = A_0$ and $\Delta_3 = A_1$. For simplicity we may assume every vertex in A_2 has some neighbor in A_0 and has some neighbor in A_1. This is because if there are some vertices in A_2 without any neighbor in A_0 then they won't appear in any OCT for ring W_1 (the same for W_2 and A_1). First suppose $G_{1,2}$ is not connected.

Suppose there are two vertices $a_2, a_2' \in A_2$. Let $a_0 \in A_0$ and $a_0' \in A_0$ so that $a_2a_0, a_2'a_0' \in E(G)$, and let $a_1, a_1' \in A_1$ so that $a_2a_1, a_2'a_1' \in E(G)$. Since the path $P = (a_0, a_2, a_1, a, a_1', a_2')$ is not induced P_6, one of the $a_0a_2', a_2a_1', a_2'a_1, a_0'a_2$ is an edge of G.

Let H_1 and H_2 be two connected component in $G_{1,2}$. Let $a_2a_1 \in E(H_1)$ and $a_2'a_1' \in E(H_2)$ where $a_1, a_1' \in A_1$ and $a_2, a_2' \in A_2$. Suppose $a_2a_0 \in E(G)$ for $a_0 \in A_0$. Now $(a_2', a_1', a, a_1, a_2, a_0)$ is an induced P_6 unless $a_0a_2' \in E(G)$. This means that A_0 is complete to A_2. Thus, MVC in $G[A_0 \cup A_2]$ contains all the vertices in A_2 (or none of them). If we take A_0 then it also break the ring W_2.

So we may assume that $G_{1,2}$ is connected. Suppose there is a $2K_2$ in $G_{1,2}$. Then by using the same argument in the proof of Lemma 4.5, $G_{1,2}$ there are B_1, B_2, B_3 where B_1 and B_2 are disjoint and B_3 is complete to both B_1 and B_2. $N(B_1) \cap A_0 = N(B_2) \cap A_0 \subseteq N(B_3) \cap A_0$ and $G[B_1 \cup (N(B_1) \cap A_0)], G[B_2 \cup (N(B_2) \cap A_0)]$ are bicliues and every vertex in B_3 is adjacent to every vertex in $N(B_1) \cap A_0$. This implies that MVC for $G_{0,2} = G[A_2 \cup A_0]$ must include B_3 and hence it must be part of OCT. Furthermore, MVC for $G_{1,2}$ (concerning A_2) must contain B_3. Furthermore, $A_2 \setminus B_3$ is complete to A_2, and hence it must be included in MVC for $G_{0,2}$ and hence part of OCT. Thus, we end up removing entire A_2, which breaks the ring W_2.

Finally we consider the case where $G_{1,2}$ is a chain graphs from right to left inclusion ordering π with respect to A_2. By using the same argument in Lemma 4.1, one can show that each connected component of $G_{0,2}$ must be a chain graph from right to left inclusion ordering π_1 with respect to A_2. Note that any MVC for $G_{1,2}$ it contains a set of vertices of A_2 from right to left in π and any MVC in $G_{0,2}$ contains a consecutive vertices according to the π_1 ordering. Since π and π_1 are consistent then the choice for $G_{1,2}$ does not affect the choice for $G_{0,2}$. □

6 Extensions to P_k-free Graphs

We propose that the structural and algorithmic techniques developed for P_6-free graphs can be generalized to P_k-free graphs for $k > 6$. While the problem remains NP-complete for $k \geq 6$, we can achieve constant-factor approximations by leveraging the tractability of graphs excluding small odd cycles.

6.1 Approximation Algorithm for P_k-free Graphs

Let $\mathcal{C}_{<L}$ denote the set of all odd cycles with length strictly less than L. Our strategy relies on a standard greedy packing approach: iteratively find and remove vertex-disjoint small odd cycles until the graph is free of them. The approximation factor is determined by the size of the largest cycle removed.

Theorem 6.1. *Let $k \geq 6$ be an integer. There is a polynomial-time algorithm that, given a P_k-free graph G, outputs an OCT, S such that $|S| \leq (k-3) \cdot \mathrm{OPT}(G)$ when k is even and $|S| \leq (k-2) \cdot \mathrm{OPT}(G)$ when k is odd. Here $\mathrm{OPT}(G)$ denotes the size of a minimum OCT of G.*

Proof. Fix $k \geq 6$ and let G be a P_k-free graph. Set $L = k - 3$ if k is even, otherwise set $L = k - 2$. We describe an algorithm that returns a set S with $|S| \leq L \cdot \mathrm{OPT}(G)$.

Phase 1: greedy packing of short odd cycles. Initialize $S_{\text{pack}} := \emptyset$ and $G_0 := G$. While G_0 contains an odd cycle of length at most L, select one such cycle C (e.g., the shortest odd cycle), add its vertex set to S_{pack}, and delete $V(C)$ from G_0. Let $G' := G \setminus S_{\text{pack}}$.

By construction, the cycles chosen in Phase 1 are vertex-disjoint and each has length at most L. Thus, if t is the number of cycles chosen, we have $|S_{\text{pack}}| \leq L \cdot t$.

Let O^* be a global minimum odd cycle transversal for G, so $|O^*| = \mathrm{OPT}(G)$. We can partition O^* into two disjoint sets: $O^*_1 = O^* \cap S_{\text{pack}}$ and $O^*_2 = O^* \setminus S_{\text{pack}}$. Because O^* is a valid OCT, it must intersect every odd cycle in G, and in particular, it must pick at least one vertex from each of the t vertex-disjoint odd cycles found in Phase 1. Therefore, $|O^*_1| \geq t$. Substituting this into our packing bound yields $|S_{\text{pack}}| \leq L \cdot |O^*_1|$.

Phase 2: exact solution on the remainder. By definition of G', it contains no odd cycle of length at most L. Equivalently:

- if k is even, then G' is $(P_k, C_3, C_5, \ldots, C_{k-3})$-free, so the smallest possible odd cycle has length at least $k - 1$;
- if k is odd, then G' is $(P_k, C_3, C_5, \ldots, C_{k-2})$-free, so the smallest possible odd cycle has length at least k.

In these two cases, we can compute a minimum odd cycle transversal of G' in polynomial time using the structural theorems proved later in the paper: when k is odd, by Theorem 7.1, and when k is even, by Theorem 8.16.

Let S_{rem} be such a minimum OCT of G'. Finally, output $S := S_{\text{pack}} \cup S_{\text{rem}}$.

Since $G \setminus S = (G' \setminus S_{\text{rem}})$ is bipartite, S is a valid OCT of G. To bound its size, observe that O_2^* is a valid odd cycle transversal for G', meaning the optimal solution on the remainder graph cannot be larger than $|O_2^*|$. Thus, $|S_{\text{rem}}| = \text{OPT}(G') \leq |O_2^*|$. Combining the bounds from both phases, and noting that $L \geq 1$ for all $k \geq 6$, we obtain:

$$|S| = |S_{\text{pack}}| + |S_{\text{rem}}| \leq L \cdot |O_1^*| + |O_2^*| \leq L(|O_1^*| + |O_2^*|) = L \cdot \text{OPT}(G)$$

So in particular $|S| \leq (k-3)\text{OPT}(G)$ for even k and $|S| \leq (k-2)\text{OPT}(G)$ for odd k, as claimed. □

7 Exact OCT on $(P_k, \mathcal{C}_{<k})$-free Graphs when k is odd

Let $k \geq 5$ be an odd integer. Let G be a $(P_k, \mathcal{C}_{<k})$-free graph, meaning G contains neither an induced path on k vertices nor any induced odd cycle of length strictly less than k. Let $C = (a_0, a_1, \ldots, a_{k-1})$ be an induced cycle of length k in G.

We define the sets V_i for $i \in \{0, \ldots, k-1\}$ (with indices taken modulo k) as:

$$V_i = \{a_i\} \cup \{v \in V(G) \setminus V(C) \mid N(v) \cap V(C) = \{a_{i-1}, a_{i+1}\}\}$$

Theorem 7.1. *The vertex set of G is partitioned into $\bigcup_{i=0}^{k-1} V_i$. Moreover, each V_i is an independent set, and edges exist only between consecutive sets V_i and V_{i+1}. Consequently, the Minimum OCT is simply the set V_i of minimum cardinality.*

Proof. **1. Classification of Neighbors:** We first show that every vertex outside C with a neighbor in C must be adjacent to at least two vertices of C. Suppose $v \in V(G) \setminus V(C)$ is adjacent to exactly one vertex on C, say a_0. Consider the path $P = (v, a_0, a_1, \ldots, a_{k-2})$. This path contains exactly k vertices. Because G is free of odd cycles of length strictly less than k (specifically $C_3, \ldots, C_{k-2}$), there can be no chords between a_0 and a_{k-2}. Furthermore, since v has no other neighbors on the cycle, it introduces no chords. Therefore, P is an induced P_k, which contradicts the assumption that G is P_k-free.

2. Adjacency Constraints: Next, suppose $v \notin V(C)$ is adjacent to two vertices a_i and a_j on C. The edges va_i and va_j, together with the two paths connecting a_i and a_j along C, create two new cycles. The sum of the lengths of these two cycles is exactly $k+2$. Since k is odd, $k+2$ is odd, meaning exactly one of these two cycles must be of odd length. Let L_{odd} denote the length of this odd cycle. By our hypothesis, we must have $L_{odd} \geq k$.

Because v is not on C, the maximum possible length for a cycle passing through v and a segment of C is $(k-1) + 2 = k+1$. The only way to achieve an odd cycle of length at least k is if the segment of C between a_i and a_j has length $k-2$. This implies that a_i and a_j must be at distance exactly 2 on the cycle (e.g., a_{i-1} and a_{i+1}). If the distance were 1, the vertices v, a_i, a_{i+1} would form an induced C_3, which is a forbidden small odd cycle. Thus, every neighbor of C must connect to exactly two vertices at distance 2, meaning every neighbor of C belongs to exactly one set V_i.

3. No Distance-2 Vertices: We now show that no vertex can be at distance 2 from C. Suppose, for the sake of contradiction, that there exists a vertex y at distance 2 from C. Let

y be adjacent to a vertex $x \in V_0 \setminus \{a_0\}$. By definition, x is adjacent to a_{k-1} and a_1. Consider the path $P = (y, x, a_1, a_2, \ldots, a_{k-2})$. This path consists of exactly k vertices. Since y is at distance 2 from C, it has no neighbors on the cycle. Furthermore, x is adjacent only to a_{k-1} (which is not in P) and a_1, so x does not introduce any chords to the path. Thus, P forms an induced P_k, yielding a contradiction. This establishes that every vertex in G belongs to $\bigcup_{i=0}^{k-1} V_i$.

4. Ring Structure: Finally, we establish the adjacencies between the sets.

- **Internal edges:** Suppose there is an edge between two vertices $u, v \in V_i$. By definition, both u and v are adjacent to a_{i-1}, which means the set $\{u, v, a_{i-1}\}$ forms an induced C_3. This is a contradiction, so each V_i must be an independent set.

- **Non-consecutive edges:** Suppose there is an edge between $u \in V_i$ and $v \in V_{i+2}$. Both vertices share the neighbor a_{i+1}, which again forms a forbidden induced C_3. Furthermore, if the distance between indices i and j on the ring is strictly greater than 2, any edge between V_i and V_j would close an induced odd cycle of length strictly less than k, which is forbidden.

Thus, edges can only exist between consecutive sets V_i and V_{i+1}.

Conclusion: The graph G is structured strictly as a ring of independent sets $V_0, V_1, \ldots, V_{k-1}, V_0$. Because k is odd, this ring structure forms a macroscopic odd cycle. The graph becomes bipartite if and only if we remove all vertices in at least one set V_i, which breaks the odd ring into a bipartite chain. Therefore, the Minimum Odd Cycle Transversal is exactly the set V_i that has the minimum cardinality. □

□

8 Structure of $(P_k, \mathcal{C}_{<k-1})$-free Graphs when k is even

Let $k \geq 8$ be an even integer. Let G be a $(P_k, \mathcal{C}_{<k-1})$-free graph, meaning G contains neither an induced path on k vertices nor any induced odd cycle of length strictly less than $k-1$. Let $C = (a_0, a_1, \ldots, a_{k-2})$ be an induced cycle of length $k-1$ in G. Since k is even, the length of C is odd.

We classify the vertices of $V(G) \setminus V(C)$ that have neighbors in C into specific sets. All index arithmetic is performed modulo $k-1$.

- A_i: The set of vertices in $G \setminus C$ adjacent to only a_i.

- B_i: The set of vertices in $G \setminus C$ adjacent exactly to a_{i-1} and a_{i+1}.

- Let G_i be the set of vertices at distance 2 from the cycle C that are adjacent to B_i. By definition vertices in G_i are independent of the vertices of C.

Lemma 8.1 (Mutual Exclusion of A_i and A_{i+2})**.** *For any index i, if $A_i \neq \emptyset$, then $A_{i+2} = \emptyset$.*

Proof. Suppose, for the sake of contradiction, that there exist vertices $x \in A_i$ and $y \in A_{i+2}$. We consider two cases based on the adjacency of x and y:

Case 1: x and y are not adjacent. Consider the path P that traverses the cycle C the "long way" around from a_i to a_{i+2}. Let $P = (x, a_i, a_{i-1}, \ldots, a_{i+3}, a_{i+2}, y)$. The segment of the cycle from a_i backwards to a_{i+2} contains exactly $(k-1) - 2 + 1 = k - 2$ vertices. Adding the endpoints x and y brings the total number of vertices in P to exactly k. By definition, x is adjacent only to a_i and y is adjacent only to a_{i+2}. Because C is an induced cycle, there are no chords between the vertices of C. Since x and y are not adjacent to each other or to any internal vertices of the cycle path, P is an induced P_k. This contradicts the assumption that G is P_k-free.

Case 2: x and y are adjacent. Consider the sequence of vertices $Z = (x, y, a_{i+2}, a_{i+1}, a_i, x)$. This sequence forms a cycle of length 5. Since x and y have no other neighbors on C, and a_i, a_{i+1}, a_{i+2} form an induced P_3, Z is an induced C_5. Because $k \geq 8$ is an even integer, we have $k - 1 \geq 7$. Thus, a cycle of length 5 is strictly smaller than $k - 1$. Since 5 is odd, Z is a forbidden small odd cycle, yielding a contradiction.

Since both cases lead to a contradiction, it is impossible for both A_i and A_{i+2} to be non-empty simultaneously. □

Lemma 8.2 (Adjacency Restrictions for B_i)**.** *Let $b \in B_i$. The vertex b can only be adjacent to vertices in the sets A_i, A_{i-2}, and A_{i+2}. It is independent of A_j for all $j \notin \{i, i-2, i+2\}$.*

Proof. Let $b \in B_i$. By definition, b is adjacent to exactly a_{i-1} and a_{i+1} on the cycle. We test the potential adjacencies between b and vertices in various A-sets:

- **Adjacency to A_i:** Suppose b is adjacent to $x \in A_i$. The sequence (b, x, a_i, a_{i-1}, b) forms an induced C_4. Because this cycle has an even length, it does not violate the $(\mathcal{C}_{<k-1})$-free property. Thus, this edge is permissible.

- **Adjacency to A_{i+2}:** Suppose b is adjacent to $y \in A_{i+2}$. The sequence $(b, y, a_{i+2}, a_{i+1}, b)$ similarly forms an induced C_4, which is permissible. By symmetry, an edge to $z \in A_{i-2}$ is also permissible via the cycle $(b, z, a_{i-2}, a_{i-1}, b)$.

- **Adjacency to A_{i+3}:** Suppose b is adjacent to $w \in A_{i+3}$. We can construct the cycle $Z = (b, w, a_{i+3}, a_{i+2}, a_{i+1}, b)$. This forms an induced cycle of length 5. As established previously, for $k \geq 8$, C_5 is a forbidden small odd cycle. Thus, b cannot be adjacent to any vertex in A_{i+3}.

By extending the logic of the third case, if b is adjacent to any vertex in A_j where the distance between a_j and the anchor vertices $\{a_{i-1}, a_{i+1}\}$ is greater than 1, it will invariably form an induced odd cycle of length at least 5 but strictly less than $k-1$. Therefore, B_i must be independent of all sets A_j except those at distance at most 1 from its anchors, namely A_i, A_{i-2}, and A_{i+2}. □

Lemma 8.3. *No vertex in G_i is adjacent to a vertex in B_{i+1}. Moreover, G_i cannot be adjacent to any B_j for $j > i + 2$.*

Proof. Suppose, for the sake of contradiction, that there exists an edge $g_i b_{i+1}$ with $g_i \in G_i$ and $b_{i+1} \in B_{i+1}$. By the definition of G_i, g_i must be adjacent to some $b_i \in B_i$. Recall the specific neighborhoods on the cycle C:

- b_i is adjacent to $\{a_{i-1}, a_{i+1}\}$.

- b_{i+1} is adjacent to $\{a_i, a_{i+2}\}$.

Consider the cycle of vertices $Z = (g_i, b_i, a_{i-1}, a_i, b_{i+1}, g_i)$, which has length 5. We exhaustively check for chords to determine if Z is induced:

- b_ib_{i+1}: If this edge exists, the vertices $\{g_i, b_i, b_{i+1}\}$ form an induced C_3. Because G is C_3-free, $b_ib_{i+1} \notin E(G)$.
- g_ia_{i-1} and g_ia_i: Since g_i is strictly at distance 2 from C, these edges cannot exist.
- b_ia_i and $b_{i+1}a_{i-1}$: By the definitions of B_i and B_{i+1}, these edges do not exist.

Since no chords can exist, Z is an induced C_5. For $k \geq 8$, C_5 is a forbidden small odd cycle. This contradiction implies that G_i must be independent of B_{i+1}. If there is an edge from a vertex in G_i to B_j with $j > i+2$ we get a shorter odd cycle $< k-1$. □

Lemma 8.4. *No vertex in G_i is adjacent to a vertex in G_j.*

Proof. Suppose, for the sake of contradiction, that there exists an edge g_ig_j with $g_i \in G_i$ and $g_j \in G_j$. By definition, there exist $b_i \in B_i$ and $b_j \in B_j$ such that $g_ib_i \in E(G)$ and $g_jb_j \in E(G)$. Observe that b_i is adjacent to a_{i-1}. Now the path

$$P = (a_{i+1}, a_{i+2}, ..., a_{k-2}, a_0, \ldots, a_{i-1}, b_i, g_i, g_j)$$

has length k, a contradiction. □

Lemma 8.5. *No vertex in G_j is adjacent to a vertex in A_i.*

Proof. Suppose, for the sake of contradiction, that there exists an edge x_ig_j with $x_i \in A_i$ and $g_j \in G_j$. By definition, there exist $a_i \in C$ such that $a_ix_i \in E(G)$. Now the path $P = (g_j, x_i, a_i, a_{i+1}, a_{i+2}, ..., a_{k-2}, a_0, \ldots, a_{i-2})$ has length k, a contradiction. □

Lemma 8.6. *No vertex in G_i is adjacent to a vertex in B_j except $i = j \pm 2$.*

Proof. suppose $g_ib_j \in E(G)$ where $g_i \in G_i$ and $b_j \in B_j$. By definition, g_i must be connected to a vertex $b_i \in B_i$. WLOG, let $i < j$. then $P_x = (a_{i-1}, b_i, g_i, b_j, a_{j+1})$ along with path in C with anchor vertices $\{a_{i-1}, a_{j+1}\}$ will invariably form an induced odd cycle of length at least 5 but strictly less than $k-1$. Therefore, G_i must be independent of all sets B_j except those at distance exactly 2 from itself. □

Lemma 8.7. *B_i is adjacent only to vertices in $B_{i\pm1}$.*

Proof. suppose $b_ib_j \in E(G)$ where $b_i \in B_i$ and $b_j \in B_j$. WLOG, let $i < j$. then $P_x = (a_{i-1}, b_i, b_j, a_{j+1})$ along with path in C with anchor vertices $\{a_{i-1}, a_{j+1}\}$ will invariably form an induced odd cycle of length at least 5 but strictly less than $k-1$ unless $j = i+1$. Therefore, B_i must be independent of all sets B_j except $B_{i\pm1}$. □

Lemma 8.8. *A_i is adjacent only to vertices in $A_{i\pm3}$.*

Proof. suppose $x_ix_j \in E(G)$ where $x_i \in A_i$ and $x_j \in A_j$. WLOG, let $i < j$. then $P_x = (a_i, x_i, x_j, a_j)$ along with path in C with anchor vertices $\{a_i, a_j\}$ will invariably form an induced odd cycle of length at least 5 but strictly less than $k-1$ unless $j = i+3$. Therefore, A_i must be independent of all sets A_j except $A_{i\pm3}$. □

Definition 8.9. *For each A_i we define A_i^+ be the set of vertices adjacent that are adjacent to a vertex in $A_{i+3} \cup B_{i+2}$ and A_i^- be the set of vertices of A_i that have some neighbors in $A_{i-3} \cup B_{i-2}$. Similarly let G_i^+ be a set of vertices in G_i that have some neighbor in B_{i+2} and let G_i^- be a set of vertices in G_i that have neighbors in B_{i-2}. Also, A_i^+ and A_i^- may both also have neighbors in B_i.*

Lemma 8.10. *For every j, there is no edge from A_j^- to $A_{j+3} \cup B_{j+2}$ and there is no edge from A_j^+ to $A_{j-3} \cup B_{j-2}$.*

Proof. Suppose, for the sake of contradiction, that there exists an edge between a vertex $x \in A_j^-$ and a vertex $z \in A_{j+3}$. By the definition of A_j^-, there must exist a vertex $w \in A_{j-3}$ such that $wx \in E(G)$. Consider the path $P_{ext} = (a_{j-3}, w, x, z, a_{j+3})$ in $G \setminus C$. This path has a length of exactly 4 edges.

We now construct a cycle C' by concatenating P_{ext} with the path along the main cycle C from a_{j+3} back to a_{j-3}. The distance between indices $j-3$ and $j+3$ along the cycle C is exactly 6 edges in the forward direction. Thus, the backward path P_{cyc} along the cycle has length:

$$L_{cyc} = (k-1) - 6 = k - 7$$

The total length of the constructed cycle C' is therefore:

$$|C'| = |P_{ext}| + |P_{cyc}| = 4 + (k-7) = k-3.$$

Since k is an even integer, $k-3$ is odd. Furthermore, because $k \geq 8$, we have $k-3 < k-1$. The graph G is defined as $(\mathcal{C}_{<k-1})$-free, which strictly forbids the existence of induced odd cycles with length strictly less than $k-1$. We must verify that C' is an induced cycle:

- **Internal Chords:** There are no chords within $\{w, x, z\}$ as $w \in A_{j-3}$ and $z \in A_{j+3}$ are at distance 6 on the cycle, and G is C_3-free.
- **Cycle Chords:** Vertices w, x, and z are only adjacent to their respective anchor vertices on C. The path P_{cyc} does not contain the anchors $a_{j-2}, \ldots, a_{j+2}$, ensuring no chords exist between the external path and the cycle segment.

Thus, C' is an induced C_{k-3}, which is a contradiction.

A symmetric contradiction arises if we assume an edge exists between $x \in A_j^-$ and $y \in B_{j+2}$. In that case, the path through the anchor vertices would similarly form a forbidden odd cycle of length $k-3$. Consequently, A_j^+ and A_j^- must be disjoint. □ □

By similar argument as in in Lemma 8.10 we have the following lemma.

Lemma 8.11. *For every j, there is no edge from G_j^- to B_{j+2} and there is no edge from G_j^+ to B_{j-2}.*

Recall that no vertex outside C can be adjacent to consecutive cycle vertices, as this would form an induced C_3. Furthermore, no vertex can be adjacent to two cycle vertices at a distance greater than 2, as this would form an induced odd cycle of length strictly less than $k-1$.

Thus, the sets A_i and B_i capture the only permissible adjacency patterns to C. The following Lemmas give structural properties of G, allowing us to prove Theorem 8.16.

Let $k \geq 8$ be an even integer. Let H be a ring graph consisting of partitions $V_0, V_1, \ldots, V_{k-2}$ where edges exist only between consecutive sets V_j and V_{j+1} (with indices evaluated modulo $k-1$). Assume every vertex in H lies on an induced $(k-1)$-cycle. Let H_j be the bipartite subgraph induced by $V_j \cup V_{j+1}$.

Lemma 8.12 (Generalized Chain)**.** *Let $a_1, a_2 \in V_j$ and $b_1, b_2 \in V_{j+1}$. Suppose $a_1b_1, a_2b_2, a_2b_1 \in E(H)$ but $a_1b_2 \notin E(H)$ (forming induced 'Z' structure). Then:*

1. $N(b_2) \cap V_{j+2} \subseteq N(b_1) \cap V_{j+2}$

2. $N(a_1) \cap V_{j-1} \subseteq N(a_2) \cap V_{j-1}$

Proof. We prove Part 1 (forward inclusion); Part 2 follows by symmetry. Suppose, for the sake of contradiction, that $N(b_2) \cap V_{j+2} \not\subseteq N(b_1) \cap V_{j+2}$. Then there exists a vertex $c \in V_{j+2}$ such that $b_2 c \in E(H)$ but $b_1 c \notin E(H)$.

Since every vertex lies on a $(k-1)$-cycle, we can extend a shortest path forward from c through the ring. Let $P_{tail} = (v_1, v_2, \ldots, v_{k-4})$ be this shortest path, where $v_1 = c \in V_{j+2}$ and $v_m \in V_{j+m+1}$ for each $1 \leq m \leq k-4$. Consider the concatenated sequence of vertices:

$$P = (a_1, b_1, a_2, b_2, v_1, v_2, \ldots, v_{k-4})$$

- **Vertex Count:** The local 'Z' structure provides 4 vertices (a_1, b_1, a_2, b_2). The tail provides $k-4$ vertices. Thus, P contains exactly k vertices.
- **Induced Property:**
  - The edges within $\{a_1, b_1, a_2, b_2, v_1\}$ are strictly defined by the hypothesis and the assumption $b_1 v_1 \notin E(H)$.
  - Since P_{tail} is a shortest path along the ring, it is chordless.
  - The path begins at $a_1 \in V_j$ and terminates at $v_{k-4} \in V_{j+(k-4)+1} = V_{j+k-3}$.
  - Because the ring length is $k-1$, the index $j+k-3$ is equivalent to $j-2 \pmod{k-1}$.
  - For $k \geq 8$, the ring length $k-1 \geq 7$. In a ring of size at least 7, the set V_{j-2} is at a distance of 2 from V_j, meaning no edges exist between V_{j-2} and V_j.

Consequently, the endpoints of P do not connect, and no wrap-around chords exist. P is an induced P_k, yielding a contradiction. □

Lemma 8.13 (Generalized $2K_2$ Forbidden)**.** *Let $a_1, a_2 \in V_j$ and $b_1, b_2 \in V_{j+1}$. Suppose the edges $a_1 b_1 \in E(H)$ and $a_2 b_2 \in E(H)$ form an induced $2K_2$ (i.e., $a_1 b_2 \notin E(H)$ and $a_2 b_1 \notin E(H)$). Then $N(a_1) \cap V_{j-1} = N(a_2) \cap V_{j-1}$ and $N(b_1) \cap V_{j+2} = N(b_2) \cap V_{j+2}$.*

Proof. Assume, for contradiction, that the neighborhoods in V_{j-1} are distinct. Without loss of generality, suppose there exists $e_1 \in N(a_1) \setminus N(a_2)$. Furthermore, assume the graphs are connected such that there exists $e_2 \in N(a_1) \cap N(a_2)$ (if no such shared neighbor exists, the components are disjoint, which immediately forces a P_k via longer paths).

Consider the sequence $S = (e_1, a_1, e_2, a_2, b_2)$. This sequence alternates between V_{j-1} and V_j, before finally stepping to V_{j+1}. It utilizes 5 vertices while only advancing the ring index from $j-1$ to $j+1$. We construct a path P by extending S forward from b_2 through $V_{j+2}, \ldots$ for exactly $k-5$ additional vertices along the ring. The total number of vertices is $5+(k-5) = k$.

The path originates in V_{j-1} and terminates in $V_{j+1+(k-5)} = V_{j+k-4}$. Modulo $k-1$, the terminal index is $j-3$. For $k \geq 8$, V_{j-3} is not adjacent to V_{j-1} (the distance is 2). Because this structure consumes vertices rapidly while advancing slowly around the ring, it avoids closing a cycle. Thus, P forms an induced P_k, a contradiction. □

Lemma 8.14 (Chain Triples Imply Bicliques)**.** *Suppose the bipartite subgraphs H_j, H_{j+1}, and H_{j+2} are proper chain graphs with alternating inclusion directions (e.g., $L \to R$, $R \to L$, and $L \to R$, respectively). Then the adjacent subgraphs H_{j-1} and H_{j+3} must be complete bipartite graphs.*

Proof. We prove the lemma for $H_{j+3} = H[V_{j+3} \cup V_{j+4}]$; the argument for H_{j-1} is symmetric.

Assume for a contradiction that H_{j+3} is not complete bipartite. Then there exist $d_1 \in V_{j+3}$ and $e_1 \in V_{j+4}$ such that $d_1e_1 \notin E(H)$.

Since $H_{j+2} = H[V_{j+2} \cup V_{j+3}]$ is a *proper* chain graph with inclusion direction $V_{j+2} \to V_{j+3}$, the neighborhoods in V_{j+3} of vertices of V_{j+2} are strictly nested. In particular, there exist vertices $c_1, c_2 \in V_{j+2}$ and $d_1, d_2 \in V_{j+3}$ forming a Z-shape:

$$c_1d_1,\ c_2d_2,\ c_2d_1 \in E(H) \qquad \text{and} \qquad c_1d_2 \notin E(H).$$

(Equivalently, the link (V_{j+2}, V_{j+3}) contains two comparable neighborhoods and hence a witness Z.)

Now apply Lemma 4.1 (the Z-inclusion lemma) to the above configuration in H_{j+2}. It yields the inclusion $N(c_1) \cap V_{j+1} \subseteq N(c_2) \cap V_{j+1}$.
Because $H_{j+1} = H[V_{j+1} \cup V_{j+2}]$ is a *proper* chain graph and the direction alternates ($V_{j+2} \to V_{j+1}$), this inclusion is strict. Hence we may choose $b_1 \in V_{j+1}$ such that $b_1c_2 \in E(H)$ but $b_1c_1 \notin E(H)$.

Next, use that $H_j = H[V_j \cup V_{j+1}]$ is a *proper* chain graph with direction $V_j \to V_{j+1}$. Since b_1 has at least one neighbor in V_j (otherwise b_1 would not lie on any cycle in the ring instance), pick any $a_1 \in V_j$ adjacent to b_1, i.e., $a_1b_1 \in E(H)$.

Finally, since every vertex lies on the macroscopic ring, e_1 has a neighbor in V_{j+5}; choose any $\hat{e} \in V_{j+5}$ with $e_1\hat{e} \in E(H)$. Consider the vertex sequence

$$P = (a_1,\ b_1,\ c_2,\ d_1,\ e_1,\ \hat{e}).$$

Claim: P is an induced P_6. Consecutive pairs in P are edges: $a_1b_1 \in E(H)$ by construction, $b_1c_2 \in E(H)$ by construction, $c_2d_1 \in E(H)$ from the chosen Z, $e_1\hat{e} \in E(H)$ by construction, and we also have $d_1e_1 \notin E(H)$ by the initial choice of (d_1, e_1). So P is a path in the underlying graph.

It remains to rule out chords among nonconsecutive vertices of P. This follows from the ring/set structure: edges exist only between consecutive sets. Indeed, $a_1 \in V_j$ is independent from $V_{j+2} \cup V_{j+3} \cup V_{j+4} \cup V_{j+5}$, $b_1 \in V_{j+1}$ is independent from $V_{j+3} \cup V_{j+4} \cup V_{j+5}$, $c_2 \in V_{j+2}$ is independent from $V_{j+4} \cup V_{j+5}$, and $d_1 \in V_{j+3}$ is independent from V_{j+5}. Also $b_1c_1 \notin E(H)$ and $c_1d_2 \notin E(H)$ were part of the chosen Z-witnessing configuration, and V_{j+4} and V_{j+5} are independent sets so there are no internal edges. Therefore there are no edges between any nonconsecutive pair of vertices in P. Hence P induces a P_6.

Now we extend P to obtain an induced P_k for any even $k \geq 8$ (contradicting P_k-freeness). Starting from $\hat{e} \in V_{j+5}$, repeatedly choose a neighbor in the next set along the ring, appending one vertex at a time: from V_{j+5} to V_{j+6} to $\ldots$. Because the ring contains no edges skipping sets, this extension cannot create chords with earlier vertices of the path; all potential chords would have to jump over at least one set and are forbidden by definition. Moreover, since we are extending forward along distinct sets and $k \geq 8$, we can append $k - 6$ additional vertices before any wrap-around adjacency could occur in the macroscopic ring. Thus we obtain an induced P_k, contradicting the assumption that the instance is P_k-free.

This contradiction shows that H_{j+3} must be complete bipartite. By symmetry, the same holds for H_{j-1}. □

8.1 The Global Ring Structure of H

Definition 8.15 (Ring Partition)**.** *We define the partition sets V_j for $0 \leq j \leq k-2$ (with index arithmetic evaluated modulo $k-1$) as follows: $V_j = B_j \cup A^+_{j-1} \cup G^+_{j-1} \cup A^-_{j+1} \cup G^-_{j+1}$.*

Theorem 8.16. *For $k \geq 8$, the graph H forms a ring of $k-1$ independent sets $V_0, V_1, \ldots, V_{k-2}$, where edges exist only between consecutive sets V_j and V_{j+1} (indices modulo $k-1$).*

Proof. Recall that for each $j \in \{0, \ldots, k-2\}$ we define

$$V_j \;=\; B_j \;\cup\; A^+_{j-1} \;\cup\; G^+_{j-1} \;\cup\; A^-_{j+1} \;\cup\; G^-_{j+1},$$

with indices taken modulo $k-1$. We prove: (i) each V_j is an independent set, and (ii) $E(H)$ is contained in $\bigcup_j E(H[V_j \cup V_{j+1}])$.

(i) Each V_j is independent. We show that no edge can have both endpoints in V_j.

No edges inside the G-parts. By Lemma 8.4, vertices belonging to G-parts form an independent set across all indices; in particular $G^+_{j-1} \cup G^-_{j+1}$ is independent.

No edges between G-parts and A/B-parts within V_j. Vertices of $G^+_{j-1} \cup G^-_{j+1}$ lie at distance 2 from the base cycle, while vertices of $B_j \cup A^+_{j-1} \cup A^-_{j+1}$ lie at distance 1. An edge between these two distance layers within the *same* partition would create a forbidden short odd cycle through the shared cycle anchors, contradicting the definition of the refined $A^{\pm}$ sets and the fact that H is C_3-free and has no induced C_5 in the regime $k \geq 8$ (the smallest allowed odd cycle length is at least $k-1 \geq 7$).

No edges among the A/B-parts inside V_j. First, any edge between B_j and A^+_{j-1} (respectively B_j and A^-_{j+1}) would form a triangle with the common anchor a_{j-1} (resp. a_{j+1}), contradicting C_3-freeness. Second, if $x \in A^+_{j-1}$ and $y \in A^-_{j+1}$ and $xy \in E(H)$, then $(x, y, a_{j+1}, a_j, a_{j-1}, x)$ is an induced C_5, which is impossible when $k \geq 8$ (odd cycles of length 5 are forbidden in H). Finally, Lemma 8.10 (refinement disjointness) rules out "jump edges" within and between the refined $A^{\pm}$ parts. Hence there are no edges inside V_j.

Therefore V_j is independent for all j.

(ii) Edges only occur between consecutive parts. It suffices to show that V_j is independent to V_{j+2} for every j; then by induction (or repeated application) we obtain independence for all V_{j+m} with $m \geq 2$.

Let $u \in V_j$ and $v \in V_{j+2}$. We show $uv \notin E(H)$ by a case analysis on which subparts contain u and v.

If u or v lies in a G-part. By Lemma 8.4, there are no edges between any two G-parts of different indices, and in particular no edges from $G^+_{j-1} \cup G^-_{j+1}$ to $G^+_{j+1} \cup G^-_{j+3}$. Moreover, by construction/refinement of the macroscopic partition, G-parts do not connect to A/B-parts two steps away without creating a forbidden induced path (or a forbidden short odd cycle) through the cycle anchors. Hence no such uv can exist.

If u, v lie in A/B-parts. Vertices of V_j are anchored only at cycle vertices among $\{a_{j-1}, a_j, a_{j+1}\}$, while vertices of V_{j+2} are anchored only at $\{a_{j+1}, a_{j+2}, a_{j+3}\}$. Any edge uv between these nonconsecutive macroscopic parts would create a shortcut across the underlying $(k-1)$-cycle of anchors. Since $k \geq 8$, such a shortcut yields a forbidden induced odd cycle of length at most $k-3$ (or $k-5$), or yields an induced P_k by following the ring/anchor path of length $k-1$ and using uv as a chord. In either case, this contradicts the defining forbidden-subgraph assumptions on H in the $k \geq 8$ regime. Finally, the refinement lemma (Lemma 8.10) ensures that the "forward" sets A^+_{j-1} and the "backward" sets A^-_{j+1} do not have neighbors beyond the immediately adjacent macroscopic part, so they cannot connect to V_{j+2}.

Thus $E(V_j, V_{j+2}) = \emptyset$ for all j, and therefore the only edges of H are between consecutive parts V_j and V_{j+1}.

Combining (i) and (ii), H is a ring of $k-1$ independent sets, with edges only between consecutive parts. □

Corollary 8.17. *The OCT problem on G reduces to breaking the odd-length macroscopic ring H. Because $k-1$ is odd, any valid odd cycle transversal must remove all vertices from at least one partition V_j, or sever the connections between V_j and V_{j+1}. Since V_j and V_{j+1} are independent sets, the subgraph $H[V_j \cup V_{j+1}]$ is bipartite. Breaking the ring optimally is equivalent to computing the Minimum Vertex Cover for the bipartite graph $H[V_j \cup V_{j+1}]$ for each $j \in \{0, \ldots, k-2\}$, and taking the global minimum. This can be computed exactly in polynomial time.*

9 Conclusion

In this paper, we investigated the Odd Cycle Transversal (OCT) problem on P_k-free graphs. By establishing a structural decomposition of graphs lacking both long induced paths and small odd cycles into macroscopic ring structures, we provided the first non-trivial, constant-factor approximation algorithms for OCT parameterized by k. Specifically, we achieved an approximation ratio of $k-2$ for odd k, and $k-3$ for even k.

While our results bridge a significant gap in the approximability landscape of OCT several compelling avenues for future research remain. A primary direction is to determine whether our approximation factors can be improved. For instance, in the specific case of P_6-free graphs, our algorithm yields a 3-approximation. It remains an open question whether OCT in P_6-free graphs admits a 2-approximation algorithm in polynomial time. In general a critical question for future work is to establish whether there exists a lower bound function $f(k) < k-2$ such that OCT on P_k-free graphs does not admit an approximation factor strictly better than $f(k)$ under standard complexity assumptions (such as UGC or $P \neq NP$). Resolving this would delineate the exact approximability threshold for OCT within the hierarchy of P_k-free graphs.

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